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A NOTE ON BEST SELECTION OF QUASI DESCARTES SYSTEMS

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Abstract. Let $\{u_0, \dots, u_n\}$ be a quasi Descartes system of $C[a, b]$ and p a positive number with $1 < p < \infty$ or ∞. In this note, we search for an $m(\leqq n)$ dimensional subspace that possesses the least distance from u_n among all $m(\leqq n)$ dimensional subspaces of $\mathrm{Span}\{u_0, \dots, u_{n-1}\}$.

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1. INTRODUCTION

Before stating a purpose of this note, we have to begin with introducing Descartes systems and quasi Descartes systems.

Let $F[a, b]$ be the space of all real-valued functions on a nondegenerate compact interval $[a, b]$ of $\mathbb{R}$ and $C[a, b]$ the space of all real-valued continuous functions on $[a, b]$. A finite subset $\{u_0, \dots, u_n\}$ of $F[a, b]$ is called a *system* if $u_0, \dots, u_n$ are linearly independent. The space spanned by $\{u_0, \dots, u_n\}$ is denoted by $\mathrm{Span}\{u_0, \dots, u_n\}$. A system $\{u_0, \dots, u_n\}$ of $F[a, b]$ is called a *Chebyshev system* if there exists a constant $\sigma = 1$ or -1 and for any $n+1$ distinct points $(a \leqq)x_0 < \cdots < x_n(\leqq b)$, the $n+1$-th order determinant

$$\sigma \cdot D\begin{pmatrix} u_0 & \dots & u_n \\ x_0 & \dots & x_n \end{pmatrix} := \sigma \cdot \det(u_i(x_j)) > 0.$$

That is to say, a Chebyshev system $\{u_0, \dots, u_n\}$ is a system satisfying that

(i) any $u \in \mathrm{Span}\{u_0, \dots, u_n\} - \{0\}$ has at most n distinct zeros in $[a, b]$;
(ii) for any $u \in \mathrm{Span}\{u_0, \dots, u_n\} - \{0\}$ there do not exist $n+2$ points $(a \leqq)x_0 < \cdots < x_{n+1}(\leqq b)$ such that $(-1)^i u(x_i)$ is positive for $i = 0, \dots, n+1$ or negative for $i = 0, \dots, n+1$.

In particular, a system with property (ii) is called a *weak Chebyshev system*. It is well known that Chebyshev systems and weak Chebyshev systems are of much use to study best approximation, interpolation and quadrature formulas in approximation theory (e.g. Karlin and Studden [3], Zielke [11] and a survey Zalik [10] and so on). If a Chebyshev system $\{u_0, \dots, u_n\}$ has a stronger

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property such that there exists a constant $\sigma = 1$ or -1 and for any nonnegative integers $(0 \leqq) i_0 < \cdots < i_m (\leqq n)$ and any $m+1$ distinct points $(a \leqq) x_0 < \cdots < x_m (\leqq b)$,

$$\sigma \cdot D \begin{pmatrix} u_{i_0} & \dots & u_{i_m} \\ x_0 & \dots & x_m \end{pmatrix} := \sigma \cdot \det(u_{i_k}(x_j)) > 0,$$

then $\{u_0, \dots, u_n\}$ is called a *Descartes system.* Descartes systems of $C[a,b]$ have remarkable properties such that Descartes' rule of signs, comparison theorem and selection theorem of best approximations and so on (see Borosh, Chui and Smith [1], Pinkus and Ziegler [7], Smith [9] and Borwein and Erdélyi [2]).

Let $\mathcal{S}$ be the set of all nondegenerate closed subintervals of $[a,b]$. For $[x_0, y_0], [x_1, y_1] \in \mathcal{S}$, if $x_0 < x_1$ and $y_0 < y_1$, then this relation is denoted by $[x_0, y_0] < [x_1, y_1]$. Furthermore, if $[x_0, y_0] < [x_1, y_1]$ or $[x_0, y_0] = [x_1, y_1]$, then we write $[x_0, y_0] \leqq [x_1, y_1]$ for this relation. We easily see that $(\mathcal{S}, \leqq)$ is a partially ordered set.

A system $\{u_0, \dots, u_n\}$ of $C[a,b]$ is called a *quasi Chebyshev system* if there exists a constant $\sigma = 1$ or -1 such that for any $n+1$ closed subintervals $I_0, \dots, I_n \in \mathcal{S}$ with $I_0 < \cdots < I_n$, the $n+1$-th order determinant

$$\sigma \cdot D \begin{pmatrix} u_0 & \dots & u_n \\ I_0 & \dots & I_n \end{pmatrix} := \sigma \cdot \det \left(\int_{I_j} u_i(x) \, \mathrm{d}x \right) > 0.$$

The definition of a quasi Chebyshev system is introduced by Shi [8]. Quasi Chebyshev systems are introduced as integral Tchebysheff systems in Kitahara [4] and $H_{\mathcal{I}}$ systems in Kitahara [5]. Furthermore, if a quasi Chebyshev system $\{u_0, \dots, u_n\}$ of $C[a,b]$ satisfies that there exists a constant $\sigma = 1$ or -1 and for any nonnegative integers $0 \leqq i_0 < \cdots < i_m \leqq n$ and any $m+1$ closed subintervals $I_0, \dots, I_m \in \mathcal{S}$ with $I_0 < \cdots < I_m$,

$$\sigma \cdot D \begin{pmatrix} u_{i_0} & \dots & u_{i_m} \\ I_0 & \dots & I_m \end{pmatrix} := \sigma \cdot \det \left(\int_{I_j} u_{i_k}(x) \, \mathrm{d}x \right) > 0,$$

then we call $\{u_0, \dots, u_n\}$ a *quasi Descartes system.* Clearly every Chebyshev system (respectively Descartes system) is a quasi Chebyshev (respectively quasi Descartes system). In the rest of this note, we suppose $\sigma = 1$ in the definitions of Chebyshev, Descartes, quasi Chebyshev or quasi Descartes systems. For a continuous function $f \in C[a,b]$, the value $\int_I f(x) \, \mathrm{d}x, I \in \mathcal{S}$ is denoted by $f[I]$, and we call a subinterval $I \in \mathcal{S}$ with $f[I] = 0$ a *vanishing subinterval of* f.

Good properties of quasi Descartes systems analogous to Descartes systems, that is, Descartes' rule of signs, comparison theorem and selection theorem of best approximations are shown in Kitahara[6]. Now we focus on selection theorem of best approximations. Let $\{u_0, \dots, u_n\}$ be a system of $C[a,b]$ and $\| \cdot \|_p$, $1 \leqq p \leqq \infty$ denote the L_p norm for $1 \leqq p < \infty$ on $C[a,b]$ and the supremum norm for $p = \infty$ on $C[a,b]$. Let $\Lambda : (0 \leqq)\lambda_0 < \lambda_1 < \cdots < \lambda_m(\leqq n)$ be given nonnegative integers and p a given positive number or ∞ with $1 \leqq p \leqq \infty$. Then, for any $f \in C[a,b]$ we define the deviation from $G = \mathrm{Span}\{u_{\lambda_0}, \ldots, u_{\lambda_m}\}$ by

$$E_\Lambda(f)_p := \inf_{u \in G} \|f - u\|_p.$$

If $\{u_0, \ldots, u_n\}$ is a Descartes system of $C[a,b]$, the following hold.

Theorem A (see Smith [9]). *Let $\{u_0, \ldots, u_n\}$ be a Descartes system of $C[a,b]$ and p a positive number with $1 \leqq p < \infty$ or ∞. If $\Lambda : (0 \leqq)\lambda_0 < \cdots < \lambda_m(< n)$ and $\Lambda' : (0 \leqq)\lambda'_0 < \cdots < \lambda'_m(< n)$ satisfy $\lambda_i \leqq \lambda'_i$ for $i = 0, \ldots, m$, then*

$$E_{\Lambda'}(u_n)_p \leqq E_\Lambda(u_n)_p\,,$$

where equality holds only when $\lambda_i = \lambda'_i$ for $i = 0, \ldots, m$.

Let $\{u_0, \ldots, u_n\}$ be a quasi Descartes system of $C[a,b]$. If each $u_i(x), i = 0, \ldots, n$ is represented as $w(x)v_i(x)$, where each $v_i(x), i = 0, \ldots, n$ is continuous on $[a,b]$ and $\{v_0, \ldots, v_n\}$ is a Descartes system on (a,b) and $w(x)$ is a nonnegative continuous function on $[a,b]$ such that $\{x \mid w(x) = 0, x \in [a,b]\}$ is nowhere dense in $[a,b]$, then $\{u_0, \ldots, u_n\}$ is said to have *pc* (product of continuous functions)-*property*. Then we have

Theorem B (see Kitahara [6]). *Let $\{u_0, \ldots, u_n\}$ be a quasi Descartes system of $C[a,b]$. Let $\Lambda : (0 \leqq)\lambda_0 < \cdots < \lambda_m(< n)$ and $\Lambda' : (0 \leqq)\lambda'_0 < \cdots < \lambda'_m(< n)$ satisfy $\lambda_i \leqq \lambda'_i$ for $i = 0, \ldots, m$. Then, the following statements hold.*

(1) $E_{\Lambda'}(u_n)_1 \leqq E_\Lambda(u_n)_1$, *where equality holds only when $\lambda_i = \lambda'_i$ for $i = 0, \ldots, m$.*

(2) *Moreover if $\{u_0, \ldots, u_n\}$ has pc-property, then*

$$E_{\Lambda'}(u_n)_p \leqq E_\Lambda(u_n)_p \text{ for each } 1 < p \leqq \infty,$$

where equality holds only when $\lambda_i = \lambda'_i$ for $i = 0, \ldots, m$.

One could ask whether Theorem B (2) holds or not for all continuous quasi Descartes systems. The purpose of this note is to give an answer of this question. That is to say, we prove

Theorem. *Let $\{u_0, \ldots, u_n\}$ be a quasi Descartes system of $C[a,b]$ and p a positive number with $1 < p < \infty$ or ∞. If $\Lambda : (0 \leqq)\lambda_0 < \cdots < \lambda_m(< n)$ and $\Lambda' : (0 \leqq)\lambda'_0 < \cdots < \lambda'_m(< n)$ satisfy $\lambda_i \leqq \lambda'_i$ for $i = 0, \ldots, m$, then*

$$E_{\Lambda'}(u_n)_p \leqq E_\Lambda(u_n)_p.$$

In section 2, we prepare auxiliary results which are necessary to prove Theorem and show a proof of Theorem in section 3.

2. AUXILIARY RESULTS

For a system $\{u_0, \ldots, u_n\}$ of $F[a, b]$, the set $\{x \mid x \in [a, b], u_i(x) = 0, i = 0, \ldots, n\}$ is denoted by $V[u_0, \ldots, u_n]$ or V for short.

LEMMA 1 (see Lemma 3.3.4 in Kitahara [5]). *Let $\{u_0, \ldots, u_n\}$ be a quasi Chebyshev system of $C[a, b]$ and $m(x) = \max_{0 \leqq i \leqq n} |u_i(x)|, x \in [a, b]$. Then, for each $x_0 \in V[u_0, \ldots, u_n] = V$*

$$\lim_{x \to x_0-, x \in [a,b]-V} u_i(x)/m(x) \text{ and } \lim_{x \to x_0+, x \in [a,b]-V} u_i(x)/m(x), \ i = 0, \ldots, n$$

exist, where if $x_0 = a$ or b, the possible case is considered.

REMARK 1. Let $\{u_0, \ldots, u_n\}$ be a quasi Chebyshev system of $C[a, b]$. By Lemma 1, we can define a system $\{s_0, \ldots, s_n\}$ such that for all $i = 0, \ldots, n$

$$s_i(x_0) = \begin{cases} \frac{u_i(x_0)}{m(x_0)}, & \text{if } x_0 \in [a, b] - V, \\ \frac{u_i}{m}(x_0+) \text{ or } \frac{u_i}{m}(x_0-), & \text{if } x_0 \in (a, b) \cap V, \\ \frac{u_i}{m}(a+), & \text{if } x_0 = a \in V, \\ \frac{u_i}{m}(b-), & \text{if } x_0 = b \in V, \end{cases}$$

where $\frac{u_i}{m}(x_0+) := \lim_{x \to x_0+, x \in [a,b]-V} u_i(x)/m(x)$ and $\frac{u_i}{m}(x_0-) := \lim_{x \to x_0-, x \in [a,b]-V} u_i(x)/m(x)$. Each $s_i, i = 0, \ldots, n$ is not always continuous but continuous from the left on (a, b) or from the right on (a, b), and continuous from the right at a and continuous from the left at b. □

Theorem 4 in Kitahara [6] still holds for a quasi Descartes system $\{u_0 = ms_0, \ldots, u_n = ms_n\}$ in which each $s_i, i = 0, \ldots, n$ is represented as Remark 1.

LEMMA 2 (see Theorem 4 in Kitahara [6]). *Let $\{u_0, \ldots, u_n\}$ be a quasi Descartes system of $C[a, b]$ and $\{s_0, \ldots, s_n\}$ the system introduced in Remark 1. Then each $u_i, i = 0, \ldots, n$, is represented as*

$$u_i(x) = m(x)s_i(x), \quad x \in [a, b]$$

and $\{s_0, \ldots, s_n\}$ is a Descartes system on (a, b).

REMARK 2. If $\{u_0, \ldots, u_n\}$ is a quasi Chebyshev system of $C[a, b]$, then the system $\{s_0, \ldots, s_n\}$ introduced in Remark 1 is a Chebyshev system on (a, b) (see Theorem 3.3.5 in Kitahara [5]). □

DEFINITION. *Let $\{u_0, \ldots, u_n\}$ be a quasi Chebyshev system of $C[a, b]$ and $u = \sum_{i=0}^n a_i u_i(x) = m(x) \sum_{i=0}^n a_i s_i(x) = m(x)s(x)$ a function in* Span$\{u_0, \ldots, u_n\} - \{0\}$. *By Lemma 2, each $u_i, i = 0, \ldots, n$ is represented as*

$$u_i(x) = m(x)s_i(x), \quad x \in [a, b].$$

One element $x_0 \in (a, b)$ is called an essential zero of u if (i) *or* (ii) *is satisfied;*

(i) $s(x_0) = 0$;

(ii) $m(x_0) = 0, s(x_0) \neq 0$ *and there exists a* $\delta > 0$ *such that* $s(x)s(y) < 0$ *for all* $x \in (x_0 - \delta, x_0)$, $y \in (x_0, x_0 + \delta)$.

LEMMA 3. *Let* $\{u_0, \dots, u_n\}$ *be a quasi Chebyshev system of* $C[a, b]$ *and* $\{s_0, \dots, s_n\}$ *the system for* $\{u_0, \dots, u_n\}$ *introduced in Remark* 1. *Let* $y_0, \dots, y_n$ *be any given* $n + 1$ *distinct points in* (a, b) *with* $y_0 < \cdots < y_n$ *and* $\{y_i^{(k)}\}, i = 0, \dots, n$ *any* $n + 1$ *sequences which satisfy that* $\lim_{k\to\infty} y_i^{(k)} = y_i, i = 0, \dots, n$ *and* $y_0^{(k)} < \cdots < y_n^{(k)}, k \in \mathbb{N}$. *Then, there exists a positive number* ρ *such that*

$$D\begin{pmatrix} s_0 & \dots & s_n \\ y_0^{(k)} & \dots & y_n^{(k)} \end{pmatrix} > \rho,\ k \in \mathbb{N}. \tag{$*$}$$

Proof. Suppose on the contrary that $(*)$ does not hold. Then, there exist subsequences $\{y_i^{(k_m)}\}$ of $\{y_i^{(k)}\}, i = 0, \dots, n$ such that

$$\lim_{k_m\to\infty} D\begin{pmatrix} s_0 & \dots & s_n \\ y_0^{(k_m)} & \dots & y_n^{(k_m)} \end{pmatrix} = 0.$$

But this contradicts the result from Lemma 2 and Remark 2. □

LEMMA 4 (see Proposition 10 in Kitahara [6]). *Let* $\{u_0, \dots, u_n\}$ *be a quasi Descartes system of* $C[a, b]$. *For any* $f \in C[a, b] - \mathrm{Span}\{u_0, \dots, u_n\}$, *if* $\tilde{u} \in \mathrm{Span}\{u_0, \dots, u_n\}$ *is the unique best* $L_p(1 \leqq p < \infty)$ *approximation to* f, *then* $f - \tilde{u}$ *changes sign at least* $n + 1$ *times in* $[a, b]$.

LEMMA 5 (see Theorem 8 (2) in Kitahara [6]). *Let* $\{u_0, \dots, u_n\}$ *be a quasi Descartes system of* $C[a, b]$. *Let*

$$p = u_\alpha + \sum_{i=1}^{k} a_i u_{\lambda_i} \text{ and } q = u_\alpha + \sum_{i=1}^{k} b_i u_{\gamma_i},$$

satisfying that $0 \leqq \lambda_1 < \cdots < \lambda_k \leqq n$, $0 \leqq \gamma_1 < \cdots < \gamma_k \leqq n$,

$$0 \leqq \gamma_i \leqq \lambda_i < \alpha\ ,\ i = 1, \dots, m, \text{ and } \alpha < \lambda_i \leqq \gamma_i \leqq n\ ,\ i = m + 1, \dots, k,$$

where at least one of the above inequalities between λ_i *and* γ_i *strictly holds. If* p *and* q *have* k *common vanishing subintervals* $I_1, \dots, I_k$ *of* $\mathcal{S}$ *with* $I_1 < \cdots < I_k$, *then for any* $J \in \mathcal{S} - \{I_1, \dots, I_k\}$ *such that* $\{I_1, \dots, I_k, J\}$ *is a linearly ordered subset of* $\mathcal{S}$

$$|p[J]| < |q[J]|.$$

3. PROOF OF THEOREM

Now we are in a position to state a proof of Theorem.

First we show that

$$E_{\Lambda'}(u_n)_p \leqq E_\Lambda(u_n)_p \text{ for all } p \in (1, \infty). \tag{1}$$

For any given p with $1 < p < \infty$, let $\tilde{u}$ be the unique best L_p approximation to u_n from $\mathrm{Span}\{u_{\lambda_0}, \dots, u_{\lambda_m}\}$. By Lemma 4, since $u_n - \tilde{u}$ changes sign just $m + 1$ times in $[a, b]$, it has $m + 1$ essential zeros $z_0, z_1, \dots, z_m$ $(a < z_0 < \cdots <$

$z_m < b$). We note that $u_n - \tilde{u}$ changes sign at $z_i, i = 0, \ldots, m$. For each $z_i, i = 0, 1, \ldots, m$, there exist a sequence of subintervals $I_i^{(k)} = [z_i - \delta_i^{(k)}, z_i + \varepsilon_i^{(k)}], k \in \mathbb{N}$ which contain z_i and satisfy that

(i) $a < z_0 - \delta_0^{(k)}\ ,\ z_m + \varepsilon_m^{(k)} < b,\ k \in \mathbb{N};$

(ii) $z_i + \varepsilon_i^{(k)} < z_{i+1} - \delta_i^{(k)}\ ,\ i = 0, 1, \ldots, m-1,\ k \in \mathbb{N};$

(iii) $\lim_{k\to\infty} \delta_i^{(k)} = 0\ ,\ \lim_{k\to\infty} \varepsilon_i^{(k)} = 0\ ,\ i = 0, 1, \ldots, m;$

(iv) $\displaystyle\int_{I_i^{(k)}} (u_n - \tilde{u})\, \mathrm{d}x = 0\ ,\ i = 0, 1, \ldots, m,\ k \in \mathbb{N}.$

For each $k \in \mathbb{N}$, let $\tilde{w}^{(k)} = \sum_{i=0}^m c_i^{(k)} u_{\lambda'_i}$ be the unique function of $\mathrm{Span}\{u_{\lambda'_0}, \ldots, u_{\lambda'_m}\}$ such that $(u_n - \tilde{w}^{(k)})[I_i^{(k)}] = 0, i = 0, 1, \ldots, m$. By the property of $\tilde{w}^{(k)}, k \in \mathbb{N}$, since $u_n - \tilde{w}^{(k)}$ changes sign just $m+1$ times in $[a, b]$, it has $m+1$ essential zeros $y_i^{(k)} \in I_i^{(k)}, i = 0, 1, \ldots, m$ at which $u_n - \tilde{w}^{(k)}$ changes sign. From (iii), noting that the sequences $\{y_i^{(k)}\}, i = 0, \ldots, m$ satisfy the condition of Lemma 3, we have

$$\sup_{0\leqq i\leqq m, k\in\mathbb{N}} |c_i^{(k)}| < +\infty.$$

Hence, we obtain

$$\sup_{k\in\mathbb{N}} \|u_n - \tilde{w}^{(k)}\|_\infty < +\infty. \tag{2}$$

For each $k \in \mathbb{N}$, we put

$$H_0^{(k)} := [a, z_0 - \delta_0^{(k)}],\ H_{m+1}^{(k)} := [z_m + \varepsilon_m^{(k)}, b]$$

and

$$H_i^{(k)} := [z_{i-1} + \varepsilon_{i-1}^{(k)}, z_i - \delta_i^{(k)}],\ i = 1, \ldots, m.$$

Furthermore, we decompose each $H_i^{(k)}, i = 0, 1, \ldots, m+1, k \in \mathbb{N}$ to subintervals $H_{i,0}^{(k)}, \ldots, H_{i,n(i,k)}^{(k)}$, i.e., $H_i^{(k)} = H_{i,0}^{(k)} \cup \cdots \cup H_{i,n(i,k)}^{(k)}$ and $H_{i,p}^{(k)} \cap H_{i,q}^{(k)}$ is a empty set or a one point set for $p \neq q$, such that $H_{i,0}^{(k)}, \ldots, H_{i,n(i,k)}^{(k)}$ have the same width $h_i^{(k)}$,

$$\left| \int_{H_i^{(k)}} |u_n - \tilde{u}|^p\, \mathrm{d}x - \sum_{\ell=0}^{n(i,k)} |(u_n - \tilde{u})(x_{i,\ell})|^p \cdot h_i^{(k)} \right| < \tfrac{1}{k} \text{ for all } x_{i,\ell} \in H_{i,\ell}^{(k)}, \tag{3}$$

$\ell = 0, \ldots, n(i,k)$ and

$$\left| \int_{H_i^{(k)}} |u_n - \tilde{w}^{(k)}|^p\, dx - \sum_{\ell=0}^{n(i,k)} |(u_n - \tilde{w}^{(k)})(x_{i,\ell})|^p \cdot h_i^{(k)} \right| < \tfrac{1}{k} \tag{4}$$

for all $x_{i,\ell} \in H_{i,\ell}^{(k)}, \ell = 0, \ldots, n(i,k)$. However, without loss of generality, we can assume that $\lim_{k\to\infty} h_i^{(k)} = 0$.

For each $k \in \mathbb{N}$, let

$$S_k = \{I_0^{(k)}, \ldots, I_m^{(k)}\} \cup \{H_{0,0}^{(k)}, \ldots, H_{0,n(0,k)}^{(k)}\} \cup \cdots \cup \{H_{m+1,0}^{(k)}, \ldots, H_{m+1,n(m+1,k)}^{(k)}\}.$$

Since S_k is a linearly ordered subset of $(\mathcal{S}, \leqq)$, from Lemma 5 we have

$$|(u_n - \tilde{w}^{(k)})[H_{i,\ell}^{(k)}]| < |(u_n - \tilde{u})[H_{i,\ell}^{(k)}]| \tag{5}$$

for all $H_{i,\ell}^{(k)}$, $\ell = 0, \dots, n(i,k)$, $i = 0, \dots, m+1$, $k \in \mathbb{N}$. Since $u_0, \dots, u_n$ are continuous functions, we easily see that

$$(u_n - \tilde{u})[H_{i,\ell}^{(k)}] = \int_{H_{i,\ell}^{(k)}} (u_n - \tilde{u})\,\mathrm{d}x = (u_n - \tilde{u})(c_{i,\ell}^{(k)}) \cdot h_i^{(k)} \text{ for some } c_{i,\ell}^{(k)} \in H_{i,\ell}^{(k)} \tag{6}$$

and

$$(u_n - \tilde{w}^{(k)})[H_{i,\ell}^{(k)}] = \int_{H_{i,\ell}^{(k)}} (u_n - \tilde{w}^{(k)})\,\mathrm{d}x = (u_n - \tilde{w}^{(k)})(d_{i,\ell}^{(k)}) \cdot h_i^{(k)} \tag{7}$$

for some $d_{i,\ell}^{(k)} \in H_{i,\ell}^{(k)}$.
From (5), (6) and (7), we have

$$\begin{aligned}\sum_{\ell=0}^{n(i,k)} |(u_n - \tilde{w}^{(k)})[H_{i,\ell}]|^p &= \sum_{\ell=0}^{n(i,k)} |(u_n - \tilde{w}^{(k)})(d_{i,\ell}^{(k)})|^p \cdot (h_i^{(k)})^p \\ < \sum_{\ell=0}^{n(i,k)} |(u_n - \tilde{u})[H_{i,\ell}]|^p &= \sum_{\ell=0}^{n(i,k)} |(u_n - \tilde{u})(c_{i,\ell}^{(k)})|^p \cdot (h_i^{(k)})^p,\end{aligned}$$

$i = 0, \dots, m, k \in \mathbb{N}$.
Furthermore, we obtain

$$\sum_{\ell=0}^{n(i,k)} |(u_n - \tilde{w}^{(k)})(d_{i,\ell}^{(k)})|^p \cdot h_i^{(k)} < \sum_{\ell=0}^{n(i,k)} |(u_n - \tilde{u})(c_{i,\ell}^{(k)})|^p \cdot h_i^{(k)}, \tag{8}$$

$i = 0, \dots, m,\ k \in \mathbb{N}$.
For each $k \in \mathbb{N}$ and any subinterval K_i of $K_0 := [a, z_0], K_1 := [z_0, z_1], \dots,$ $K_{m+1} := [z_m, b]$, without loss of generality say $K_i = [z_{i-1}, z_i]$, we have

$$\begin{aligned}\int_{K_i} |u_n - \tilde{u}|^p\,\mathrm{d}x = \int_{H_i^{(k)}} |u_n - \tilde{u}|^p\,\mathrm{d}x &+ \int_{z_{i-1}}^{z_{i-1}+\varepsilon_{i-1}^{(k)}} |u_n - \tilde{u}|^p\,\mathrm{d}x \\ &+ \int_{z_i - \delta_i^{(k)}}^{z_i} |u_n - \tilde{u}|^p\,\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\int_{K_i} |u_n - \tilde{w}^{(k)}|^p\,\mathrm{d}x = \int_{H_i^{(k)}} |u_n - \tilde{w}^{(k)}|^p\,\mathrm{d}x &+ \int_{z_{i-1}}^{z_{i-1}+\varepsilon_{i-1}^{(k)}} |u_n - \tilde{w}^{(k)}|^p\,\mathrm{d}x \\ &+ \int_{z_i - \delta_i^{(k)}}^{z_i} |u_n - \tilde{w}^{(k)}|^p\,\mathrm{d}x.\end{aligned}$$

Then we obtain an estimation of $\int_{H_i^{(k)}} |u_n - \tilde{w}^{(k)}|^p\,\mathrm{d}x$:

$$
\begin{aligned}
&\text{(by (4) and (7))} \quad \int_{H_i^{(k)}} |u_n - \tilde{w}^{(k)}|^p \,\mathrm{d}x \;<\; \sum_{\ell=0}^{n(i,k)} |(u_n - \tilde{w}^{(k)})(d_{i,\ell}^{(k)})|^p \cdot h_i^{(k)} + \tfrac{1}{k} \\
&\text{(by (8))} \qquad\qquad\qquad\qquad\qquad\quad <\; \sum_{\ell=0}^{n(i,k)} |(u_n - \tilde{u})(c_{i,\ell}^{(k)})|^p \cdot h_i^{(k)} + \tfrac{1}{k} \\
&\text{(by (3))} \qquad\qquad\qquad\qquad\qquad\quad <\; \int_{H_i^{(k)}} |u_n - \tilde{u}|^p \,\mathrm{d}x + \tfrac{2}{k}.
\end{aligned}
$$

From this inequality stated above, we observe that for each $i = 0, \ldots, m+1$ and each $k \in \mathbb{N}$

$$
\begin{aligned}
\int_{K_i} |u_n - \tilde{w}^{(k)}|^p \,\mathrm{d}x &< \int_{H_i^{(k)}} |u_n - \tilde{u}|^p \,\mathrm{d}x + \tfrac{2}{k} + \int_{z_{i-1}}^{z_{i-1}+\varepsilon_{i-1}^{(k)}} |u_n - \tilde{w}^{(k)}|^p \,\mathrm{d}x \\
&\quad + \int_{z_i-\delta_i^{(k)}}^{z_i} |u_n - \tilde{w}^{(k)}|^p \,\mathrm{d}x \\
&< \int_{K_i} |u_n - \tilde{u}|^p \,\mathrm{d}x \;+\; \tfrac{2}{k} \;+\; \int_{z_{i-1}}^{z_{i-1}+\varepsilon_{i-1}^{(k)}} |u_n - \tilde{w}^{(k)}|^p \,\mathrm{d}x \\
&\quad + \int_{z_i-\delta_i^{(k)}}^{z_i} |u_n - \tilde{w}^{(k)}|^p \,\mathrm{d}x.
\end{aligned}
$$

Hence we see that for each $k \in \mathbb{N}$

$$
\begin{aligned}
\|u_n - \tilde{w}^{(k)}\|_p^p &< \|u_n - \tilde{u}\|_p^p + \tfrac{2(m+2)}{k} \\
&\quad + \sum_{i=0}^{m+1} \left(\int_{z_{i-1}}^{z_{i-1}+\varepsilon_{i-1}^{(k)}} |u_n - \tilde{w}^{(k)}|^p \,\mathrm{d}x + \int_{z_i-\delta_i^{(k)}}^{z_i} |u_n - \tilde{w}^{(k)}|^p \,\mathrm{d}x \right),
\end{aligned}
$$

where $z_{-1} = a, z_{m+1} = b$. If we put

$$
A(k) := \sum_{i=0}^{m+1} \left(\int_{z_{i-1}}^{z_{i-1}+\varepsilon_{i-1}^{(k)}} |u_n - \tilde{w}^{(k)}|^p \,\mathrm{d}x + \int_{z_i-\delta_i^{(k)}}^{z_i} |u_n - \tilde{w}^{(k)}|^p \,\mathrm{d}x \right),
$$

then $A(k)$ satisfies $A(k) \to 0$ $(k \to \infty)$ by the condition (iii) and (2). On the other hand, if $\tilde{w}$ is the unique best L_p approximation to u_n from $\mathrm{Span}\{u_{\lambda'_0}, \ldots, u_{\lambda'_m}\}$, then we have

$$
\|u_n - \tilde{w}\|_p^p \leqq \inf_{k \in \mathbb{N}} \|u_n - \tilde{w}^{(k)}\|_p^p.
$$

Consequently, we obtain that

$$\|u_n - \tilde{w}\|_p^p \leqq \underline{\lim}_{k\to\infty} \|u_n - \tilde{w}^{(k)}\|_p^p$$
$$\leqq \underline{\lim}_{k\to\infty} \left(\|u_n - \tilde{u}\|_p^p + \tfrac{2(m+2)}{k} + A(k) \right) = \|u_n - \tilde{u}\|_p^p.$$

This means that (1) holds.

Finally we show that

(9) $$E_{\Lambda'}(u_n)_\infty \leqq E_\Lambda(u_n)_\infty.$$

To show this, we observe some general results. For each p with $1 < p \leqq \infty$, let $\tilde{u}_p = \sum_{i=0}^m c_i^{(p)} u_{\lambda_i}$ be a best L_p approximation to u_n from $\mathrm{Span}\{u_{\lambda_0}, \dots, u_{\lambda_m}\}$. Since $\|u_n - \tilde{u}_p\|_p \leqq \|u_n\|_p$, $\{\|\tilde{u}_p\|_p\}$ is bounded in $\mathbb{R}$. Noting that $u_0, \dots, u_n$ is linearly independent, we have

$$\sup_{0\leqq i\leqq m, p\in(1,\infty)} |c_i^{(p)}| < +\infty.$$

Then, there exists a sequence $p_k \in (1,\infty), k \in \mathbb{N}$ with $p_k \to \infty$ $(k \to \infty)$ and each sequence $\{c_i^{(p_k)}\}, i = 0, \dots, m$ converges to a c_i^*. Since we easily see that

$$\|u_n - \tilde{u}_{p_k}\|_{p_k} \leqq \|u_n - \tilde{u}_\infty\|_{p_k},$$

we obtain

$$\left\| u_n - \sum_{i=0}^m c_i^* u_{\lambda_i} \right\|_\infty = \lim_{p_k\to\infty} \|u_n - \tilde{u}_{p_k}\|_{p_k} \leqq \|u_n - \tilde{u}_\infty\|_\infty.$$

Here we use the result that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$ for all $f \in C[a,b]$ and the convergence of the sequences $\{c_i^{(p_k)}\}, i = 0, \dots, m$. This means that $\sum_{i=0}^m c_i^* u_{\lambda_i}$ is a best L_∞ approximation to u_n from $\mathrm{Span}\{u_{\lambda_0}, \dots, u_{\lambda_m}\}$.

Now we turn to the proof of (9). Suppose on the contrary that $E_{\Lambda'}(u_n)_\infty > E_\Lambda(u_n)_\infty$. From the result stated above, we can find a sufficiently large positive number $p \in (1,\infty)$ such that

(10) $$|E_{\Lambda'}(u_n)_\infty - E_{\Lambda'}(u_n)_p| < \varepsilon \text{ and } |E_\Lambda(u_n)_\infty - E_\Lambda(u_n)_p| < \varepsilon,$$

where $\varepsilon := \frac{E_{\Lambda'}(u_n)_\infty - E_\Lambda(u_n)_\infty}{3}$. By (10), we get $E_{\Lambda'}(u_n)_p > E_\Lambda(u_n)_p$, which contradicts (1). This completes the proof. □

We have proven Theorem, but the following problem is still open.

Problem. *Let p be a positive number p with $1 < p \leqq \infty$ or ∞ and $\{u_0, \dots, u_n\}$ a quasi Descartes system of $C[a,b]$. If $\Lambda : (0 \leqq)\lambda_0 < \cdots < \lambda_m(< n)$ and $\Lambda' : (0 \leqq)\lambda'_0 < \cdots < \lambda'_m(< n)$ satisfy $\lambda_i \leqq \lambda'_i, i = 0, \dots, m$ and $\lambda_j < \lambda'_j$ for some j with $0 \leqq j \leqq m$, then is it true that*

$$E_{\Lambda'}(u_n)_p < E_\Lambda(u_n)_p \ ?$$

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