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ON LOWER SEMICONTINUITY OF THE RESTRICTED CENTER MULTIFUNCTION*

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Abstract. Given a finite dimensional subspace V and a certain family $\mathcal{F}$ of nonempty closed and bounded subsets of $\mathcal{C}_0(T,U)$, where T is a locally compact Hausdorff space and U is a strictly convex Banach space, we investigate here lower semicontinuity of the *restricted center multifunction* $C_V:\mathcal{F}\rightrightarrows V$. In particular, we establish a Haar-like *intrinsic characterization* of finite dimensional subspaces V of $\mathcal{C}_0(T,U)$ which yields lower semicontinuity of C_V.

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1. INTRODUCTION

Let us be given a family $\mathcal{F}$ of nonempty closed and bounded subsets of a normed linear space X, and a finite dimensional subspace V of X. For $F\in\mathcal{F}$ and $x\in X$, let

$$r(F;x):=\sup\{\|x-y\|:y\in F\}$$

denote the radius of the smallest closed ball centered at x covering F and let

$$\begin{aligned} r_V(F) &:= \inf\{r(F;v):v\in V\},\\ C_V(F) &:= \{v_0\in V: r(F;v_0)=r_V(F)\}. \end{aligned}$$

The number $r_V(F)$ is called the *restricted (Chebyshev) radius* of F in V. It is easily seen that the set $C_V(F)$ is nonempty, closed and convex. A typical element $v_0\in C_V(F)$ is usually called a *restricted (Chebyshev) center* or a *best simultaneous approximant* of F in V. The multifunction $C_V:\mathcal{F}\rightrightarrows V$, with values $C_V(F), F\in\mathcal{F}$, is called the *restricted center multifunction*.

Let us note that in case F is a singleton $\{x\}, r_V(F)$ is the distance of x from V, denoted by $d(x,V)$, and $C_V(F)$ is the set

$$P_V(x):=\{v_0\in V:\|x-v_0\|=d(x,V)\}$$

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of best approximants to x in V. The multifunction $P_V : X \rightrightarrows V$, in this case, is usually called the *metric projection* onto V. The problems concerning various continuities of metric projection in the Banach space $\mathcal{C}_0(T)$ of real-valued continuous functions vanishing at infinity have been significantly investigated by a number of authors (cf., e.g., [2], [3], [5], [9], [6], [7], [12], [16]). Some of these results pertaining to lower semicontinuity of metric projection have been generalized to the space $\mathcal{C}_0(T, U)$, where U is a strictly convex Banach space in ([4], [11], [12]).

The problems concerning various continuities of the restricted center multifunction have received some attention recently (cf., e.g., [14], [20], [1], ([15] Ch. 8, §5), [18], [8], [19]).

Given a finite dimensional subspace V of an arbitrary real normed linear space X, we investigate here a sufficient condition for lower semicontinuity of the restricted center multifunction C_V defined on a certain family $\mathcal{F}$ of subsets of X. This extends certain results in [4] for lower semicontinuity of metric projection. In [8], a characterization of lower semicontinuity of restricted center multifunction defined on a certain family $\mathcal{F}$ of subsets of the space $\mathcal{C}_0(T)$ (Theorem 3.2) was studied. This result was in the same spirit as that of the particular case of ([11], Theorem 4.1) for metric projections. Here we extend this investigation to the space $\mathcal{C}_0(T, U)$. This approach naturally leads us to our main goal of exploring an intrinsic Haar-like characterization of finite-dimensional subspaces V of $\mathcal{C}_0(T, U)$, for which the restricted center multifunction $C_V : \mathcal{F} \rightrightarrows V$, with values $C_V(F), F \in \mathcal{F}$, is lower semicontinuous.

2. PRELIMINARIES

Throughout the following, X will be a real normed linear space which for the most part will be the Banach space $\mathcal{C}_0(T, U)$, where T is a locally compact Hausdorff space and U is a strictly convex (real) Banach space, and V will be a finite dimensional subspace of X.

Let us recall that $\mathcal{C}_0(T, U)$ consists of all continuous functions $f : T \to U$ vanishing at infinity, i.e, a continuous function f is in $\mathcal{C}_0(T, U)$ if and only if, for every $\epsilon > 0$, the set $\{t \in T : \|f(t)\| \geq \epsilon\}$ is compact. The space $\mathcal{C}_0(T, U)$ is endowed with the norm:

$$\|f\| := \max\{\|f(t)\| : t \in T\}, \quad f \in \mathcal{C}_0(T, U).$$

Throughout the remainder, V will be a finite dimensional subspace of X.

Let $\mathrm{CLB}(X)$ denote the family of all nonempty closed and bounded subsets of X equipped with the Hausdorff metric H defined by

$$H(A, B) := \max\{e(A, B), e(B, A)\}, \quad A, B \in \mathrm{CLB}(X),$$

where $e(A, B) := \sup\{d(a, B) : a \in A\}$ denotes the excess of A over B.

If $\mathcal{F} \subseteq \mathrm{CLB}(X)$, we regard $\mathcal{F}$ as a metric space equipped with the induced Hausdorff metric topology. By a *multifunction* $T : \mathcal{F} \rightrightarrows V$ we mean a set-valued function whose values $T(F), F \in \mathcal{F}$ are nonempty closed subsets of V. Recall that a multifunction $T : \mathcal{F} \rightrightarrows V$ is said to be *lower semicontinuous* (resp. *upper semicontinuous*) abbreviated lsc (resp. usc) if the set $\{F \in \mathcal{F} : T(F) \cap O \neq \emptyset\}$ (resp. $\{F \in \mathcal{F} : T(F) \cap K \neq \emptyset\}$) is open (resp. closed) whenever O (resp K) is an open (resp. a closed) subset of V. Let us also recall the notion of the *derived submultifunction* $T^* : \mathcal{F} \rightrightarrows V$ of T defined by

$$\begin{aligned} T^*(F) := \{&v \in T(F) : \lim_n d(v, T(F_n)) = 0,\\ &\text{for every sequence } F_n \text{ in } \mathcal{F} \text{ convergent to } F\}. \end{aligned}$$

It follows immediately from the definitions that T is lsc if and only if $T = T^*$. Next, let us recall [17] that a set $F \in \mathrm{CLB}(X)$ is said to be *sup-compact* w.r.t. V if for each $v_0 \in V$, every maximizing sequence $\{f_n\}$, i.e., a sequence $\{f_n\} \subseteq F$ such that $\lim_n \|f_n - v_0\| = r(F; v_0)$, has a convergent subsequence converging in F. Clearly, if F is sup-compact w.r.t. V, then the set

$$\mathcal{Q}_{F,v_0} := \{f_0 \in F : \|f_0 - v_0\| = r(F; v_0)\}$$

of all *remotal points* of v_0 in F is non-void for each $v_0 \in V$. Sets which are sup-compact (w.r.t. X) are called M-compact in [21]. Examples of sets which are sup-compact but not compact are also given there. Let

$$\begin{aligned} s\text{-}K_V(X) := \{F \in \mathrm{CLB}(X) : &\ F \text{ is sup-compact w.r.t } V\\ &\text{and } r_V(F) > r_X(F)\}. \end{aligned}$$

In the sequel, for some of the results to follow, we will take $\mathcal{F} = s\text{-}K_V(X)$ which contains the family $K_V(X)$ of all nonempty compact subsets F of X satisfying the same restriction $r_V(F) > r_X(F)$.

3. A SUFFICIENT CONDITION FOR LOWER SEMICONTINUITY OF THE MULTIFUNCTION C_V

Throughout this section X will be a (real) normed linear space whose normed dual will be denoted by X^*, and V will be a finite dimensional subspace of X. The weak* or $\sigma(X^*, X)$-topology of X^* will be denoted by w^*. Let $\mathrm{Ext}(B(X^*))$ denote the set of all extreme points of the closed unit ball $B(X^*)$ of X^*. For the sake of brevity, let us denote

$$\mathcal{E}_{X^*} := \overline{\mathrm{Ext}}^{w^*}(B(X^*)),$$

the closure being taken in the w^*-topology. Also, for $x \in X$, let

$$\mathcal{E}_x := \{x^* \in \mathcal{E}_{X^*} : |x^*(x)| = \|x\|\},$$

denote the set of all *critical functionals*. Clearly, $\mathcal{E}_x$ is nonempty and w^*-compact subset of X^* for each $x \in X$. For $A \subseteq X$, we denote by $A^\perp$ the *annihilator* of A :

$$A^\perp := \{x^* \in X^* : x^*(A) = \{0\}\}.$$

For $f \in X$, let

$$\begin{aligned}\mathcal{E}_{f-A} &= \cap_{\alpha\in A}\mathcal{E}_{f-\alpha}\\ &= \{x^* \in \mathcal{E}_{X^*} : |x^*(f-\alpha)| = \|f-\alpha\|, \forall \alpha \in A\}.\end{aligned}$$

For F in CLB(X), let us denote by $\mathcal{G}_F$, the subspace

$$\mathcal{G}_F := \text{span}\{v_1 - v_2 : v_1, v_2 \in C_V(F)\}.$$

Note that

$$\mathcal{G}_F^{\perp} = \bigcup_{v\in C_V(F)} \{v - v_0\}^{\perp},$$

for any fixed $v_0 \in C_V(F)$. Hence, for $F \in \text{CLB}(X)$ such that $0 \in C_V(F)$, we have $\mathcal{G}_F^{\perp} = C_V(F)^{\perp}$. We will denote the relative interior of $C_V(F)$ by relint$C_V(F)$.

Lemma 1. *Let V be a finite dimensional subspace of a normed space X, and let $F \in \text{CLB}(X)$ be sup-compact w.r.t. V. If $v_0 \in \text{relint}C_V(F)$, then*

$$\mathcal{E}_{f-v_0} = \mathcal{E}_{f_0-C_V(F)} \subseteq \mathcal{G}_F^{\perp} \cap \mathcal{E}_{X^*}, \tag{1}$$

for every $f_0 \in Q_{F,v_0}$. Also

$$\mathcal{Q}_{F,v_0} = \bigcap_{v\in C_V(F)} \mathcal{Q}_{F,v}. \tag{2}$$

Proof. Let $v_0 \in \text{relint}C_V(F)$. Then there exists $\epsilon > 0$ such that for every $v \in C_V(F)$, whenever $|\lambda| < \epsilon, v_0 + \lambda(v - v_0) \in C_V(F)$. Let $f_0 \in Q_{F,v_0}$ and $x^* \in \mathcal{E}_{f_0-v_0}$. Then for any λ with $|\lambda| < \epsilon$,

$$\begin{aligned}|x^*(f_0 - v_0 - \lambda(v - v_0))| &\leq \|f_0 - v_0 - \lambda(v - v_0)\|\\ &\leq \sup_{f\in F} \|f - v_0 - \lambda(v - v_0)\|\\ &= r_V(F) = \|f_0 - v_0\| = |x^*(f_0 - v_0)|.\end{aligned}$$

Strict convexity of $\mathbb{R}$ entails that $x^*(v - v_0) = 0$. Hence, $x^* \in \mathcal{G}_F^{\perp} \cap \mathcal{E}_{X^*}$. Therefore,

$$\mathcal{E}_{f_0-v_0} \subseteq \mathcal{G}_F^{\perp} \cap \mathcal{E}_{X^*}.$$

Also, if $x^* \in \mathcal{E}_{f_0-v_0}$ and $v \in C_V(F)$, then

$$\begin{aligned}\|f_0 - v_0\| &\geq |x^*(f_0 - v)| = |x^*(f_0 - v_0)| = r_V(F)\\ &\geq \|f_0 - v\|.\end{aligned}$$

This implies that $|x^*(f_0 - v)| = \|f_0 - v\| = r_V(F)$. Therefore, $x^* \in \mathcal{E}_{f_0-v}, f_0 \in Q_{F,v}$, and we conclude that

$$\mathcal{E}_{f_0-v_0} = \mathcal{E}_{f_0-C_V(F)}, \text{ and } \mathcal{Q}_{F,v_0} = \bigcap_{v\in C_V(F)} \mathcal{Q}_{F,v}.$$

$\square$

THEOREM 2. *Let X, V and F be as in Lemma 1. If*

$$\mathcal{E}_{f_0 - v_0} \subseteq \text{int}\,(\mathcal{G}_F^{\perp} \cap \mathcal{E}_{X^*}), \tag{3}$$

the interior being taken in the induced w^*-topology of $\mathcal{E}_{X^*}$, for every $f_0 \in \mathcal{Q}_{F,v_0}$, whenever $v_0 \in \text{relint}C_V(F)$, then the multifunction $C_V : s\text{-}K_V(X) \rightrightarrows V$ is lsc at F.

Proof. It would be enough to prove that $C_V^*(F) = C_V(F)$. Let us denote $\text{int}\,(\mathcal{G}_F^{\perp} \cap \mathcal{E}_{X^*})$ by $\mathcal{M}$. By hypothesis, for every $f \in F$ and $x^* \in \mathcal{E}_{X^*} \setminus \mathcal{M}$, $|x^*(f - v_0)| < r_V(F)$. Since $\mathcal{M}$ is open in $\mathcal{E}_{X^*}$,

$$\sup_{f \in F} \sup_{x^* \in \mathcal{E}_{X^*} \setminus \mathcal{M}} |x^*(f - v_0)| < r_V(F).$$

Let

$$0 < \epsilon < \tfrac{1}{4} \left\{ r_V(F) - \sup_{f \in F} \sup_{x^* \in \mathcal{E}_{X^*} \setminus \mathcal{M}} |x^*(f - v_0)| \right\}.$$

Since V is finite dimensional, C_V is usc. Therefore, there exists a $\delta > 0$ such that $d(v', C_V(F)) < \epsilon$ for every $v' \in C_V(G)$ whenever $G \in s\text{-}K_V(X)$ is such that $H(G, F) < \delta$. Pick $v \in C_V(F)$ such that $\|v - v'\| < \epsilon$. For $G \in s\text{-}K_V(X)$ such that $H(G, F) < \min\{\epsilon, \delta\}$ and any $g \in G$, we have

$$\sup_{x^* \in \mathcal{M}} |x^*(g - (v_0 + v' - v))| = \sup_{x^* \in \mathcal{M}} |x^*(g - v')| \leq r_V(G). \tag{4}$$

For $g \in G \cap F$,

$$\begin{aligned} &\sup_{x^* \in \mathcal{E}_{X^*} \setminus \mathcal{M}} |x^*(g - (v_0 + v' - v))| \leq \\ \leq\ & \sup_{x^* \in \mathcal{E}_{X^*} \setminus \mathcal{M}} |x^*(g - v_0)| + \sup_{x^* \in \mathcal{E}_{X^*} \setminus \mathcal{M}} |x^*(v' - v)| \\ \leq\ & r_V(F) - 4\epsilon + \epsilon \\ \leq\ & r_V(F) - \epsilon \\ <\ & r_V(F) - H(G, F) \\ \leq\ & r_V(G). \end{aligned} \tag{5}$$

Also, for $g \in G$ with $g \notin F$, pick $f \in F$ such that $\|f - g\| < d(g, F) + \epsilon$. Then

$$\begin{aligned} &\sup_{x^* \in \mathcal{E}_{X^*} \setminus \mathcal{M}} |x^*(g - (v_0 + v' - v))| \leq \\ \leq\ & \sup_{x^* \in \mathcal{E}_{X^*} \setminus \mathcal{M}} |x^*(f - g)| + \sup_{x^* \in \mathcal{E}_{X^*} \setminus \mathcal{M}} |x^*(f - v_0)| + \sup_{x^* \in \mathcal{E}_{X^*} \setminus \mathcal{M}} |x^*(v' - v)| \\ \leq\ & H(G, F) + \epsilon + r_V(F) - 4\epsilon + \epsilon \\ \leq\ & r_V(F) - \epsilon < r_V(F) - H(G, F) \\ \leq\ & r_V(G). \end{aligned} \tag{6}$$

From (4), (5) and (6), it follows that $r(G, v_0 + v' - v) \leq r_V(G)$. Hence, $v_0 + v' - v \in C_V(G)$. Therefore, $d(v_0, C_V(G)) \leq \|v_0 - (v_0 + v' - v)\| < \epsilon$. Hence,

$v_0 \in C_V^*(F)$. Since $C_V^*(F)$ is closed and relint$C_V(F)$ is dense in $C_V(F)$, it follows that $C_V^*(F) = C_V(F)$. □

REMARK 3. Since $C_V(F - v_0) = C_V(F) - v_0$ for $v_0 \in V$, we can say that if

$$\mathcal{E}_{f_0} \subseteq \text{int}\,(C_V^{\perp}(F) \cap \mathcal{E}_{X^*}) \tag{7}$$

for every $f_0 \in \mathcal{Q}_{F,0}$ whenever $0 \in \text{relint}C_V(F)$, then the multifunction $C_V : s\text{ - }K_V(X) \rightrightarrows V$ is lsc at F. □

4. LOWER SEMICONTINUITY OF C_V IN THE SPACE $\mathcal{C}_0(T, U)$

Let $X = \mathcal{C}_0(T, U)$, where T is a locally compact Haussdorff space, and U is a strictly convex (real) Banach space. Throughout the remainder, V will be a finite dimensional subspace of X. For $X = \mathcal{C}_0(T, U), f \in X$ and $\mathcal{A} \subseteq X$, let

$$Z(\mathcal{A}) := \{t \in T : \alpha(t) = 0 \text{ for all } \alpha \in \mathcal{A}\}.$$

For $\alpha \in \mathcal{A}$, let

$$E(f - \alpha) := \{t \in T : \|f(t) - \alpha(t)\| = \|f - \alpha\|\},$$

denote the set of all *critical points* of the function $f - \alpha$. Also let

$$\begin{aligned} E(f - \mathcal{A}) &:= \cap\{E(f - \alpha) : \alpha \in \mathcal{A}\} \\ &= \{t \in T : \|f(t) - \alpha(t)\| = \|f - \alpha\| \text{ for all } \alpha \in \mathcal{A}\}. \end{aligned}$$

We note that in case $X = \mathcal{C}_0(T, U)$ the set of extreme points of the closed unit ball of X^* is given by (cf., e.g., [15], p.422),

$$\text{Ext}B(X^*) = \{x^*_{u^*,t} : u^* \in \text{Ext}B(U^*),\ t \in T\},$$

where

$$x^*_{u^*,t}(x) = u^*(x(t)),\ \ x \in X.$$

Also note that in this case if U^* is also assumed to be strictly convex, then in the above representation of Ext$B(X^*)$, we may take u^* in $S(U^*)$, the unit sphere of U^*.

The following theorem for characterization of restricted centers whose proof follows easily from ([17], Theorem 2.6) and the above representation of the extreme points of $B(X^*)$ will be required as a tool in the sequel.

THEOREM 4. *Let $X = \mathcal{C}_0(T, U), V = \text{span}\{v_1, \ldots, v_n\}$ be an n-dimensional subspace of X, and $v_0 \in V$. Let $F \in K(X)$. The following statements are equivalent.*

(i) $v_0 \in C_V(F)$.
(ii) *For each $v \in V$,*

$$\max\{u^*(f_0(t) - v_0(t))u^*(v(t)) : f_0 \in \mathcal{Q}_{F,v_0}, t \in E(f_0 - v_0) \text{ and } u^* \in \mathcal{E}_{f_0(t)-v_0(t)}\} \geq 0.$$

(iii) *The origin* $(0,\ldots,0)$ *of* $\mathbb{R}^n$ *belongs to the convex hull* $\mathrm{co}(S)$ *of* S*, where*

$$S := \{(u^*((f_0(t)-v_0(t))u^*(v_1(t)),\ldots,u^*(f_0(t)-v_0(t))u^*(v_n(t))) : \\ f_0 \in \mathcal{Q}_{F,v_0}, t \in E(f_0-v_0) \text{ and } u^* \in \mathcal{E}_{f_0(t)-v_0(t)}\}.$$

(iv) *There exist* $f_i \in \mathcal{Q}_{F,v_0}, t_i \in E(f_0-v_0), u_i^* \in \mathcal{E}_{f_0(t)-v_0(t)}, i=1,\ldots,m$ *and* $\lambda_1,\ldots,\lambda_m > 0$ *with* $\sum_{i=1}^m \lambda_i = 1$*, where* $m \le n+1$*, such that for every* $v \in V$*,*

$$\sum_{i=1}^m \lambda_i u_i^*(f_i(t_i)-v_0(t_i))u_i^*(v(t_i)) = 0.$$

The next lemma is an analogue of Lemma 1 for the present case. Its proof is exactly identical. Let us recall that we are denoting by $\mathcal{G}_F$ the subspace $\mathrm{span}\{v_2-v_1 : v_1,v_2 \in C_V(F)\}$ of V, and by $\mathrm{relint}C_V(F)$, the relative interior of $C_V(F)$.

LEMMA 5. *Let* X, V *and* F *be as in the last theorem. If* $v_0 \in \mathrm{relint}C_V(F)$*, then*

$$E(f-v_0) = E(f-C_V(F)) \subseteq Z(\mathcal{G}_F)$$

for every $f \in \mathcal{Q}_{F,v_0}$*. Also* $\mathcal{Q}_{F,v_0} = \bigcap_{v\in C_V(F)} \mathcal{Q}_{F,v}$.

REMARK 6. Note that $Z(\mathcal{G}_F) = \cap\{Z(v-v_0) : v \in C_V(F)\}$ for any fixed $v_0 \in C_V(F)$. Hence, the conclusion of the lemma can be restated as follows: If $0 \in \mathrm{relint}C_V(F)$, then

$$E(f-v_0) = E(f-C_V(F)) \subseteq Z(C_V(F))$$

for every $f_0 \in \mathcal{Q}_{F,0}$. □

4.1. An intrinsic characterization of lower semicontinuity of the multifunction C_V. As before, let $X = \mathcal{C}_0(T,U)$ and V be a finite dimensional subspace of X. The next lemma involves perturbation of sets. For F, G in $K(X)$, and $S \subseteq T$, we write $F|_S = G|_S$ if for every $f \in F$, there is a $g \in G$ such that $f|_S = g|_S$, and conversely. The proof is a verbatim reproduction of Lemma 2 in [8] which was given for $\mathcal{C}_0(T)$. However, for the convenience of the reader, we give it once again here.

LEMMA 7. *Let* $F \in K(X)$ *be such that* $0 \in \mathrm{relint}C_V(F)$ *and* $r_V(F) = 1$*. Let* O *be any open neighborhood of* $Z(C_V(F))$*. If* $G \in K(X)$ *is such that* $G|_O = F|_O$ *and* $\sup_{g\in G}\|g\| = 1$*, then* $0 \in C_V(G)$ *and* $C_V(G) \subseteq \mathrm{span}C_V(F)$*.*

Proof. Since $0 \in \mathrm{relint}C_V(F)$, by Lemma 5, $E(f_0) \subseteq Z(C_V(F))$ for every $f_0 \in \mathcal{Q}_{F,0}$. Also $r_V(F) = \sup_{f\in F}\|f\| = 1$. Hence $\|f\|_{T\setminus O} < 1$ for all $f \in F$, whenever O is an open neighborhood of $Z(C_V(F))$. Let $0 < \lambda < \frac{1}{2}(1 - \sup_{f\in F}\|f\|_B)$, where $B = T \setminus O$. Let $p \in C_V(G)$. Then,

$$\sup_{g\in G}\|g-p\| \le \sup_{g\in G}\|g\| = 1. \tag{8}$$

For any $f \in F$,

$$\begin{aligned} \|f - \lambda p\|_B &\leq \|f\|_B + \lambda \|p\|_B \\ &\leq \|f\|_B + 2\lambda \\ &< \sup_{f \in F} \|f\|_B + 1 - \sup_{f \in F} \|f\|_B \\ &= 1. \end{aligned}$$

Therefore, $\sup_{f \in F} \|f - \lambda p\|_B \leq 1$.
For any $f \in F$ and $t \in O$,

$$\begin{aligned} \|f(t) - \lambda p(t)\| &= \|f(t) - \lambda f(t) + \lambda f(t) - \lambda p(t)\| \\ &\leq (1-\lambda)\|f(t)\| + \lambda \|f(t) - p(t)\| \\ &\leq (1-\lambda) + \lambda \\ &= 1. \end{aligned}$$

Therefore,

$$\sup_{f \in F} \|f - \lambda p\|_O \leq 1. \tag{9}$$

Hence,

$$\sup_{f \in F} \|f - \lambda p\| \leq 1 = r_V(F). \tag{10}$$

The relation (10) implies that $\lambda p \in C_V(F)$. Since this is true for every $p \in C_V(G)$, we get $C_V(G) \subseteq \mathrm{span} C_V(F)$.

Also strict inequality in (8) gives strict inequality in (9) and hence in (10), which is not possible. Thus $0 \in C_V(G)$. □

Let us now recall the following well known result for lower semicontinuity of metric projection due to Blatter, Morris and Wulbert [2].

THEOREM 8. *Let $X = \mathcal{C}(T)$ and V be a finite dimensional subspace of X. The metric projection multifunction $P_V : X \rightrightarrows V$ is lsc if and only if $Z(P_V(f))$ is open for every f in $\mathcal{C}(T)$ for which $0 \in P_V(f)$.*

We are now ready to state and prove our first main characterization theorem for lower semicontinuity of C_V. This extends Theorem 2 of [8] and Theorems 6 and 9 of [4].

THEOREM 9. *Let V be a finite dimensional subspace of $\mathcal{C}_0(T, U)$.*

(i) *If the multifunction $C_V := K(X) \rightrightarrows V$ is lsc for all $F \in K(X)$ with $0 \in \mathrm{relint} C_V(F)$, then*

$$E(g - v_0) \subseteq \text{ int } Z(\mathcal{G}_G)$$

for every $G \in K(X), v_0 \in \mathrm{relint} C_V(G)$ and $g \in \mathcal{Q}_{G,v_0}$.

(ii) *The multifunction* $C_V := K(X) \rightrightarrows V$ *is lsc at* $F \in K(X)$ *if*

$$E(f - v_0) \subseteq \text{ int } Z(\mathcal{G}_F)$$

for every $f \in \mathcal{Q}_{F,v_0}$, *whenever* $v_0 \in \text{relint} C_V(F)$.

Proof. (i) For every $F \in K(X)$ with $0 \in \text{relint}C_V(F), C_V$ is given to be lsc at F. If possible let (i) be not true, i.e., suppose there exists $G \in K(X), v_0 \in \text{relint}C_V(G)$ and an element $g_0 \in \mathcal{Q}_{G,v_0}$ such that

$$E(g_0 - v_0) \not\subseteq \text{ int } \bigcap_{v \in C_V(G)} Z(v_0 - v).$$

Let $F = G - v_0$. Then $0 \in \text{relint}C_V(F)$, and by hypothesis C_V is lsc at F. Also let $f_0 = g_0 - v_0$. Then $\|f_0\| = \|g_0 - v_0\| = r_V(G) = r_V(F)$. Therefore, $f_0 \in Q_{F,0}$ and $E(f_0) \not\subseteq \text{ int } Z(C_V(F))$.

By Lemma 5, $E(f_0) \subseteq Z(C_V(F))$. Hence there exists a $t_0 \in E(f_0)$ such that $t_0 \in \text{bd}Z(C_V(F))$, i.e., there exists a net $\{t_\lambda : \lambda \in \Lambda\}$ such that $t_\lambda \notin Z(C_V(F)), \lambda \in \Lambda$, and $t_\lambda \to t_0$. Let $\{v_1, \dots, v_m\} \subseteq C_V(F)$ be such that $V_1 = \text{span}\{v_1, \dots, v_m\} = \text{span}C_V(F)$. Two cases arise as follows.

Case (i) Since V is finite-dimensional, if necessary by passing to a subnet, we may assume that for each $\lambda \in \Lambda$ there exists some $x^*_\lambda \in \mathcal{E}_{f_0(t_0)}$ such that $x^*_\lambda(v_1(t_\lambda)) \neq 0$.

If necessary by passing once more to a subnet, it can be ensured that there are signs $\epsilon_k \in \{-1, 1\}$, $k = 1, \dots, m$, such that we have

$$\begin{aligned} \epsilon_1 x^*_\lambda(v_1(t_\lambda)) &< 0 \quad \text{and} \\ \epsilon_k x^*_\lambda(v_k(t_\lambda)) &\leq 0, \quad k = 2, \dots, m. \end{aligned}$$

For each $\delta > 0$, the set $B_\delta = \{t \in T : |f_0(t_0) - f_0(t)| < \delta\}$ is a neighborhood of t_0. Hence there exists $\lambda \in \Lambda$ such that $t_\lambda \in B_\delta$. Since $t_\lambda \notin Z(C_V(F)), t_\lambda \notin E(f_0)$. Since $Z(C_V(F))$ is a closed set, there exists a compact neighborhood W of t_λ such that $Z(C_V(F)) \cap W = \emptyset$. Without loss of generality, we may assume that $W \subset B_\delta$. Let ρ be a continuous function such that $0 \leq \rho(t) \leq 1$ for $t \in T, \rho(t_\lambda) = 1$ and $\rho(t) = 0$ for $t \in T \setminus W$. Define

$$f_\delta(t) := \rho(t) f_0(t_0) + (1 - \rho(t)) f_0(t).$$

Then $f_\delta \in \mathcal{C}_0(T, U)$ and $\|f_\delta - f_0\| < \delta$. Also, for $0 < \delta < \|f_0\| - \max_{t \in W} \|f_0(t)\|$, it is easily seen that $\|f_\delta\| = \|f_0\|$. Let $F_\delta = F \cup \{f_\delta\}$. Then $F_\delta \in K(X)$ for each $\delta > 0$ and $H(F, F_\delta) < \delta$.

We now have an $F \in K(X)$ with $0 \in \text{relint}C_V(F)$ and $T \setminus W$ is a neighborhood of $Z(C_V(F))$. Also $F_\delta \in K(X)$ is such that $F_\delta|_{T \setminus W} = F|_{T \setminus W}$ and $\sup_{f' \in F_\delta} \|f'\| = \sup_{f \in F} \|f\|$. Hence by Lemma 7, $0 \in C_V(F_\delta)$ and $C_V(F_\delta) \subseteq \text{span}C_V(F) = V_1$. Since $0 \in C_V(F_\delta)$,

$$r_V(F_\delta) = \sup_{f' \in F_\delta} \|f'\| = \sup_{f \in F} \|f\| = r_V(F).$$

We now define an open set A in V_1 such that $C_V(F)\cap A\neq\emptyset$, but $C_V(F_\delta)\cap A=\emptyset$. Define the set A by

$$A:=\left\{\sum_{k=1}^{m}a_k\epsilon_k v_k : a_k>0,\quad k=1,\ldots,m\right\}.$$

Then $0\in$ bd A (w.r.t. V_1) and $0\in \text{relint}C_V(F)$. Hence $C_V(F)\cap A\neq\emptyset$. But for any $u\in A$,

$$\begin{aligned}\sup_{f\in F_\delta}\|f-u\| &\geq \|f_\delta-u\|\\ &\geq \|f_\delta(t_\lambda)-u(t_\lambda)\|\\ &\geq x_\lambda^*(f_\delta(t_\lambda)-u(t_\lambda))\\ &= x_\lambda^*(f_0(t_0))-\sum_{k=1}^{m}a_k\epsilon_k x_\lambda^*(v_k(t_\lambda)\\ &= \|f_0\|-\sum_{k=1}^{m}a_k\epsilon_k x_\lambda^*(v_k(t_\lambda))\\ &> r_V(F)=r_V(F_\delta).\end{aligned}$$

In the above set of inequalities, t_λ is a fixed element of the net $\{t_\lambda:\lambda\in\Lambda\}$ such that $t_\lambda\in B_\delta$ and $u=\sum_{k=1}^{m}a_k\epsilon_k v_k$ for some $a_k>0, k=1,\ldots,m$. Hence for any $u\in A, u\notin C_V(F_\delta)$, i.e., $C_V(F_\delta)\cap A=\emptyset$.

If the net $\{t_\lambda:\lambda\in\Lambda\}$ does not satisfy the conditions that we assumed in the first case, then we need to consider the following alternative.

Case (ii) By passing to a subnet, if necessary, we may assume that

$$u^*(v_1(t_\lambda))=0 \text{ for all } \lambda\in\Lambda \text{ and } u^*\in\mathcal{E}_{f_0(t_0)}.$$

Let $u_\lambda^*\in\mathcal{E}_{f_0(t_0)+v_1(t_\lambda)}$. Since U is strictly convex and $v_1(t_\lambda)$ is not proportional to $f_0(t_0), u_\lambda^*\notin\mathcal{E}_{f_0(t_0)}$. Therefore,

$$u_\lambda^*(f_0(t_0))<\|f_0(t_0)\|.$$

Also, for $z^*\in\mathcal{E}_{f_0(t_0)}$, we have

$$\|f_0(t_0)+v_1(t_\lambda)\|\geq z^*(f_0(t_0))+z^*(v_1(t_\lambda))=\|f_0(t_0)\|.$$

Therefore,

$$\begin{aligned}\|f_0(t_0)\| &\leq \|f_0(t_0)+v_1(t_\lambda)\|=u_\lambda^*(f_0(t_0)+v_1(t_\lambda))\\ &< \|f_0(t_0)\|+u_\lambda^*(v_1(t_\lambda)).\end{aligned}$$

Hence, it follows that

$$\|f_0(t_0)+v_1(t_\lambda)\|\geq r_V(F) \tag{11}$$

and $u_\lambda^*(v_1(t_\lambda))>0$ for each $\lambda\in\Lambda$.

If necessary by passing once more to a subnet, it can be ensured that there are signs $\epsilon_k \in \{-1, 1\},\ k = 1, \ldots, m$, such that we have

$$\begin{array}{rcl} \epsilon_1 u_\lambda^*(v_1(t_\lambda)) & < & 0 \quad \text{and} \\ \epsilon_k u_\lambda^*(v_k(t_\lambda)) & \le & 0, \quad k = 2, \ldots, m. \end{array}$$

For $\delta > 0$, the set

$$\begin{array}{rl} B_\delta := \{t \in T : & \|f_0(t_0) - f_0(t)\| < \delta/3, \|v_1(t)\| < \delta/3 \text{ and} \\ & |\ \|f_0(t_0) + v_1(t)\| - r_V(F)\ | < \delta/3\} \end{array}$$

is a neighborhood of t_0. Hence there exists $\lambda \in \Lambda$ such that $t_\lambda \in B_\delta$. Also, there exists a compact neighborhood W of t_λ such that $W \cap Z(C_V(F)) = \emptyset$. We may assume, without loss of generality, that $W \subset B_\delta$. Let ρ be a continuous function such that $0 \le \rho(t) \le 1$ for $t \in T, \rho(t_\lambda) = 1$ and $\rho(t) = 0$ for $t \in T \setminus W$. Let us define

$$f_\delta(t) := r_V(F)\left(\rho(t)\tfrac{f_0(t_0)+v_1(t_\lambda)}{\|f_0(t_0)+v_1(t_\lambda)\|} + (r_V(F))^{-1}(1-\rho(t))f_0(t)\right).$$

Then $f_\delta \in \mathcal{C}_0(T, U)$, and using (11), for $t \in B_\delta$, we have

$$\begin{aligned} \|f_\delta(t) - f_0(t)\| &= r_V(F)\rho(t)\left\|\tfrac{f_0(t_0)+v_1(t_\lambda)}{\|f_0(t_0)+v_1(t_\lambda)\|} - (r_V(F))^{-1}f_0(t)\right\| \\ &= \tfrac{r_V(F)\rho(t)}{\|f_0(t_0)+v_1(t_\lambda)\|}\|f_0(t_0) - f_0(t) + v_1(t_\lambda) \\ &\quad + (1 - \|f_0(t_0) + v_1(t_\lambda)\|(r_V(F))^{-1})f_0(t)\| \\ &\le \|f_0(t_0) - f_0(t)\| + \|v_1(t_\lambda)\| + |\ r_V(F) - \|f_0(t_0)+v_1(t_\lambda)\|\ | < \delta. \end{aligned}$$

For $t \notin B_\delta$, we have $f_\delta(t) = f_0(t)$. Hence, $\|f_\delta - f_o\| < \delta$. Also, for

$$0 < \delta < \|f_0\| - \max_{t\in W}\|f_0(t)\|,$$

it is easy to see that $\|f_\delta\| = \|f_0\|$. Thus again defining $F_\delta := F \cup \{f_\delta\}$, we have $F_\delta \in K(X)$ for each $\delta > 0$ and $H(F, F_\delta) < \delta$.

Exactly as in case(i) we again have an $F \in K(X)$ with $0 \in \text{relint}C_V(F)$ and $T \setminus W$ is a neighborhood of $Z(C_V(F))$. Also $F_\delta \in K(X)$ is such that $F_\delta|_{T\setminus W} = F|_{T\setminus W}$ and $\sup_{f' \in F_\delta}\|f'\| = \sup_{f\in F}\|f\|$. Hence by Lemma 7, $0 \in C_V(F_\delta)$ and $C_V(F_\delta) \subseteq \text{span}C_V(F) = V_1$. Again since $0 \in C_V(F_\delta)$,

$$r_V(F_\delta) = \sup_{f' \in F_\delta}\|f'\| = \sup_{f\in F}\|f\| = r_V(F).$$

Consider again the open set A in V_1 defined by

$$A := \left\{\sum_{k=1}^m a_k\epsilon_k v_k : a_k > 0, \quad k = 1, \ldots, m\right\}.$$

Note that $0 \in$ bd A (w.r.t. V_1) and $0 \in \mathrm{relint} C_V(F)$. Hence $C_V(F) \cap A \neq \emptyset$. However, for each $u \in A$,

$$\begin{aligned} \sup_{f \in F_\delta} \|f - u\| &\geq \|f_\delta - u\| \\ &\geq \|f_\delta(t_\lambda) - u(t_\lambda)\| \\ &\geq u_\lambda^* \left(f_\delta(t_\lambda) - \sum_{k=1}^m a_k \epsilon_k v_k(t_\lambda) \right) \\ &= u_\lambda^* \left(\frac{r_V(F)(f_0(t_0)+v_1(t_\lambda))}{\|f_0(t_0)+v_1(t_\lambda)\|} \right) - \sum_{k=1}^m a_k \epsilon_k u_\lambda^*(v_k(t_\lambda)) \\ &> r_V(F). \end{aligned}$$

Therefore, $C_V(F_\delta) \cap A = \emptyset$. Thus in either case the hypothesis that the multifunction C_V is lsc at F is contradicted, and we conclude that contrary to our assumption, $E(g_0 - v_0) \subseteq$ int $Z(\mathcal{G}_G)$ must hold. This completes the proof of (i).
(ii) The proof of Theorem 2 (ii) of [8] extends verbatim to the present case. □

We can now state a global necessary and sufficient condition for lower semicontinuity of C_V as follows.

THEOREM 10. *Let V be a finite dimensional subspace of $X = \mathcal{C}_0(T, U)$. Then the multifunction $C_V : K(X) \rightrightarrows V$ is lsc if and only if for each $F \in K(X)$, we have*

$$E(f - v_0) \subseteq \ int\ Z(\mathcal{G}_F)$$

for every $f \in \mathcal{Q}_{F,v_0}$, whenever $v_0 \in \mathrm{relint} C_V(F)$.

The next theorem partially extends Theorem 4.5 of Blatter, Morris, and Wulbert [2].

THEOREM 11. *Let X, V as in the last theorem. If for every $F \in K(X)$ with $0 \in \mathrm{relint} C_V(F)$, the set $Z_F := Z(C_V(F))$ is open, then the multifunction $C_V : K(X) \rightrightarrows V$ is lsc.*

Proof. Let $F \in K(X)$ and $v_0 \in \mathrm{relint} C_V(F)$. Since $C_V(F - v_0) = C_V(F) - v_0$ and C_V is lsc at F if and only if it is lsc at $F - v_0$, we may assume, without loss of generality, that $0 \in \mathrm{relint} C_V(F)$. Let $f_0 \in \mathcal{Q}_{F,v_0}$. Since Z_F is both open as well as closed, the function $g_0 : T \to U$ defined by

$$g_0(t) := \begin{cases} f_0(t), & \text{if } t \in Z_F \\ 0, & \text{if } t \in T \setminus Z_F, \end{cases}$$

is in $\mathcal{C}_0(T, U)$. Let $\sum := \{x^* \in X^* : x^*(g_0) > 0\}$. Then $\sum$ is w^*-open. Also, if $x^*_{u^*,t} \in \mathcal{E}_{X^*} \cap \sum$, then

$$x^*_{u^*,t}(v) = u^*(v(t)) = u^*(0) = 0 \text{ for all } v \in C_V(F).$$

Hence, $\mathcal{E}_{X^*} \cap \sum \subseteq C_V^{\perp}(F) \cap \mathcal{E}_{X^*}$. Since

$$\mathcal{E}_{f_0} \subseteq \mathcal{E}_{X^*} \cap \sum \subseteq C_V^{\perp}(F) \cap \mathcal{E}_{X^*},$$

$\mathcal{E}_{f_0} \subseteq$ int $(C_V^{\perp}(F) \cap \mathcal{E}_{X^*}$, and by Theorem 2, we conclude that C_V is lsc at F. □

4.2. Characterization of lower semicontinuity of C_V using Haar-like condition. Our goal here is to give an intrinsic characterization of finite dimensional subspaces V of $\mathcal{C}_0(T, U)$ for which the restricted center multifunction $C_V : K(X) \rightrightarrows V$ is lsc.

Let us recall that a finite dimensional subspace V of $\mathcal{C}_0(T)$ is called a *Haar subspace* (or that it is said to satisfy *Haar condition*) if for each $v \in V \setminus \{0\}$, $\mathrm{card}Z(v) \leq \dim V - 1$. It is easily seen that V is a Haar subspace of dimension n if and only if $\dim V|_S = n$ for every subset S of T such that $\mathrm{card}(S) = n$. Here $\mathrm{card}(S)$ denotes the cardinality of S and $V|_S := \{v|_S : v \in V\}$. Let us also recall the generalized Haar condition introduced by Zukhovitskii and Stechkin [22] for a finite dimensional subspace V of $\mathcal{C}_0(T, U)$. Consider the following properties of V.

(T_m) For each $v \in V \setminus \{0\}$, there are at most m zeros in T.

(P_m) For each set of m distinct points $t_i \in T$ and m elements $u_i \in U$, there exists at least one $v \in V$, such that

$$v(t_i) = u_i, \;\; i = 1, \ldots, m.$$

An n-dimensional subspace V of $\mathcal{C}_0(T, U)$ is said to satisfy the *generalized Haar condition* if either $\dim U = k \leq n$ and V satisfies conditions (T_m) and (P_m) where $m \in N$ is the unique integer satisfying

$$mk < n \leq (m + 1)k,$$

or $\dim U > n$, and V satisfies condition (T_0). It is easily seen that in case $\dim U = k < \infty, V$ satisfies conditions (T_m) and (P_m) if and only if for any finite set $S \subseteq T$,

$$\dim V|_S \geq \min\{\dim V, k.\mathrm{card}(S)\}.$$

For finite dimensional subspaces V of $\mathcal{C}_0(T)$, the following extension of *Haar condition* is due to W. Li [10].

Definition 12. *V is said to satisfy property* (Li) *if for every $v \in V \setminus \{0\}$,*

$$\mathrm{cardbd}Z(v) \leq \; \dim\{p \in V : p|_{\mathrm{int}\ Z(v)} = 0\} - 1.$$

For finite dimensional subspaces V of $\mathcal{C}_0(T, U)$, the following variant of the *generalized Haar condition* is also due to W. Li [12].

Definition 13. *A finite dimensional subspace V of $\mathcal{C}_0(T, U)$ is said to satisfy property* (Li′) *if for every $v \in V \setminus \{0\}$,*

$$\mathrm{cardbd}Z(v) \leq (\dim U)^{-1} \cdot \dim\{p|_{bdZ(v)} : p \in V \;\; and \;\; p|_{\mathrm{int}\ Z(v)} = 0\}.$$

Note that if T is connected, then property (Li) coincides with the Haar condition. Moreover, in case $\dim U = k \leq n = \dim V$, the property (Li′) is implied by the generalized Haar condition.

We require the following lemma due to W. Li [12] to prove the next theorem.

Lemma 14. *For a finite dimensional subspace V of $\mathcal{C}_0(T,U)$, the following statements are equivalent.*

(i) *The metric projection multifunction $P_V : X \rightrightarrows V$ is lsc.*
(ii) *V satisfies property* (Li′)*.*
(iii) *For any set $\{t_i : 1 \leq i \leq m\} \subseteq T$, if there exist $u_i^* \in U^* \setminus \{0\}, 1 \leq i \leq m$ such that*

$$\sum_{i=1}^{m} u_i^*(v(t_i)) = 0, \quad v \in V,$$

then for any $v \in V$ with $\{t_i : 1 \leq i \leq m\} \subseteq Z(v)$, we have

$$\{t_i : 1 \leq i \leq m\} \subseteq \operatorname{int} Z(v).$$

The next theorem gives an intrinsic characterization of finite dimensional subspaces V of $\mathcal{C}_0(T,U)$ for which the restricted center multifunction C_V is lsc. It generalizes Theorem 3 of [8].

Theorem 15. *For a finite dimensional subspace V of $\mathcal{C}_0(T,U)$, the following statements are equivalent.*

(i) *The multifunction $C_V : K_V(X) \rightrightarrows V$ is lsc.*
(ii) *V satisfies property* (Li′)*.*

Proof. We imitate here the proof of Thorem 3 of [8].
(i) $\Rightarrow$ (ii) : The statement (i) restricted to singletons is nothing but the lower semicontinuity of metric projection P_V. It follows from Lemma 14 that the lower semicontinuity of metric projection P_V is equivalent to V satisfying property (Li′). Hence (ii) is true.
(ii) $\Rightarrow$ (i): In view of Theorem 9 (ii), without loss of generality, it is enough to prove that property (Li′) gives,

$$E(f_0) \subseteq \operatorname{int} Z(C_V(F)) \tag{12}$$

for all $f_0 \in \mathcal{Q}_{F,0}$, whenever $F \in K_V(X)$ and $0 \in \operatorname{relint} C_V(F)$. We prove this by the method of contradiction. Assume (12) does not hold for some $F \in K_V(X)$ and some $f_0 \in \mathcal{Q}_{F,0}$, where $0 \in \operatorname{relint} C_V(F)$. For simplicity, we denote $\operatorname{int} Z(C_V(F))$ by M. Therefore, we have $A_{f_0} := E(f_0) \setminus M \neq \emptyset$, for some $f_0 \in \mathcal{Q}_{F,0}$. Let $T_0 := T \setminus M, F_0 := (F)|_{T_0}$, and $V_0 = \{v|_{T_0} : v \in V \text{ and } v|_M = 0\}$.

We claim that $r_V(F) = r_{V_0}(F_0), 0 \in C_{V_0}(F_0)$, and

$$A_{f_0} = \{t \in T_0 : \|f_0(t)\| = \|f_0\|_{T_0}\}.$$

If $v \in V_0$, then $v|_M = 0$. Hence

$$\begin{aligned}&\sup_{f\in F}\sup_{t\in T}\|f(t)-v(t)\| \leq \\ &\leq \sup_{f\in F}\max\left\{\sup_{t\in T_0}\|f(t)-v(t)\|, \sup_{t\in M}\|f(t)\|\right\}.\end{aligned}$$

Since $0 \in C_V(F), \sup_{f\in F}\|f(t)\| \leq r_V(F)$. If for some $v \in V_0$,

$$\sup_{f\in F}\sup_{t\in T_0}\|f(t)-v(t)\| \leq \sup_{f\in F}\sup_{t\in M}|f(t)\|, \tag{13}$$

then $\sup_{f\in F}\sup_{t\in T}\|f(t)-v(t)\| \leq r_V(F)$. Hence $v \in C_V(F)$. By Lemma 1, f_0 which is a farthest point in F of 0 is also a farthest point of v. By our assumption, $A_{f_0} \subset T_0$, and it contains points of $E(f_0 - C_V(F))$. Therefore,

$$\sup_{f\in F}\sup_{t\in T_0}\|f(t)-v(t)\| = r_V(F),$$

and we get equality in (13). Hence we have,

$$\begin{aligned} r_V(F) &= \inf_{v\in V}\sup_{f\in F}\sup_{t\in T}\|f(t)-v(t)\| \\ &\leq \inf_{v\in V_0}\sup_{f\in F}\sup_{t\in T}\|f(t)-v(t)\| \\ &= \inf_{v\in V_0}\sup_{f\in F}\sup_{t\in T_0}\|f(t)-v(t)\| \\ &= r_{V_0}(F_0).\end{aligned}$$

Also for every $f \in F$,

$$\begin{aligned} r_V(F) &= \|f_0\| = \|f_0\|_{T_0} \\ &\geq \|f\| \\ &\geq \|f\|_{T_0},\end{aligned}$$

i.e., $r_V(F) \geq \sup_{f\in F}\|f\|_{T_0} \geq r_{V_0}(F_0)$. Hence $r_V(F) = r_{V_0}(F_0)$, and $\|f_0\|_{T_0} = r_{V_0}(F_0)$, which in turn implies that $0 \in C_{V_0}(F_0)$.

From the above, it also follows that $A_{f_0} = \{t \in T_0 : \|f_0(t)\| = \|f_0\|_{T_0}\}$, i.e., $A_{f_0} = E(f_0) \cap T_0$. Hence A_{f_0} is none other than the set of all critical points of 0 with $f_0|_{T_0}$ as the farthest point.

Let $X_0 = \mathcal{C}_0(T_0, U)$. By the assumption $r_V(F) > r_V(X)$, and condition (iv) in the characterization theorem (Theorem 4) applied to $0 \in C_{V_0}(F_0)$, there exist scalars $\lambda_i > 0$, functionals $v_i^* \in \overline{\text{Ext}}^{w^*} B(U^*)$, points $t_i \in T_0$, and elements $f_i^0 \in F_0$, $i = 1, \ldots, m$ such that

$$\sum_{i=1}^{m}\lambda_i = 1,\ |v_i^*(f_i^0(t_i))| = r_{V_0}(F_0) \tag{14}$$

and

$$\sum_{i=1}^{m} \lambda_i v_i^*(f^0(t_i)) v_i^*(v(t_i)) = 0 \text{ for all } v \in V. \tag{15}$$

Since f_i^0 is of the form $f_i|_{T_0}$ for some $f_i \in F$, we have, in fact, $\{t_1, \ldots, t_m\} \subseteq E(f_i)$. By Lemma 5, $\{t_1, \ldots, t_m\} \subseteq Z(v)$ for all $v \in C_V(F)$. Since V satisfies property (Li′), taking $u_i^* = \lambda_i v_i^*(f^0(t_i))v_i^*$ in Lemma 14 (iii), we get, for every $v \in C_V(F)$,

$$\{t_1, \ldots, t_m\} \subseteq \operatorname{int} Z(v) \cap T_0.$$

Since V is finite dimensional, this in turn implies that,

$$\{t_1, \ldots, t_m\} \subseteq \operatorname{int} \bigcap_{v \in C_V(F)} (Z(v) \cap T_0).$$

This contradicts the definition of T_0. Hence $E(f_0) \subseteq \operatorname{int} Z(C_V(F))$ must hold for every $f_0 \in \mathcal{Q}_{F,0}$, $0 \in \operatorname{relint} C_V(F)$ and $F \in K_V(X)$. We can now apply Theorem 9 (ii) to conclude that the multifunction C_V is lsc. □

As observed in Theorem D and Theorem 3 of Li [12], in case U is a strictly convex Banach space such that $\dim U = k \leq n$, and V is an n-dimensional subspace of X, the generalized Haar condition consisting of (P_m) and (T_m) where $m \in N$ is the unique integer satisfying $mk < n \leq (m+1)k$ is equivalent to the condition

$$\operatorname{card} Z(v)) \leq (\dim U)^{-1} \cdot \dim\{p|_{bdZ(v)} : p \in V \text{ and } p|_{\operatorname{int} Z(v)} = 0\}. \tag{16}$$

In conjunction with Theorem 3.5 of [17], this result yields the next theorem.

THEOREM 16. *Let U be a k-dimensional Euclidean space and V be an n-dimensional subspace of $X = \mathcal{C}_0(T, U)$. If $k \leq n$, then in order that $C_V(F)$ be a singleton for each $F \in K(X)$ it is necessary and sufficient that condition* (16) *be satisfied.*

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