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AN ITERATIVE METHOD FOR APPROXIMATING FIXED POINTS OF PRESIĆ NONEXPANSIVE MAPPINGS

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Abstract. The main result of the paper unifies two important fixed point theorems published in the same year, 1965, the first one discovered independently by Browder [F.E. Browder, *Nonexpansive nonlinear operators in Banach spaces*, Proc. Nat. Acad. Sci. U.S.A., **54** (1965), 1041–1044], Göhde [D. Göhde, *Zum Prinzip der kontraktiven Abbildung*, Math. Nachr., **30** (1965), 251–258] and Kirk [W.A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, **72** (1965), 1004–1006], while the second one is due to Presić [S.B. Presić, *Sur une classe d' inéquations aux différences finite et sur la convergence de certaines suites*, Publ. Inst. Math. (Beograd)(N.S.), **5(19)** (1965), 75–78]. In this way we show how amazingly two apparently different beautiful results in mathematics can meet after almost half a century!

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1. INTRODUCTION

One of the most interesting generalizations of Banach's contraction mapping principle has been obtained in 1965 by S. Presić [19]:

THEOREM 1 (S. Presić [19], 1965). *Let (X,d) be a complete metric space, k a positive integer, $\alpha_1,\alpha_2,\dots,\alpha_k\in\mathbb{R}_+$, $\sum\limits_{i=1}^{k}\alpha_i=\alpha<1$ and $f:X^k\to X$ a mapping satisfying*

$$d(f(x_0,\dots,x_{k-1}),f(x_1,\dots,x_k))\leq \alpha_1 d(x_0,x_1)+\dots+\alpha_k d(x_{k-1},x_k), \tag{1.1}$$

for all $x_0,\dots,x_k\in X$.

Then:

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1) *f has a unique fixed point x^*, that is, there exists a unique $x^* \in X$ such that $f(x^*, \ldots, x^*) = x^*$;*
2) *the sequence $\{x_n\}_{n\geq 0}$ defined by*

$$x_{n+1} = f(x_{n-k+1}, \ldots, x_n), \quad n = k-1, k, k+1, \ldots \tag{1.2}$$

converges to x^*, for any $x_0, \ldots, x_{k-1} \in X$.

It is easy to see that, in the particular case $k = 1$, from Theorem 1 we get exactly the well-known contraction mapping principle of Banach, while the k-step iterative method (1.2) reduces to the one step method of successive approximations for the self-mapping $f : X \to X$, i.e., to

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, 3, \ldots, \tag{1.3}$$

also known as *Picard iteration.*

For this reason, in this paper, a mapping satisfying the contraction condition (1.1) in Theorem 1 will be called a *Presić contraction.*

Theorem 1 and other similar results, like the ones in [5], [14], [15], [21], have important applications in the iterative solution of nonlinear equations, see [17] and [18], as well as in the study of global asymptotic stability of the equilibrium for nonlinear difference equations, see the very recent paper [3].

An important generalization of Theorem 1 was proved in I.A. Rus [21], see also [22], for operators f fulfilling the more general condition

$$d(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \leq \varphi(d(x_0, x_1), \ldots, d(x_{k-1}, x_k)), \tag{1.4}$$

for any $x_0, \ldots, x_k \in X$, where $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$ satisfies certain conditions.

Another important generalization of Presić's result was recently obtained by L. Cirić and S. Presić in [5], where, instead of (1.1) and its generalization (1.4), the following contraction condition is considered:

$$d(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \leq \lambda \max\{d(x_0, x_1), \ldots, d(x_{k-1}, x_k)\}, \tag{1.5}$$

for any $x_0, \ldots, x_k \in X$, where $\lambda \in (0, 1)$.

Other general Presić type fixed point results have been very recently obtained by the second author in [13]-[16].

The main result in [14] is the following fixed point theorem.

THEOREM 2. *Let (X, d) be a complete metric space, k a positive integer, $a \in \mathbb{R}$ a constant such that $0 \leq ak(k+1) < 1$ and $f : X^k \to X$ an operator satisfying the following condition:*

$$d(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \leq a \sum_{i=0}^{k} d(x_i, f(x_i, \ldots, x_i)), \tag{1.6}$$

for any $x_0, x_1, \ldots, x_k \in X$. Then

1) *f has a unique fixed point x^*, that is, there exists a unique $x^* \in X$ such that $f(x^*, \ldots, x^*) = x^*$;*

2) *the sequence* $\{y_n\}_{n\geq 0}$ *defined by* $y_{n+1} = f(y_n, y_n, \ldots, y_n), n \geq 0$, *converges to* x^*;
3) *the sequence* $\{x_n\}_{n\geq 0}$ *with* $x_0, \ldots, x_{k-1} \in X$ *and* $x_n = f(x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}), n \geq k$, *also converges to* x^*, *with a rate estimated by:*

$$d(x_{n+1}, x^*) \leq M\theta^n, n \geq 0,$$

for a positive constant M *and a certain* $\theta \in (0, 1)$.

Notice that the proofs of the main results in [13]-[19] for mappings $f : X^k \to X$ are essentially based on some known (common) fixed point theorems for usual Banach contractions and, respectively, Kannan type contractions of the form $F : X \to X$.

Some important results related to Presić contractions and their applications to multi-step iterative methods have been obtained in [17] and [18].

As nonexpansive mappings are obvious generalizations of usual contraction mappings $f : X \to X$, it is the main aim of this paper to obtain Presić fixed point theorems for *nonexpansive* type mappings of the form $f : X^k \to X$.

To this end we shall present in the next section a brief introduction to fixed point theory for nonexpansive mappings.

2. BASIC FIXED POINT THEORY FOR NONEXPANSIVE MAPPINGS

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an α-*contraction* if there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y), \forall x, y \in X. \tag{2.1}$$

In the case $\alpha = 1$ in (2.1), T is said to be *nonexpansive*.

Nonexpansive mappings, although are generalizations of contractions, do not inherit more from contraction mappings. More precisely, if K is a nonempty closed subset of a Banach space E and $T : K \to K$ is nonexpansive, it is known, see [1], that T may not have a fixed point (unlike the case when T is α-contraction), see the examples in [7].

Even in the cases when T has a fixed point, the Picard iteration associated to T may fail to converge to the fixed point.

For the above and many other reasons, a much more richer geometrical structure of the ambient space is needed in order to ensure the existence of a fixed point and / or the convergence of an iterative method (generally more elaborated than Picard iteration) to a fixed point of a nonexpansive mapping T.

Definition 3. *A normed linear space is called uniformly convex if, for any* $\epsilon \in (0, 2]$, *there exists* $\delta = \delta(\epsilon) > 0$ *such that if* $x, y \in E$ *with* $\|x\| = \|y\| = 1$ *and* $\|x - y\| \geq \epsilon$, *then*

$$\left\|\tfrac{1}{2}(x + y)\right\| \leq 1 - \delta.$$

Definition 4. *A normed linear space is called strictly convex if, for all $x, y \in E$, $x \neq y$, $\|x\| = \|y\| = 1$, we have*

$$\|\lambda x + (1-\lambda)y\| < 1,\ \forall \lambda \in (0,1).$$

If we denote by $S_r(a)$ the *sphere* centered at a in E with radius r, that is,

$$S_r(a) = \{x \in E :\ \|x - a\| = r\},$$

then E is uniformly convex, see [7] if, for any two distinct points x, y on the unit sphere centered at origin, the midpoint of the line segment joining x and y is never on the sphere but is close to sphere only if x and y are closed enough to each other.

Similarly, E is strictly convex if, for any two distinct points x, y on the unit sphere centered at origin, any point of the line segment joining x and y is never on the sphere, except for its endpoints.

Remark 5. Similar to the considerations above, it follows that any uniformly convex space is strictly convex. The converse is generally not true, see for example [7]. □

We can now formulate one of the most influential fixed point theorems for nonexpansive mappings, which was discovered independently by F.E. Browder, D. Göhde and W.A. Kirk in 1965, cf. [2], [8], [10].

Theorem 6. *Let K be a nonempty closed convex and bounded subset of a uniformly Banach space E and let $T : K \to K$ be a nonexpansive mapping. Then T has a fixed point.*

Remark 7. Under the assumptions of Theorem 6, no information on the approximation of the fixed points of T is available. Actually, Picard iteration does not resolve this situation, in general. Due to this fact, for the class of nonexpansive mappings other fixed point iteration procedures have been considered, see [1] and [4]. The two most usual ones are defined in the following. □

Let K be a convex subset of a normed linear space E and let $T : K \to K$ be a mapping. For $x_0 \in K$ and $\lambda \in [0,1]$ the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n = 0, 1, 2, \cdots \tag{2.2}$$

is usually called *Krasnoselskij iteration*, or *Krasnoselskij-Mann iteration*. Clearly, (2.2) reduces to the Picard iteration (1.3) for $\lambda = 1$.

For $x_0 \in K$ the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n,\ n = 0, 1, 2, \cdots, \tag{2.3}$$

where $\{\lambda_n\} \subset [0,1]$ is a sequence of real numbers satisfying some appropriate conditions, is called *Mann iteration*.

It was shown by Krasnoselskij [11], in the case $\lambda = 1/2$, and then by Schaefer [24], for $\lambda \in (0,1)$ arbitrary, that if E is a uniformly convex Banach space and

K is a convex and compact subset of E (and so having the set of fixed points nonempty, by Theorem 6), then the Krasnoselskij iteration converges to a fixed point of T.

Moreover, Edelstein [6] proved that strict convexity of E suffices for the same conclusion, see also [4].

The question of whether strict convexity can be removed or not has been practically answered in the affirmative by Ishikawa [9], who proved the following result.

THEOREM 8. *Let K be a subset of a Banach space E and let $T : K \to K$ be a nonexpansive mapping. For arbitrary $x_0 \in K$, consider the Mann iteration process $\{x_n\}$ given by* (2.3) *under the following assumptions*

(a) $x_n \in K$ *for all positive integers;*
(b) $0 \leq \lambda_n \leq b < 1$ *for all positive integers;*
(c) $\sum_{n=0}^{\infty} \lambda_n = \infty$.

If $\{x_n\}$ is bounded, then $x_n - Tx_n \to 0$ as $n \to \infty$.

The following corollaries of Theorem 8 will be particularly important for our considerations in the paper.

COROLLARY 9. [4, Th. 6.17] *Let K be a convex and compact subset of a Banach space E and let $T : K \to K$ be a nonexpansive mapping. If the Mann iteration process $\{x_n\}$ given by* (2.3) *satisfies assumptions* (a)–(c) *in Theorem* 8 *then $\{x_n\}$ converges strongly to a fixed point of T.*

COROLLARY 10. *(*[4]*, Corollary 6.19) Let K be a closed bounded convex subset of a real normed space E, and let $T : K \to K$ be a nonexpansive mapping. If $I - T$ maps closed bounded subsets of E into closed subsets of E and $\{x_n\}$ is the Mann iteration defined by* (2.3)*, with $\{\lambda_n\}$ satisfying assumptions* (b)–(c) *in Theorem* 8*, then $\{x_n\}$ converges strongly to a fixed point of T in K.*

3. APPROXIMATING FIXED POINTS OF PRESIĆ NONEXPANSIVE OPERATORS

DEFINITION 11. *Let (X, d) be a metric space, k a positive integer and $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}_+$ such that $\sum\limits_{i=1}^{k} \alpha_i = \alpha \leq 1$. A mapping $f : X^k \to X$ satisfying*

$$d(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \leq \sum_{i=1}^{k} \alpha_i d(x_{i-1}, x_i), \tag{3.1}$$

for all $x_0, \ldots, x_k \in X$, is called a ***Presić nonexpansive operator****.*

Since in Definition 11 the constant α is allowed to be less or equal to 1, we can see that the class of Presić nonexpansive operators strictly includes the class of Presíc contractions appearing in Theorem 1. Also note that in the case of a normed linear space X, condition (3.1) will be

$$\|f(x_0,\dots,x_{k-1})-f(x_1,\dots,x_k)\|\le\sum_{i=1}^{k}\alpha_i\,\|x_{i-1}-x_i\|\,, \tag{3.2}$$

which, in the case $k=1$, reduces to the Banach's contractive condition (2.1), if $\alpha<1$, and to the nonexpansiveness condition if $\alpha=1$.

The next theorem is our main result in this paper.

THEOREM 12. *Let C be a nonempty closed convex and bounded subset of a uniformly Banach space E, k a positive integer, and let $f:C^k\to C$ be a Presić nonexpansive mapping. Then f has a fixed point x^* in C, that is, there exists $x^*\in C$ such that $f(x^*,\dots,x^*)=x^*$.*

Proof. Let $F:C\to C$, be defined by $F(x)=f(x,x,\dots,x), x\in C$. For any $x,y\in C$ one has:

$$\begin{aligned}\|F(x)-F(y)\|&=\|f(x,x,\dots,x)-f(y,y,\dots,y)\|\\&\le\|f(x,\dots,x)-f(x,\dots,x,y)\|+\\&\quad+\|f(x,\dots,x,y)-f(x,\dots,x,y,y)\|+\\&\quad+\dots+\|f(x,y,\dots,y)-f(y,\dots,y)\|\,.\end{aligned}$$

By (3.2) it follows that

$$\|f(x,x,\dots,x)-f(x,\dots,x,y)\|\le\alpha_k\cdot\|x-y\|\,,$$

$$\|f(x,\dots,x,y)-f(x,\dots,x,y,y)\|\le\alpha_{k-1}\cdot\|x-y\|\,,$$

$$\dots$$

$$\|f(x,y,\dots,y)-f(y,\dots,y)\|\le\alpha_1\cdot\|x-y\|\,,$$

and hence

$$\|F(x)-F(y)\|\le\alpha_k\cdot\|x-y\|+\alpha_{k-1}\cdot\|x-y\|+\dots+\alpha_1\cdot\|x-y\|\,.$$

Using the fact that $\sum_{i=1}^{k}\alpha_i=\alpha\le1$, we get

$$\|F(x)-F(y)\|\le\alpha\cdot\|x-y\|\le\|x-y\|\,,$$

which shows that F is nonexpansive. Now we apply the Browder-Göhde-Kirk fixed point theorem (Theorem 6) to F to get the conclusion of the theorem. □

REMARK 13. Note that Theorem 12 is a generalization of Theorem 1 and that, in the particular case $k=1$, Theorem 12 reduces to Theorem 6. As in the case of Theorem 6, a Presić nonexpansive mapping f has generally more than one fixed point. The next example gives a Presić nonexpansive mapping f whose set of fixed points is an interval and also shows that Theorem 12 is an effective generalization of Theorem 1. □

Example 14. Let $[0,1]$ be the unit interval with the usual Euclidian norm and let $f:[0,1]\times[0,1]\to[0,1]$ be given by $f(x,y)=\dfrac{x+y}{2}$, for all $x,y\in[0,1]$.

Then: a) f is Presić nonexpansive b) f is not a Presić contraction. $\square$

Proof. a) In this case ($k=2$), the Presić nonexpansive condition (3.2) reads as follows: there exist $\alpha_1,\alpha_2\in\mathbb{R}_+$ with $\alpha_1+\alpha_2=\alpha\leq 1$ such that for all $x_0,x_1,x_2\in[0,1]$:

$$|f(x_0,x_1)-f(x_1,x_2)|\leq\alpha_1|x_0-x_1|+\alpha_2|x_1-x_2|. \tag{3.3}$$

By the definition of f, (3.3) becomes

$$\left|\tfrac{x_0-x_2}{2}\right|\leq\alpha_1|x_0-x_1|+\alpha_2|x_1-x_2|,$$

which obviously holds for $\alpha_1=\alpha_2=\dfrac{1}{2}$, in view of the triangle inequality.

b) The Presić contraction condition (1.1) will be in this case: there exist $\alpha_1,\alpha_2\in\mathbb{R}_+$ with $\alpha_1+\alpha_2<1$ such that for all $x_0,x_1,x_2\in[0,1]$:

$$|f(x_0,x_1)-f(x_1,x_2)|\leq\alpha_1|x_0-x_1|+\alpha_2|x_1-x_2|. \tag{3.4}$$

We shall prove now that we can find a triple $x_0,x_1,x_2\in[0,1]$ for which (3.4) cannot be true under the strict inequality $\alpha_1+\alpha_2<1$.

Indeed, let $x_0=1$, $x_2=0$ and $x_1=x\in[0,1]$, when condition (3.4) becomes

$$\tfrac{1}{2}-\alpha_1\leq(\alpha_2-\alpha_1)\,x. \tag{3.5}$$

We have to discuss tree cases.

Case 1. $\alpha_2-\alpha_1>0$. Then by (3.5),

$$x\geq\frac{\frac{1}{2}-\alpha_1}{\alpha_2-\alpha_1}. \tag{3.6}$$

As $x\in[0,1]$, the inequality (3.6) holds for all x only if

$$\frac{\frac{1}{2}-\alpha_1}{\alpha_2-\alpha_1}\leq 0,$$

which implies that $\alpha_1\geq\frac{1}{2}$. Since $\alpha_2>\alpha_1$, we conclude that $\alpha_1+\alpha_2>1$, which contradicts the contraction condition $\alpha_1+\alpha_2<1$.

Case 2. $\alpha_2-\alpha_1=0$. Then by (3.5) we get $\alpha_1\geq\frac{1}{2}$ which shows that $\alpha_1+\alpha_2=2\alpha_1\geq 1$, which again contradicts $\alpha_1+\alpha_2<1$.

Case 3. $\alpha_2-\alpha_1<0$. Then by (3.5) we obtain

$$x\leq\frac{\frac{1}{2}-\alpha_1}{\alpha_2-\alpha_1}=\frac{\alpha_1-\frac{1}{2}}{\alpha_1-\alpha_2}. \tag{3.7}$$

Having in view the fact that $x\in[0,1]$, the inequality (3.7) holds for all x only if

$$\frac{\alpha_1-\frac{1}{2}}{\alpha_1-\alpha_2}\geq 1,$$

which implies $\alpha_2\geq\frac{1}{2}$. Since $\alpha_1>\alpha_2$, we obtain that $\alpha_1+\alpha_2>1$, a contradiction. Therefore f is not a Presić contraction.

An indirect proof of part b) easily follows by Theorem 1 and the fact that $Fix\,(f) = [0,1]$. □

Theorem 12 ensures merely the existence of a fixed point of f. We can further obtain a method for approximating the fixed point of Presić nonexpansive mappings f, as in the next two theorems.

THEOREM 15. *Let C be a convex and compact subset of a Banach space E, k a positive integer, and let $f : C^k \to C$ be a Presić nonexpansive mapping. If the sequence $\{\lambda_n\}$ satisfies assumptions* (b)–(c) *in Theorem* 8*, then the Mann type iteration process $\{x_n\}$ defined by $x_0 \in C$ and*

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n f(x_n, x_n, \dots, x_n),\ n = 0, 1, 2, \dots \tag{3.8}$$

converges strongly to a fixed point of f, that is, to a point $x^* \in C$ for which $f(x^*, \dots, x^*) = x^*$.

Proof. We use the same arguments as in the proof of Theorem 12 to show that the mapping $F : C \to C$ given by $F(x) = f(x, x, \dots, x), x \in C$ is nonexpansive. Then apply Corollary 9 to get the conclusion. □

THEOREM 16. *Let E be a real normed space, C a closed bounded convex subset of E, k a positive integer, and let $f : C^k \to C$ be a Presić nonexpansive mapping. If T given by $T(x) = x - f(x, x, \dots, x)$ maps closed bounded subsets of E into closed subsets of E and $\{x_n\}$ is the Mann iteration defined by* (3.8)*, with $\{\lambda_n\}$ satisfying assumptions* (b)–(c) *in Theorem* 8*, then $\{x_n\}$ converges strongly to a fixed point of f in C.*

Proof. We use similar arguments to those in the proof of Theorem 12 to show that the mapping $F : C \to C$ given by $F(x) = f(x, x, \dots, x), x \in C$ is nonexpansive. Then apply Corollary 10 to get the conclusion. □

4. CONCLUSIONS AND AN OPEN PROBLEM

Note that in the proof of Theorem 12 we basically used Theorem 6. Due to this fact Theorem 12 is, strictly speacking, not a generalization of Theorem 6, but we can give a direct proof of the former which actually follows the main steps of the latter.

It was shown, see Theorem 1 and 2 as well as the related results in [5], [13]-[16], that if f is Presić contraction or a Presić-Kannan contraction and so on, then a k-step iterative method $\{x_n\}_{n\geq 0}$ defined by $x_0, \dots, x_{k-1} \in X$ and

$$x_n = f(x_{n-k}, x_{n-k+1}, \dots, x_{n-1}), n \geq k,$$

can be used to approximate the unique solution x^* of the equation $x = f(x, \dots, x)$.

On the other hand, the Mann type iteration process $\{x_n\}$ given by (3.8), that is

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n f(x_n, x_n, \dots, x_n),\ n = 0, 1, 2, \dots$$

used in Theorems 15 and 16 to approximate a solution x^* of the equation $x = f(x, \ldots, x)$ for Presić nonexpansive mappings f, is a one step iterative method.

The following question then naturally arises: is it still possible to approximate the fixed points in Theorems 12, 15 and 16, by means of a k-step iterative method of the form

$$x_n = (1 - \lambda_n)x_p + \lambda_n f(x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}), n \geq k,$$

where $n - 1 \leq p \leq n - k$?

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