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SOME APPROXIMATION PROPERTIES OF MODIFIED SZASZ-MIRAKYAN-KANTOROVICH OPERATORS

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Abstract. In this paper we consider the modified Szasz-Mirakyan-Kantorovich operators for functions f integrable in the sense of Denjoy-Perron. Moreover, we estimate the rate of pointwise convergence of $M_n f(x)$ at the Lebesgue-Denjoy points x of f .

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1. PRELIMINARIES

In [6], Z. Walczak introduced and considered some approximation properties of the modifed Szasz-Mirakyan operators defined by

$$M_n f(x) = \sum_{k=0}^{\infty} p_{n,k}(x;a_n) f(k/b_n) \quad (x \in R_0, n \in N),$$

where $p_{n,k}(x;a_n) = \exp(-a_n x)\frac{(a_n x)^k}{k!}$, $R_0 = [0;\infty)$, $N = \{1,2,...\}$ and $(a_n)_1^\infty$, $(b_n)_1^\infty$ are given increasing and unbounded sequences of positive numbers, such that

$$\lim_{n\to\infty} \tfrac{1}{b_n} = 0 \quad \text{and} \quad \tfrac{a_n}{b_n} = 1 + O\left(\tfrac{1}{b_n}\right). \tag{1}$$

If $a_n = b_n = n$, for all $n \in N$, then M_n become the classical Szasz-Mirakyan operator examined for continuous and bounded function in [4]. Some approximation properties of the operators M_n, for continuous functions f on R_0, can be found e.g. in [1] and [6].

Denote by M_n^* the Kantorovich type modification of operator M_n and define it for measurable function f on R_0 as:

$$M_n^* f(x) = b_n \sum_{k=0}^{\infty} p_{n,k}(x;a_n) \int_{k/b_n}^{(k+1)/b_n} f(t)\mathrm{d}t \quad x \in R_0, n \in N, \tag{2}$$

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where the integral is taken in the sense of Lebesgue or Denjoy-Perron. Convergence Theorems for the classical Szasz-Mirakyan-Kantorovich operators ($a_n = b_n = n$) are presented in [2], [3]. Some approximation properties of $M_n^* f$ can be found in [7].

The aim of this paper is to examine the rate of the convergence of operators $M_n^* f$, mainly, at those points $x \in R_0$ at which

$$\lim_{h\to 0} \tfrac{1}{h} \int_0^h (f(x+t) - f(x)\mathrm{d}t = 0.$$

The general estimate is expressed in terms of the quantity

$$w_x(\delta; f) = \sup_{0<|h|\leq\delta} \left| \tfrac{1}{h} \int_0^h (f(x+t) - f(x))\mathrm{d}t \right| \quad (\delta > 0).$$

Clearly, if f is locally integrable in the sense of Denjoy-Perron on R_0, then

$$\lim_{\delta\to 0+} w_x(\delta; f) = 0 \quad \text{for almost every } x.$$

In view of this property, we deduce that in some classes of functions,

$$\lim_{n\to\infty} M_n^* f(x) = f(x) \quad \text{almost everywhere.}$$

Moreover, using some other properties of $w_x(\delta; f)$, we present the estimate of the rate of $M_n^* f(x)$ the pointwise convergence of in terms of the weighted moduli of continuity.

Throughout the paper, the symbol $K(...)$, $K_j(...)$, $(j = 1, 2, ...)$ will mean some positive constants, not necessarily the same at each occurrence, depending only on the parameters indicated in parentheses.

2. AUXILIARY ESTIMATES

It is easy to see, that for every $x \in R_0$ and for all integers $n \in N$,

$$\sum_{k=0}^{\infty} p_{n,k}(x; a_n) = 1, \tag{3}$$

$$\sum_{k=0}^{\infty} \left(\tfrac{k}{b_n} - x\right) p_{n,k}(x; a_n) = x\left(\tfrac{a_n}{b_n} - 1\right), \sum_{k=0}^{\infty} \left(\tfrac{k}{a_n} - x\right) p_{n,k}(x; a_n) = 0. \tag{4}$$

By properties (1), we have that exists a positive, absolute constants K, K_1, such that

$$\left|\tfrac{a_n}{b_n} - 1\right| \leq \tfrac{K}{b_n}, \quad \tfrac{1}{a_n} \leq \tfrac{K}{b_n} \quad \text{and} \quad \tfrac{a_n}{b_n} \leq K_1 \tag{5}$$

for all $n \in N$.

For $q \in N_0 = N \cup \{0\}$, $l \in N_0$, $n \in N$ we define

$$S_{q,l}^{(n)}(x; b_n) = b_n \sum_{k=0}^{\infty} \left(\tfrac{k}{b_n} - x\right)^q \left(\tfrac{k}{a_n} - x\right)^l p_{n,k}(x; a_n). \tag{6}$$

LEMMA 1. *Suppose that* $q \in N$ *and* $(a_n)_1^\infty$,$(b_n)_1^\infty$ *are fixed. Then, for* $n \in N$, $x \in R_0$ *we have*

$$S_{q,0}^{(n)}(x;b_n) = xa_n \sum_{j=0}^{q-2} \binom{q-1}{j} \frac{1}{b_n^{q-j}} S_{j,0}^{(n)}(x;b_n) + x\left(\frac{a_n}{b_n} - 1\right) S_{q-1,0}^{(n)}(x;b_n), \tag{7}$$

$$S_{q,1}^{(n)}(x;b_n) = x \sum_{j=0}^{q-1} \binom{q}{j} \frac{1}{b_n^{q-j}} S_{j,0}^{(n)}(x;b_n), \tag{8}$$

$$\begin{aligned} S_{q,2}^{(n)}(x;b_n) = {} & \frac{x}{a_n} \sum_{j=0}^{q} \binom{q}{j} \frac{1}{b_n^{q-j}} S_{j,0}^{(n)}(x;b_n) \\ & + x^2 \sum_{j=1}^{q-1} \binom{q}{j} \frac{1}{b_n^{q-j}} \sum_{i=0}^{j-1} \binom{j}{i} \frac{1}{b_n^{j-i}} S_{i,0}^{(n)}(x;b_n). \end{aligned} \tag{9}$$

(Note that the symbol $\sum_{j=0}^{-1}$ ... denotes zero.)

Proof. In view of the definition (6)

$$\begin{aligned} S_{q,0}^{(n)}(x;b_n) &= \sum_{k=0}^{\infty} \exp(-a_n x) \frac{(a_n x)^k}{k!} \left(\frac{k}{b_n} - x\right)^q \\ &= \frac{a_n x}{b_n} \sum_{k=1}^{\infty} \exp(-a_n x) \frac{(a_n x)^{k-1}}{(k-1)!} \left(\frac{k}{b_n} - x\right)^{q-1} - xS_{q-1,0}^{(n)}(x;b_n) \\ &= \frac{a_n x}{b_n} \sum_{k=0}^{\infty} \exp(-a_n x) \frac{(a_n x)^k}{k!} \left(\frac{k+1}{b_n} - x\right)^{q-1} - xS_{q-1,0}^{(n)}(x;b_n) \\ &= \sum_{j=0}^{q-1} \binom{q-1}{j} \frac{a_n x}{b_n^{q-1}} \sum_{k=0}^{\infty} p_{n,k}(x;a_n) \left(\frac{k}{b_n} - x\right)^j - xS_{q-1,0}^{(n)}(x;b_n) \\ &= xa_n \sum_{j=0}^{q-2} \binom{q-1}{j} \frac{1}{b_n^{q-j}} S_{j,0}^{(n)}(x;b_n) + x\left(\frac{a_n}{b_n} - 1\right) S_{q-1,0}^{(n)}(x;b_n). \end{aligned}$$

This mean that (7) is true.

Next, assuming (6), we have

$$\begin{aligned} S_{q,1}^{(n)}(x;b_n) &= \sum_{k=0}^{\infty} \exp(-a_n x) \frac{(a_n x)^k}{k!} \left(\frac{k}{b_n} - x\right)^q \left(\frac{k}{a_n} - x\right) \\ &= \sum_{k=0}^{\infty} \exp(-a_n x) \frac{(a_n x)^{k+1}}{k!} \frac{1}{a_n} \left(\frac{k+1}{b_n} - x\right)^q - xS_{q,0}^{(n)}(x;b_n) \end{aligned}$$

$$
\begin{aligned}
&= x\sum_{j=0}^{q}\binom{q}{j}\tfrac{1}{b_n^{q-j}}S_{j,0}^{(n)}(x;b_n) - xS_{q,0}^{(n)}(x;b_n)\\
&= x\sum_{j=0}^{q-1}\binom{q}{j}\tfrac{1}{b_n^{q-j}}S_{j,0}^{(n)}(x;b_n).
\end{aligned}
$$

So (8) is true. Hence we get

$$
\begin{aligned}
S_{q,2}^{(n)}(x;b_n) \quad &= \sum_{k=0}^{\infty}\exp(-a_nx)\tfrac{(a_nx)^k}{k!}\left(\tfrac{k}{b_n}-x\right)^q\left(\tfrac{k}{a_n}-x\right)^2\\
&= x\sum_{k=0}^{\infty}\exp(-a_nx)\tfrac{(a_nx)^k}{k!}\left(\tfrac{k+1}{b_n}-x\right)^q\left(\tfrac{k+1}{a_n}-x\right)\\
&\quad -xS_{q,1}^{(n)}(x;b_n)\\
&= x\sum_{j=0}^{q}\binom{q}{j}\tfrac{1}{b_n^{q-j}}\left(S_{j,1}^{(n)}(x;b_n)+\tfrac{1}{a_n}S_{j,0}^{(n)}(x;b_n)\right) - xS_{q,1}^{(n)}(x;b_n)\\
&= x\sum_{j=0}^{q-1}\binom{q}{j}\tfrac{1}{b_n^{q-j}}\left(S_{j,1}^{(n)}(x;b_n)+\tfrac{1}{a_n}S_{j,0}^{(n)}(x;b_n)\right) + \tfrac{x}{a_n}S_{q,0}^{(n)}(x;b_n).
\end{aligned}
$$

Applying (8) the desired equality (9) follows. □

LEMMA 2. *For every $n \in N$, $q \in N$ and $q \in R_0$ the following inequality is true*

$$
|S_{q,2}^{(n)}(x;b_n)| \le K(q)\left(x^2(1+x)^{q-1}\tfrac{1}{b_n^{[(q+3)/2]}} + x(x+1)\tfrac{1}{b_n^{q+1}}\right), \tag{10}
$$

where $(a_n)_1^{\infty}$, $(b_n)_1^{\infty}$ are fixed.

Proof. Using (4), (7) and the method of induction one can easily verify that for all $n \in N$, $q \in N$, $x \in R_0$ there holds

$$
|S_{q,0}^{(n)}(x;b_n)| \le K(q)x(1+x)^{q-1}\tfrac{1}{b_n^{[(q+1)/2]}} \tag{11}
$$

Simple calculation, (3), (4) give us

$$
\begin{aligned}
S_{0,2}^{(n)}(x;b_n) \quad &= \sum_{k=0}^{\infty}\exp(-a_nx)\tfrac{(a_nx)^k}{k!}\left(\tfrac{k}{a_n}-x\right)^2\\
&= \tfrac{1}{a_n}\sum_{k=1}^{\infty}\exp(-a_nx)\tfrac{(a_nx)^k}{(k-1)!}\left(\tfrac{k}{a_n}-x\right) - xS_{0,1}^{(n)}(x;b_n)\\
&= x\sum_{k=0}^{\infty}\exp(-a_nx)\tfrac{(a_nx)^k}{k!}\left(\tfrac{k+1}{a_n}-x\right) - xS_{0,1}^{(n)}(x;b_n)\\
&= \tfrac{x}{a_n}S_{0,0}^{(n)}(x;b_n) = \tfrac{x}{a_n},
\end{aligned}
$$

and

$$
\begin{aligned}
S_{1,2}^{(n)}(x;b_n) &= \sum_{k=0}^{\infty} \exp(-a_n x)\tfrac{(a_n x)^k}{k!}\left(\tfrac{k}{a_n}-x\right)^2\left(\tfrac{k}{b_n}-x\right)\\
&= \tfrac{1}{b_n}\sum_{k=1}^{\infty} \exp(-a_n x)\tfrac{(a_n x)^k}{(k-1)!}\left(\tfrac{k}{a_n}-x\right)^2 - xS_{0,2}^{(n)}(x;b_n)\\
&= \tfrac{xa_n}{b_n}\sum_{k=0}^{\infty} \exp(-a_n x)\tfrac{(a_n x)^k}{k!}\left(\left(\tfrac{k}{a_n}-x\right)^2+\tfrac{1}{a_n^2}\right.\\
&\quad \left.+\tfrac{2}{a_n}\left(\tfrac{k}{a_n}-x\right)\right) - xS_{0,2}^{(n)}(x;b_n)\\
&= \tfrac{xa_n}{b_n}\left(S_{0,2}^{(n)}(x;b_n)+\tfrac{1}{a_n^2}S_{0,0}^{(n)}(x;b_n)+\tfrac{2}{a_n}S_{0,1}^{(n)}(x;b_n)\right)\\
&\quad -xS_{0,2}^{(n)}(x;b_n)\\
&= \tfrac{xa_n}{b_n}\left(\tfrac{x}{a_n}+\tfrac{1}{a_n^2}\right) - x\tfrac{x}{a_n} = \tfrac{x^2}{a_n}\left(\tfrac{a_n}{b_n}-1\right)+\tfrac{x}{a_n b_n}.
\end{aligned}
$$

Applying (5) we obtained (10) for $q=1$. Hence

$$
\begin{aligned}
S_{2,2}^{(n)}(x;b_n) &= \tfrac{x}{a_n}\sum_{j=0}^{2}\tbinom{2}{j}\tfrac{1}{b_n^{2-j}}S_{j,0}^{(n)}(x;b_n) + x^2\tfrac{2}{b_n}\tfrac{1}{b_n}S_{0,0}^{(n)}(x;b_n)\\
&= \tfrac{x}{a_n}\left(\tfrac{1}{b_n^2}S_{0,0}^{(n)}(x;b_n)+\tfrac{2}{b_n}S_{1,0}^{(n)}(x;b_n)+S_{2,0}^{(n)}(x;b_n)\right)\\
&\quad +\tfrac{2x^2}{b_n^2}S_{0,0}^{(n)}(x;b_n).
\end{aligned}
$$

Clearly, in view of (3), (5) and (11)

$$
\begin{aligned}
|S_{2,2}^{(n)}(x;b_n)| &\le K\tfrac{x}{a_n}\left(\tfrac{1}{b_n^2}+\tfrac{x}{b_n^2}+x(1+x)\tfrac{1}{b_n}\right)+\tfrac{2x^2}{b_n^2}\\
&\le K_1\left\{x^2(1+x)\tfrac{1}{b_n}\left(\tfrac{1}{a_n}+\tfrac{1}{b_n}\right)+\tfrac{x(1+x)}{a_n b_n^2}\right\}\\
&\le K_2\left\{x^2(1+x)\tfrac{1}{b_n^2}+\tfrac{x(1+x)}{b_n^3}\right\}.
\end{aligned}
$$

Hence for $q=2$ the inequality (10) is true.

Analogously we have (10) for $q=3$

$$
\begin{aligned}
&S_{3,2}^{(n)}(x;b_n) =\\
&= \left|\tfrac{x}{a_n}\sum_{j=0}^{3}\tbinom{3}{j}\tfrac{1}{b_n^{3-j}}S_{j,0}^{(n)}(x;b_n) + x^2\sum_{j=1}^{2}\tbinom{3}{j}\tfrac{1}{b_n^{3-j}}\sum_{i=0}^{j-1}\tbinom{j}{i}\tfrac{1}{b_n^{j-i}}S_{i,0}^{(n)}(x;b_n)\right|
\end{aligned}
$$

$$
\begin{aligned}
&= \Big| \tfrac{x}{a_n} \left(\tfrac{1}{b_n^3} S_{0,0}^{(n)}(x;b_n) + \tfrac{3}{b_n^2} S_{1,0}^{(n)}(x;b_n) + \tfrac{3}{b_n} S_{2,0}^{(n)}(x;b_n) + S_{3,0}^{(n)}(x;b_n) \right) \\
&\quad + x^2 \tfrac{3}{b_n^2} \tfrac{1}{b_n} S_{0,0}^{(n)}(x;b_n) + x^2 \tfrac{3}{b_n} \left(\tfrac{1}{b_n^2} S_{0,0}^{(n)}(x;b_n) + \tfrac{2}{b_n} S_{1,0}^{(n)}(x;b_n) \right) \Big| \\
&\le \tfrac{x}{a_n} \left(\tfrac{1}{b_n^3} + Kx \tfrac{1}{b_n^3} + x(1+x) \tfrac{1}{b_n^2} + x(1+x)^2 \tfrac{1}{b_n^2} \right) + K \left(x^2 \tfrac{1}{b_n^3} + x^3 \tfrac{1}{b_n^3} \right) \\
&\le K_1 \left(\tfrac{x^2(1+x)^2}{b_n^3} + \tfrac{x(1+x)}{b_n^4} \right).
\end{aligned}
$$

If $q \ge 4$ then, in view of (9),

$$
\begin{aligned}
&|S_{q,2}^{(n)}(x;b_n)| \le \\
&\le \tfrac{x}{a_n} \sum_{j=0}^{q} \tbinom{q}{j} \tfrac{1}{b_n^{q-j}} |S_{j,0}^{(n)}(x;b_n)| + x^2 \sum_{j=1}^{q-1} v \tfrac{1}{b_n^{q-j}} \sum_{i=0}^{j-1} \tbinom{j}{i} \tfrac{1}{b_n^{j-i}} |S_{i,0}^{(n)}(x;b_n)| \\
&= \tfrac{x}{a_n} \left(\tfrac{1}{b_n^q} |S_{0,0}^{(n)}(x;b_n)| + \tfrac{q}{b_n^{q-1}} |S_{1,0}^{(n)}(x;b_n)| + \sum_{j=2}^{q} \tbinom{q}{j} \tfrac{1}{b_n^{q-j}} |S_{j,0}^{(n)}(x;b_n)| \right) \\
&\quad + x^2 \Bigg(\tfrac{q}{b_n^{q-1}} \tfrac{1}{b_n} |S_{0,0}^{(n)}(x;b_n)| + \tbinom{q}{2} \tfrac{1}{b_n^{q-2}} \left(\tfrac{1}{b_n^2} |S_{0,0}^{(n)}(x;b_n)| + \tfrac{2}{b_n} |S_{1,0}^{(n)}(x;b_n)| \right) \\
&\quad + \sum_{j=3}^{q-1} \tbinom{q}{j} \tfrac{1}{b_n^{q-j}} \Bigg(\sum_{i=2}^{j-1} \tbinom{j}{i} \tfrac{1}{b_n^{j-i}} |S_{i,0}^{(n)}(x;b_n)| \\
&\quad + \tfrac{1}{b_n^j} |S_{0,0}^{(n)}(x;b_n)| + j \tfrac{1}{b_n^{j-1}} |S_{1,0}^{(n)}(x;b_n)| \Bigg) \Bigg) \\
&\le \tfrac{x}{a_n} \left(\tfrac{1}{b_n^q} + \tfrac{q}{b_n^{q-1}} x \left| \tfrac{a_n}{b_n} - 1 \right| + \sum_{j=2}^{q} \tbinom{q}{j} \tfrac{1}{b_n^{q-j}} K(j) x (1+x)^{j-1} \tfrac{1}{b_n^{[(j+1)/2]}} \right) \\
&\quad + x^2 K_1(q) \Bigg(\tfrac{1}{b_n^q} + \tfrac{1}{b_n^{q-1}} x \left| \tfrac{a_n}{b_n} - 1 \right| \\
&\quad + \sum_{j=3}^{q-1} \tfrac{1}{b_n^{q-j}} \left(\sum_{i=2}^{j-1} \tfrac{1}{b_n^{j-1}} x(1+x)^{i-1} \tfrac{1}{b_n^{[(i+1)/2]}} + \tfrac{1}{b_n^j} + x \tfrac{1}{b_n^{j-1}} \left| \tfrac{a_n}{b_n} - 1 \right| \right) \Bigg).
\end{aligned}
$$

Consequently, from (5)

$$
\begin{aligned}
&|S_{q,2}^{(n)}(x;b_n)| \le \\
&\le K(q) \tfrac{x}{a_n} \left(\tfrac{1}{b_n^q} + \tfrac{1}{b_n^q} x + x(1+x)^{q-1} \tfrac{1}{b_n^{[(q+1)/2]}} \right) \\
&\quad + x^2 K(q) \left(\tfrac{1}{b_n^q} + \tfrac{1}{b_n^q} x + \sum_{j=3}^{q-1} \tfrac{1}{b_n^{q-j}} \left(x(1+x)^{j-2} \tfrac{1}{b_n^{[j/2+1]}} + \tfrac{1}{b_n^j} + x \tfrac{1}{b_n^j} \right) \right)
\end{aligned}
$$

$$\leq K_1(q)\left(x^2(1+x)^{q-1}\frac{1}{b_n^{[(q+3)/2]}}\right.$$
$$\left.+x^2\left(\frac{1+x}{b_n^q}+\frac{1+x}{b_n^{q-1}}+\frac{x(1+x)^{q-3}}{b_n^{[(q+1)/2]+1}}\right)+\frac{x(1+x)}{b_n^{q+1}}\right)$$
$$\leq K_2(q)\left(x^2(1+x)^{q-1}\frac{1}{b_n^{[(q+3)/2]}}+\frac{x(1+x)}{b_n^{q+1}}\right),$$

and inequality (10) follows now immediately. □

Identity (3), estimate (10) and the known Schwarz inequality lead to

LEMMA 3. *Let $q \in N_0$, $x \in R_0$. Then, for $n \in N$*

(12)
$$\sum_{k=0}^{\infty}\left|\frac{k}{a_n}-x\right|\left|\frac{k}{b_n}-x\right|^{q+1}p_{n,k}(x;a_n)\leq$$
$$\leq K(q)\left(\frac{x(1+x)^{q+1/2}}{b_n^{q/2+1}}+\frac{\sqrt{x(1+x)}}{b_n^{q+3/2}}\right),$$

where $(a_n)_1^\infty$, $(b_n)_1^\infty$, are fixed.

3. MAIN RESULT

In this Section, we consider only the points $x \in [0;\infty)$ at which $w_x(\delta;f) < \infty$ for all $\delta > 0$.

THEOREM 4. *Let $f : R_0 \to R$ is integrable in the Lebesgue or Denjoy-Perron sesnse on every compact interval contained in R_0 and let $n \in N$, $x \in R_0$. Given any number $q \in N$, we have*

(13)
$$|M_n^* f(x)-f(x)| \leq K(q)\left((1+x)^{q+1/2}+\left(\frac{1+x}{xb_n^{q+1}}\right)^{1/2}\right)$$
$$\sum_{v=0}^{\infty} w_x\left(\frac{v+1}{\sqrt{b_n}}\right)\frac{1}{(v+1)^q}.$$

Proof. For the sake of brevity we will write $f(x+t)-f(x)=\varphi_x(t)$ and $w_x(\delta;f)=w_x(\delta)$. In view of (3) we have

$$\begin{aligned}M_n^* f(x)-f(x) &= b_n\sum_{k=0}^{\infty}p_{n,k}(x;a_n)\int_{k/b_n}^{(k+1)/b_n}(f(t)-f(x))\mathrm{d}t\\ &= b_n\sum_{k=0}^{\infty}p_{n,k}(x;a_n)\int_{k/b_n-x}^{(k+1)/b_n-x}\varphi_x(t)\mathrm{d}t\\ &= b_n\sum_{k=1}^{\infty}(p_{n,k-1}(x;a_n)-p_{n,k}(x;a_n))\int_0^{k/b_n-x}\varphi_x(t)\mathrm{d}t\\ &\quad -b_n\exp(-a_n x)\int_0^{-x}\varphi_x(t)\mathrm{d}t.\end{aligned}$$

It is easy to see that

$$x(p_{n,k-1}(x;a_n)-p_{n,k}(x;a_n))=p_{n,k}(x;a_n)\left(\tfrac{k}{a_n}-x\right).$$

Consequently

$$x(M_n^*f(x)-f(x))=b_n\sum_{k=0}^{\infty}p_{n,k}(x;a_n)\left(\tfrac{k}{a_n}-x\right)\int_0^{k/b_n-x}\varphi_x(t)\mathrm{d}t.$$

Applying the obviously inequality $\left|\int_0^h\varphi_x(t)dt\right|\le|h|w_x(|h|)$, we obtain

$$\begin{aligned} x|M_n^*f(x)-f(x)| &\le b_n\sum_{k=0}^{\infty}p_{n,k}(x;a_n)\left|\tfrac{k}{a_n}-x\right|\left|\tfrac{k}{b_n}-x\right|w_x\left(\left|\tfrac{k}{b_n}-x\right|\right)\\ &\le\sum_{v=0}^{\infty}T_v^{(n)}(\lambda;x)w_x((v+1)\lambda), \end{aligned}$$

where λ is an arbitrary positive number and

$$T_v^{(n)}(\lambda;x)=\sum_{v\lambda<|k/b_n-x|\le(v+1)\lambda}b_n\left|\tfrac{k}{a_n}-x\right|\left|\tfrac{k}{b_n}-x\right|p_{n,k}(x;a_n).$$

We shall estimate the factors $T_v^{(n)}(\lambda;x)$ for $v=0$ and $v\ge1$, respectively. Clearly, in view of (12)

$$\begin{aligned} T_0^{(n)}(\lambda;x) &\le b_n\sum_{k=0}^{\infty}\left|\tfrac{k}{a_n}-x\right|\left|\tfrac{k}{b_n}-x\right|p_{n,k}(x;a_n)\\ &\le K\left(x(1+x)^{1/2}+\left(\tfrac{x(1+x)}{b_n}\right)^{1/2}\right). \end{aligned}$$

Next, if $v\ge1$, we have, by Lemma 3,

$$\begin{aligned} T_v^{(n)}(\lambda;x) &\le\tfrac{b_n}{v^q\lambda^q}\sum_{k=0}^{\infty}\left|\tfrac{k}{a_n}-x\right|\left|\tfrac{k}{b_n}-x\right|^{q+1}p_{n,k}(x;a_n)\\ &\le\tfrac{K(q)}{v^q\lambda^q}\left(\tfrac{x(1+x)^{q+1/2}}{b_n^{q/2}}+\tfrac{x^{1/2}(1+x)^{1/2}}{b_n^{q+1/2}}\right). \end{aligned}$$

Now, taking $\lambda=1/b_n^{1/2}$, we easily get

$$\begin{aligned} x|M_n^*f(x)-f(x)| &\le\left(x(1+x)^{q+1/2}+\tfrac{x^{1/2}(1+x)^{1/2}}{b_n^{(q+1)/2}}\right)\\ &\cdot\left(Kw_x\left(\tfrac{1}{\sqrt{b_n}}\right)+K(q)\sum_{v=1}^{\infty}w_x\left(\tfrac{v+1}{\sqrt{b_n}}\right)\tfrac{1}{(v+1)^q}\right). \end{aligned}$$

This last relation is equivalent to (13). □

REMARK 5. Let $D^*_{loc}(R)$ be the class of all functions integrable in the Denjoy-Perron sense on every compact interval contained in R. Clearly, if $f \in D^*_{loc}(R)$, then the function

$$F(x) = \int_0^x f(t)\mathrm{d}t$$

is ACG^* on every $[a;b] \subset R_0$ and $F'(x) = f(x)$ almost everywhere [5]. $\square$

Consequently,

$$\lim_{\delta \to 0+} w_x(\delta; f) = 0 \quad \text{a.e. on } R. \tag{14}$$

Suppose that $f \in D^*_{loc}(R)$ and that

$$||f|| \equiv \sup_{-\infty<v<\infty} \left(\sup_{0\le u\le 1} \left| \int_v^{v+u} f(t)\mathrm{d}t \right| \right) < \infty.$$

The operators $M^*_n f(x)$ are well-defined for all $n \in N$. This follows at once from the inequality

$$\left| \int_{k/n}^{(k+1)/n} f(t)\mathrm{d}t \right| \le ||f||.$$

In view of (14), given any $\epsilon > 0$ there is a $\delta_0 > 0$ such that $w_x(\delta; f) < \epsilon$ whenever $0 < \delta \le \delta_0$. In case $\delta \ge \delta_0$ we have

$$w_x(\delta; f) < \epsilon + |f(x)| + \sup_{\delta_0\le|h|\le\delta} \left| \tfrac{1}{h} \int_0^h f(x+t)\mathrm{d}t \right| \le \epsilon + |f(x)| + \tfrac{1}{\delta_0}||f||$$

provided that $\delta \le 1$. If $\delta > 1$, then putting $\mu = [\delta] + 1$ and $x_j = x + jh/\mu$ $(j = 0, 1, ..., \mu)$, we get

$$\left| \int_0^h f(x+t)\mathrm{d}t \right| \le \sum_{j=0}^{\mu-1} \left| \int_{x_j}^{x_{j+1}} f(t)\mathrm{d}t \right| \le \mu||f|| \le 2\delta||f||.$$

Hence

$$w_x(\delta; f) < \epsilon + |f(x)| + \tfrac{1}{\delta_0}(1+2\delta)||f|| \quad \text{for all} \quad \delta > 0.$$

This inequality and the condition (14) ensure that the right-hand side of the estimate (13) (with arbitrary $q \ge 3$) converges to zero as $n \to \infty$.

Let $m \in N_0$. Denote by $C_m(R_0)$ the class of all measurable functions f on R_0, such that

$$||f||_m = \sup_{x\in R} \tfrac{|f(x)|}{1+x^{2m}} < \infty.$$

It is easy to see, that operators $M^*_n f$ are well-defined for every function $f \in C_m(R_0)$.

Further, for continuous $f \in C_m(R_0)$, let us introduce the weighted modulus of continuity

$$\omega(\delta; f)_m = \sup_{|h|\le\delta} ||f(\cdot + h) - f(\cdot)|| \quad (\delta > 0).$$

Theorem 6. *Let $m \in N_0$, $x \in R_0$, $f \in C_m(R_0)$ and $(a_n)_1^\infty$, $(b_n)_1^\infty$ are fixed. Then*

$$(1+x^{2m})^{-1}|M_n^* f(x) - f(x)| \leq$$
$$\leq K(m)\left((1+x)^{2m+3} + \left(\tfrac{1+x}{x}\right)^{1/2} \tfrac{1}{b_n^{m+5/4}}\right)\omega(b_n^{-1/2}; f)_m,$$

for all $n \in N$.

Proof. It is easy to see, that for any $x \in R_0$, $r \in N$

$$\begin{aligned} |f(x+rt) - f(x)| &\leq \sum_{v=0}^{r-1} \tfrac{|f(x+vt+t)-f(x+vt)|}{1+(x+vt)^{2m}}(1+(x+vt)^{2m}) \\ &\leq \omega(|t|; f)_m \sum_{v=0}^{r-1}(1+(|x|+v|t|)^{2m}) \\ &\leq (1+(|x|+(r-1)|t|)^{2m} r\omega(|t|; f)_m. \end{aligned}$$

Therefore, for any $x \in R$, $\delta > 0$, $r \in N$

$$w_x(r\delta; f) \leq (1+(2x)^{2m} + (2(r-1)\delta)^{2m})r\omega(\delta; f)_m.$$

This inequality and Theorem 4, with $q = 2m + 5/2$, lead to Theorem 6. $\square$

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