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MODIFIED BETA APPROXIMATING OPERATORS OF THE FIRST AND SECOND KIND

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Abstract. By using the modified beta distributions of the first kind (MB1) and of the second kind (MB2) we obtain a general class of modified beta first kind operators and modified beta second kind operators. We obtain several positive linear operators as a special case of this modified beta operators.

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1. INTRODUCTION

We continue our earlier investigations [5], [6], [7], [8], [9], [10], [11], [12] concerning to use Euler's beta function for constructing linear positive operators.

We first review the definitions of the generalized beta distributions of the first kind (GB1) and the second kind (GB2) and special cases.

The Euler beta distribution of the first kind is defined for $p, q > 0$ by the following formula

$$\mathcal{B}(t; p, q) = \tfrac{1}{B(p,q)} t^{p-1}(1-t)^{q-1} \quad \text{for} \quad t \in (0,1), \tag{1.1}$$

and zero otherwise, where $B(p, q)$ is the beta function.

The generalized beta distribution of the first kind is defined by the probability density function (pdf) (see [4])

$$GB1(y; e, d, p, q) = \tfrac{|e| y^{ep-1}(1-(y/d)^e)^{q-1}}{d^{ep} B(p,q)} \quad \text{for} \quad 0 < y^e < d^e \tag{1.2}$$

and zero otherwise, where the parameters d, p, q are positive.

The defined kth-order moments of GB1 random variables are given by [4]

$$E_{GB1}(y^k) = \tfrac{d^k B(p+k/e,q)}{B(p,q)} \quad \text{for} \quad p + k/e > 0. \tag{1.3}$$

This four-parameter pdf is very flexible and includes the modified beta distribution of the first kind for $e = 1$.

$$MB1(y; d, p, q) = GB1(y; e = 1, d, p, q) = \tfrac{y^{p-1}(d-y)^{q-1}}{d^{p+q-1} B(p,q)}, \quad 0 < y < d. \tag{1.4}$$

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The distribution $MB1(y;d,p,q)$ with $p,q>0$ and $0<d\le 1$ is said to be the Beta-Stacy (BS) distribution (see [13]).

The standard beta distribution of the first kind (1.1) corresponds to (1.4) with $b=1$.

The Euler beta distribution of the second kind is defined for $p,q>0$ by the following formula

$$\mathcal{B}(u;p,q)=\frac{1}{B(p,q)}\cdot\frac{u^{p-1}}{(1+u)^{p+q}},\quad u>0 \tag{1.5}$$

and zero otherwise.

The generalized beta distribution of the second kind is defined by the pdf (see [4])

$$GB2(v;e,d,p,q)=\frac{|e|v^{ep-1}}{d^{ep}B(p,q)(1+(v/d)^{e})^{p+q}}\quad\text{for}\quad v>0 \tag{1.6}$$

and zero otherwise.

The defined kth-order moments of the GB2 are given by [4]

$$E_{GB2}(v^k)=\frac{d^kB(p+k/e,q-k/e)}{B(p,q)}\quad\text{for}\quad -p<k/e<q \tag{1.7}$$

Letting $e=1$ in (1.6) gives the modified beta distribution of the second kind (MB2)

$$MB2(v;d,p,q)=GB2(v,e=1,d,p,q)=\frac{d^qv^{p-1}}{B(p,q)(d+v)^{p+q}},\quad v>0. \tag{1.8}$$

The standard beta distribution of the second kind (1.5) is obtained by (1.8) for $d=1$.

2. THE MODIFIED BETA FIRST KIND OPERATORS

Let us denote by $M[0,\infty)$ the linear space of functions defined on $[0,\infty)$, bounded and Lebesgue measurable in each interval $[c,d]$, $0<d<c<\infty$.

By using the modified beta distribution of the first kind (MB1), defined by (1.4) we can define the following general transform:

$$\mathcal{B}_{p,q}^{(a,b)}f=\frac{1}{d^{p+q-1}B(p,q)}\int_0^d y^{p-1}(d-y)^{q-1}f(y^a(d-y)^b)\mathrm{d}y$$

or, equivalent

$$\mathcal{B}_{p,q}^{(a,b)}f=\frac{1}{B(p,q)}\int_0^1 t^{p-1}(1-t)^{q-1}f(d^{a+b}t^a(1-t)^b)\mathrm{d}t \tag{2.1}$$

where $a,b\in\mathbb{R}$ and $f\in M[0,\infty)$ such that $\mathcal{B}_{p,q}^{(a,b)}|f|<\infty$.

We determine d such that $\mathcal{B}_{p,q}^{(a,b)}e_1=e_1$, that is $d^{a+b}=\frac{B(p,q)x}{B(p+a,q+b)}$, and we obtain from (2.1) the (a,b)-modified beta operator (see also [9])

$$(\mathcal{B}_{p,q}^{(a,b)}f)(x)=\frac{1}{B(p,q)}\int_0^1 t^{p-1}(1-t)^{q-1}f\left(\frac{B(p,q)x}{B(p+a,q+b)}t^a(1-t)^b\right)\mathrm{d}t \tag{2.2}$$

where $f\in M[0,\infty)$ such that $(\mathcal{B}_{p,q}^{(a,b)}|f|)(x)<\infty$.

If we put in (2.2) $b=0$ we obtain the modified beta first kind operator

$$(2.3)\qquad (\mathcal{B}_{p,q}^{(a)}f)(x)=\tfrac{1}{B(p,q)}\int_0^1 t^{p-1}(1-t)^{q-1}f\left(\tfrac{B(p,q)}{B(p+a,q)}t^a x\right)\mathrm{d}t$$

where $f\in M[0,\infty)$ such that $(\mathcal{B}_{p,q}^{(a)}|f|)(x)<\infty$.

One observe that $\mathcal{B}_{p,q}^{(a)}$ is a linear positive operator.

I. If we choose in (2.3) $a=1$ we obtain the modified beta first kind operator

$$(2.4)\qquad (\mathcal{B}_{p,q}f)(x)=(\mathcal{B}_{p,q}^{(1)}f)(x)=\tfrac{1}{B(p,q)}\int_0^1 t^{p-1}(1-t)^{q-1}f\left(\tfrac{p+q}{p}tx\right)\mathrm{d}t.$$

Theorem 2.1. *The moments of order k of the operator $\mathcal{B}_{p,q}$ have the following values*

$$(\mathcal{B}_{p,q}e_k)(x)=\left(\tfrac{p+q}{p}x\right)^k\tfrac{(p)_k}{(p+q)_k}.$$

Proof. The result follows from (1.3) for $e=1$. □

Consequently, we obtain

$$(2.5)\qquad \mathcal{B}_{p,q}((t-x)^2;x)=\tfrac{qx^2}{p(p+q+1)}.$$

Special cases 1. If we choose in (2.4) $p>0$, $q>0$ such that $\frac{p}{p+q}=x$, or $p=\frac{\beta}{\alpha}x$, $q=\frac{\beta}{\alpha}(1-x)$, $x\in(0,1)$, $\alpha,\beta>0$ then we obtain the following linear positive operators, considered by the author in [11] (see also [7])

$$(2.6)\qquad (\mathcal{B}^{(\alpha,\beta)}f)(x)=\frac{1}{B\left(\frac{\beta}{\alpha}x,\frac{\beta}{\alpha}(1-x)\right)}\int_0^1 t^{\frac{\beta}{\alpha}x-1}(1-t)^{\frac{\beta}{\alpha}(1-x)-1}f(t)\mathrm{d}t.$$

2. If we put in (2.4) $p=\frac{n}{n+1}$, $q=\frac{1}{n+1}$, $n\in\mathbb{N}$, we obtain a new positive linear operator

$$(2.7)\qquad (\mathcal{B}_nf)(x)=\frac{1}{B\left(\frac{n}{n+1},\frac{1}{n+1}\right)}\int_0^1 t^{-\frac{1}{n+1}}(1-t)^{-\frac{n}{n+1}}f\left(\tfrac{n+1}{n}tx\right)\mathrm{d}t.$$

Corollary 2.2. *The following relation holds*

$$\mathcal{B}_n((t-x)^2;x)=\tfrac{x^2}{2n}.$$

Proof. It is obtained from (2.5) for $p=\frac{n}{n+1}$ and $q=\frac{1}{n+1}$. □

For another special cases of the operator (2.4) see [9].

II. If we put $a=-1$ in (2.3) we obtain the modified beta first kind operator

$$(2.8)\qquad (B_{p,q}f)(x)=(\mathcal{B}_{p,q}^{(-1)}f)(x)=\tfrac{1}{B(p,q)}\int_0^1 t^{p-1}(1-t)^{q-1}f\left(\tfrac{p-1}{p+q-1}\cdot\tfrac{x}{t}\right)\mathrm{d}t$$

where $f\in M[0,\infty)$ such that $(B_{p,q}|f|)(x)<\infty$.

THEOREM 2.3. *The moments of order k $(1 \le k < p)$ of the operator $B_{p,q}$ have the following values*

$$(B_{p,q}e_k)(x) = \frac{(p+q-1)\ldots(p+q-k)}{(p-1)\ldots(p-k)}\left(\frac{p-1}{p+q-1}\cdot x\right)^k.$$

Proof. The result follows from (1.3) for $e=-1$. □

Consequently, we obtain

$$B_{p,q}((t-x)^2;x) = \frac{q}{(p-2)(p+q-1)}x^2, \quad p>2. \tag{2.9}$$

Special cases

1. If we choose in (2.8) $p>0$ and $q>0$ such that $\frac{p+q-1}{p-1}=x$, $x>1$ or $p=1+\frac{\beta}{\alpha}$, $q=\frac{\beta}{\alpha}(x-1)$, $x>1$, $\alpha,\beta>0$, $\beta>\alpha$ then we obtain the following linear positive operators, considered by the author in [11] (see also [7])

$$(B^{(\alpha,\beta)}f)(x) = \frac{1}{B\left(1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}(x-1)\right)}\int_0^1 t^{\frac{\beta}{\alpha}}(1-t)^{\frac{\beta}{\alpha}(x-1)-1}f\left(\frac{1}{t}\right)\mathrm{d}t. \tag{2.10}$$

2. If we put in (2.7) $p=2+\frac{n}{n+1}$, $q=\frac{1}{n+1}$, $n\in\mathbb{N}$ we obtain a new positive linear operator

$$(B_nf)(x) = \frac{1}{B\left(2+\frac{n}{n+1},\frac{1}{n+1}\right)}\int_0^1 t^{1+\frac{n}{n+1}}(1-t)^{-\frac{n}{n+1}}f\left(\frac{2n+1}{2n+2}\cdot\frac{1}{t}\right)\mathrm{d}t. \tag{2.11}$$

COROLLARY 2.4. *The following relation holds*

$$B_n((t-x)^2;x) = \frac{x^2}{2n}.$$

Proof. It is obtained from (2.9) for $p=2+\frac{n}{n+1}$ and $q=\frac{1}{n+1}$. □

For another special cases of the operator (2.8) see [9].

3. THE MODIFIED BETA SECOND KIND OPERATORS

We shall define a linear transform by using the modified beta distribution of the second kind (MB2), defined by (1.8)

$$\mathcal{T}_{p,q}^{(a,b)}f = \frac{d^q}{B(p,q)}\int_0^\infty \frac{v^{p-1}}{(d+v)^{p+q}}f\left(\frac{v^a}{(d+v)^{a+b}}\right)\mathrm{d}v$$

or, equivalent

$$\mathcal{T}_{p,q}^{(a,b)}f = \frac{1}{B(p,q)}\int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}}f\left(\frac{1}{d^b}\cdot\frac{u^a}{(1+u)^{a+b}}\right)\mathrm{d}u \tag{3.1}$$

where $a,b\in\mathbb{R}$ and $f\in M[0,\infty)$, such that $\mathcal{T}_{p,q}^{(a,b)}|f|<\infty$.

We determine d such that $\mathcal{T}_{p,q}^{(a,b)}e_1=e_1$, that is $d^{-b}=\frac{B(p,q)}{B(p+a,q+b)}x$ and we obtain the (a,b)-modified beta operator (see also [10])

$$(\mathcal{T}_{p,q}^{(a,b)}f)(x) = \frac{1}{B(p,q)}\int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}}f\left(\frac{B(p,q)}{B(p+a,q+b)}\cdot\frac{u^a x}{(1+u)^{a+b}}\right)\mathrm{d}u \tag{3.2}$$

where $f \in M[0,\infty)$, such that $(\mathcal{T}_{p,q}^{(a,b)}|f|)(x) < \infty$.

If we put in (3.2) $a+b=0$ we obtain the modified beta second kind operator

$$(\mathcal{T}_{p,q}^{(a)} f)(x) = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f\left(\frac{B(p,q)u^a x}{B(p+a,q-a)}\right) \mathrm{d}u \tag{3.3}$$

where $f \in M[0,\infty)$, such that $(\mathcal{T}_{p,q}^{(a)}|f|)(x) < \infty$.

One observe that $\mathcal{T}_{p,q}^{(a)}$ is a linear positive operator.

If we choose in (3.3) $a=1$ we obtain the modified beta second kind operator

$$(\mathcal{T}_{p,q} f)(x) = (\mathcal{T}_{p,q}^{(1)} f)(x) = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f\left(\frac{q-1}{p} ux\right) \mathrm{d}u. \tag{3.4}$$

THEOREM 3.1. *The moment of order k $(1 \le k < q)$ of the operator $\mathcal{T}_{p,q}$ has the following value*

$$(\mathcal{T}_{p,q} e_k)(x) = \left(\frac{q-1}{p} x\right)^k \frac{p(p+1)\ldots(p+k-1)}{(q-1)\ldots(q-k)}, \quad 1 \le k < q.$$

Proof. The result follows from (1.7) for $e=1$. □

Consequently, we obtain

$$\mathcal{T}_{p,q}((t-x)^2;x) = \frac{p+q-1}{p(q-2)} x^2, \quad q > 2.$$

Special cases

1. If we choose in (3.4) $p > 0$, $q > 0$ such that $\frac{p}{q-1} = x$, or $p = \frac{\beta}{\alpha} x$, $q = 1 + \frac{\beta}{\alpha}$, $x > 0$, $\alpha, \beta > 0$ then we obtain the following linear positive operator, considered by author in [12] (see also [8])

$$(\mathcal{T}^{(\alpha,\beta)} f)(x) = \frac{1}{B\left(\frac{\beta}{\alpha} x, 1 + \frac{\beta}{\alpha}\right)} \int_0^\infty \frac{u^{\frac{\beta}{\alpha} - 1}}{(1+u)^{1+\frac{\beta}{\alpha}(x+1)}} f(u) \mathrm{d}u. \tag{3.5}$$

2. For $p=q=n$ we obtain from (3.4) the operator

$$(T_n f)(x) = \frac{1}{B(n,n)} \int_0^\infty \frac{u^{n-1}}{(1+u)^{2n}} f\left(\frac{n-1}{n} ux\right) \mathrm{d}u \tag{3.6}$$

considered by V. Totik in [14].

3. A double sequence of positive linear operators (names beta operators by Uprety [15], see also [3]) defined as

$$(F_{m,n} f)(x) = \frac{1}{B(m,n)} \int_0^\infty \frac{u^{m-1}}{(1+u)^{m+n}} f\left(\frac{n}{m} ux\right) \mathrm{d}u \tag{3.7}$$

is similar with (3.4).

4. For $q=p+1$ we obtain by (3.4) the positive linear operator

$$(\mathcal{T}_p f)(x) = \frac{1}{B(p,p+1)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{2p+1}} f(ux) \mathrm{d}u$$

considered by S.M. Mazhar for $p=n$ (see also [5]).

5. For $q = p + 2$ we obtain by (3.4) the positive linear operator

$$(T_p f)(x) = \frac{1}{B(p,p+2)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{2p+2}} f(ux)\mathrm{d}u$$

considered by H. Karski in [1], [2], for $p = n + 1$.

6. For $q = p + 3$ we obtain by (3.4) the positive linear operator

$$(T_p f)(x) = \frac{1}{B(p,p+3)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{2p+3}} f(ux)\mathrm{d}u.$$

For $p = n$ we obtain a new positive linear operator

$$(T_n f)(x) = \frac{1}{B(n,n+3)} \int_0^\infty \frac{u^{n-1}}{(1+u)^{2n+3}} f(ux)\mathrm{d}u$$

and

$$T_n((t-x)^2; x) = \tfrac{2}{n} x^2.$$

Remark 3.2. If we put $a = -1$ in (3.3) we obtain the following modified beta second kind operator

$$(\mathcal{T}_{p,q}^{(-1)} f)(x) = (\mathcal{T}_{p,q} f)(x) = \frac{1}{B(p,q)} \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} f\left(\frac{p-1}{q} \cdot \frac{x}{u}\right) \mathrm{d}u$$

and we observe that $\mathcal{T}_{p,q}$ is similar with T_{pq} (see [10]). □

REFERENCES

[1] Karsli, H., *Rate of convergence of new Gamma type operators for functions with derivatives of bounded variation*, Mathematical and Computer Modeling, **45**, pp. 617–624, 2007.

[2] Karsli, H. and Gupta, V., *Rate of pointwise convergence of a new kind of gamma operators for functions of bounded variation*, Appl. Math. Lett., 2007, doi:10.1016.

[3] Khan, M.R., *Approximation properties of beta operators*, in Progress in Approximation Theory, Academic Press, New York, pp. 483–495, 1991.

[4] McDonald, J.B., *Some generalized functions for the size distributions of icome*, Econometrica, **52**, pp. 647–663, 1984.

[5] Miheşan, V., *Approximation of continuous functions by means of positive linear operators*, Ph. D. Thesis, "Babeş-Bolyai" University, Cluj-Napoca, 1997.

[6] Miheşan, V., *On the gamma and beta approximation operators*, Gen. Math., **6**, pp. 61–64, 1998.

[7] Miheşan, V., *The beta approximating operators of first kind*, Studia Univ. Babeş-Bolyai, Mathematica, vol. XLIX, **2**, pp. 65–77, 2004.

[8] Miheşan, V., *The beta approximating operators of second kind*, Studia Univ. "Babeş-Bolyai", Mathemativa, vol. XLIX, **2**, pp. 79–88, 2004.

[9] Miheşan, V., *On the modified beta approximating operators of first kind*, Rev. Anal. Numér. Théor. Approx., **33**, no. 1, pp. 67–71, 2004.

[10] Miheşan, V., *On the modified beta approximating operators of second kind*, Rev. Anal. Numér. Théor. Approx., **34**, no. 2, pp. 135–138, 2005.

[11] Miheşan, V., *On a general class of beta approximating operators of first kind*, (in press).
[12] Miheşan, V., *On a general class of beta approximating operators of second kind*, (in press).
[13] Mihram, G.A. and Hultquist, R.A., *A bivariate warning-time/failure-time distribution*, Journal of the American Statistical Association, **62**, pp. 589–599, 1967.
[14] Totik, V., *Uniform approximation by positive operators on infinite interval*, Analysis Mathematica, **10**, pp. 162–182, 1984.
[15] Upreti, R., *Approximation properties of beta operators*, J. Approx. Theory, **45**, pp. 85–89, 1985.

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