EXTENSIONS OF SEMI-HÖLDER REAL VALUED FUNCTIONS ON A QUASI–METRIC SPACE

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Abstract. In this note the semi-Hölder real valued functions on a quasi–metric (asymmetric metric) space are defined. An extension theorem for such functions and some consequences are presented.

Keywords. Semi-Hölder functions, extensions.

1. PRELIMINARIES

Let $X$ be a non-empty set. A function $d : X \times X \to [0, \infty)$ is called a quasi–metric on $X$ [9] (see also [1]) if the following conditions hold:

AM1) $d(x, y) = d(y, x) = 0 \iff x = y$
AM2) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

When $X$ is non-empty set and $d$ a quasi–metric on $X$, the pair $(X, d)$ is called a quasi–metric space.

The function $\overline{d} : X \times X \to [0, \infty)$ defined by $\overline{d}(x, y) = d(y, x)$ for all $x, y \in X$ is also a quasi–metric on $X$, called the conjugate quasi–metric of $d$.

Obviously, the function $d^*(x, y) = \max\{d(x, y), \overline{d}(x, y)\}$ is a metric on $X$. Each quasi–metric $d$ on $X$ induces a topology $\tau(d)$, which has as a base the family of balls (forward open balls [4]).

\begin{equation}
B^+(x, \varepsilon) : \{y \in X : d(x, y) < \varepsilon\}, \ x \in X, \ \varepsilon > 0.
\end{equation}

This topology is called the forward topology of $X$ ([4], [1]) and is denoted also by $\tau_+$.

The topology induced by the quasi–metric $\overline{d}$ is called the backward topology and is denoted by $\tau_-$.

The topology $\tau_+$ is a $T_0$ topology. If the condition AM1) is replaced by the condition

AM0) $d(x, y) \geq 0$ and $d(x, y) = 0$, for all $x, y \in X$

then the topology $\tau_+$ is a $T_1$ topology.

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Let \((X, d)\) be quasi-metric space. A sequence \((x_k)_{k \geq 1}\) \(d\)-converge to \(x_0 \in X\), respectively \(\overline{d}\)-converge to \(x_0 \in X\), iff
\[
\lim_{k \to \infty} d(x_0, x_k) = 0, \quad \text{respectively} \\
\lim_{k \to \infty} \overline{d}(x_0, x_k) = \lim_{k \to \infty} d(x_k, x_0) = 0.
\]
A set \(K \subset X\) is called \(d\)-compact if every open cover of \(K\) with respect to the topology \(\tau_+\) has a finite subcover. We say that \(K\) is \(d\)-sequentially compact if every sequence in \(K\) has a \(d\)-convergent subsequence with limit in \(K\) (Definition 4.1 in [1]).

Finally, the set \(Y\) in \((X, d)\) is called \((d, \overline{d})\)-sequentially compact if every sequence \((y_n)_{n \geq 1}\) in \(Y\) has a subsequence \((y_{n_k})_{k \geq 1}\) \(d\)-convergent to \(u \in Y\) and \(d\)-convergent to \(v \in Y\). By Lemma 3.1 in [1] if the topology of \(X\) is \(T_1\), i.e. \(d\) verifies the axioms AM0) and AM2), it follows that \(u = v\).

The following definition of a \(d\)-semi-Hölder function (of exponent \(\alpha \in (0, 1)\)) is inspired by the definition of semi-Lipschitz function in [9].

**Definition 1.** Let \(Y\) be a non-empty subset of a quasi-metric space \((X, d)\), and \(\alpha \in (0, 1)\) arbitrarily chosen, but fixed. A function \(f : Y \to \mathbb{R}\) is called \(d\)-semi-Hölder (of exponent \(\alpha\)) if there exists \(L = L(f, Y) \geq 0\) (named a \(d\)-semi-Hölder constant for \(f\)) such that
\[
f(x) - f(y) \leq Ld^{\alpha}(x, y),
\]
for all \(x, y \in Y\).

The smallest \(d\)-semi-Hölder constant for \(f\), verifying (1.3), is denoted by \(\|f\|_{\alpha, Y}\) and
\[
\|f\|_{\alpha, Y} = \sup \left\{ \frac{(f(x) - f(y))d^{\alpha}(x, y)}{d(x, y)} : d(x, y) > 0, \quad x, y \in Y \right\}.
\]
This means that \(\|f\|_{\alpha, Y} = \inf \{L \geq 0 : L\) verifying (1.3)\}.

The set of all \(d\)-semi-Hölder function \(f : Y \to \mathbb{R}\) is denoted by \(\Lambda_\alpha(Y)\), i.e.
\[
\Lambda_\alpha(Y) := \{f : Y \to \mathbb{R}, \text{ } f \text{ is } d\text{-semi-Hölder of exponent } \alpha\}.
\]
This set is a cone in the linear space \(\mathbb{R}^Y\) of all functions \(f : Y \to \mathbb{R}\), i.e. \(\Lambda_\alpha(Y)\) is closed with respect to pointwise operations of multiplication with nonnegative real numbers of a function in \(\Lambda_\alpha(Y)\), and of addition of two functions in \(\Lambda_\alpha(Y)\).

The functional \(\|\cdot\|_{\alpha, Y} : \Lambda_\alpha(Y) \to [0, \infty)\) is nonnegative and sublinear, and the pair \((\Lambda_\alpha(Y), \|\cdot\|_{\alpha, Y})\) is called the asymmetric normed cone of \(d\)-semi Hölder functions on \(Y\) (compare with [10]).

The cone \((\Lambda_\alpha(Y), \|\cdot\|_{\alpha, Y})\) is different from the cone of \(d\)-semi-Lipschitz function \((\alpha = 1)\) considered in [9]. For example, if one considers \(Y = [0, 1]\) \(d(x, y) = |x - y|\) and \(f : [0, 1] \to \mathbb{R}, f(x) = x \sin \frac{x}{2}, x \in (0, 1]; f(0) = 0\), then it is known that \(f \in \Lambda_\alpha(X, d)\) if and only if \(\alpha \in (0, 1/2]\) (see [14], Problem 153) and in this case \(\|f\|_{\alpha, Y} \leq \{1 + 2 \ln(1 + 2\pi) + 2\pi\}^{1/2}\).
2. EXTENSIONS OF $d$-SEMI-HÖLDER FUNCTIONS

Let $(X, d)$ be a quasi–metric space, $Y \subset X$ and $f \in \Lambda_\alpha(Y)$. A function $F \in \Lambda_\alpha(X)$ is called an extension of $f$ (preserving the semi-Hölder constant $L(f, Y)$ if

\[(2.1) \quad F|_Y = f \text{ and } L(F, X) = L(f, Y)\]

The existence of extension in $\Lambda_\alpha(X)$ for each $f \in \Lambda_\alpha(X)$ is assured by the following theorem.

**Theorem 2.** Let $(X, d)$ be a quasi–metric space, $Y \subset X$ and $f \in \Lambda_\alpha(Y)$ with $d$-semi-Hölder constant $L(f, Y)$. Then there exist $F \in \Lambda_\alpha(X)$ such that $F|_Y = f$ and $L(F, X) = L(f, Y)$.

**Proof.** Let $G : X \to \mathbb{R}$ defined by

\[(2.2) \quad G(x) = \sup_{y \in Y} \{f(y) - L(f, Y) \cdot d^\alpha(y, x)\}, \quad x \in X.\]

Let $y_0 \in Y$ be a fixed element, and $x \in X$. For every $y \in Y$,

\[
f(y) - L(f, Y) \cdot d^\alpha(y, x) = f(y) - f(y_0) - L(f, Y) \cdot d^\alpha(y, x) + f(y_0) \\
\leq L(f, Y)[d^\alpha(y, y_0) - L(f, Y) \cdot d^\alpha(y, x)] + f(y_0) \\
= f(y_0) + L(f, Y)[d^\alpha(y, y_0) - d^\alpha(y, x)] \\
\leq f(y_0) + L(f, Y) \cdot d^\alpha(x, y_0).
\]

Then it follows that the set

\[
\{f(y) - L(f, Y) \cdot d^\alpha(y, x) : y \in Y\}
\]

is bounded from above, and $G(x)$ exists for every $x \in X$. By the definition of $G(x)$, for every $y \in Y$

\[
G(x) \geq f(y) - L(f, Y) \cdot d^\alpha(y, x), \quad x \in X,
\]

and for $x = y$ one obtains

\[
G(y) \geq f(y).
\]

On the other hand, for $y \in Y$ and every $y' \in Y$,

\[
f(y') - f(y) \leq L(f, Y) \cdot d^\alpha(y', y).
\]

It follows

\[
f(y') - L(f, Y) \cdot d^\alpha(y', y) \leq f(y),
\]

and taking the supremum with respect to $y' \in Y$ one obtains

\[
G(y) \leq f(y), \quad y \in Y.
\]

Consequently $G|_Y = f$.

Now, let $u, v \in X$ and $\varepsilon > 0$. Choosing $y \in Y$ such that

\[
G(u) \leq f(y) - L(f, Y) \cdot d^\alpha(y, u) + \varepsilon,
\]

and
it follows
\[ G(u) - G(v) \leq f(y) - L(f, Y) d^a(y, u) + \varepsilon - f(y) + L(f, Y) d^a(y, v) \]
\[ = L(f, Y) [d^a(y, v) - d^a(y, u)] + \varepsilon \]
\[ \leq L(f, Y) d^a(u, v) + \varepsilon. \]
Because \( \varepsilon > 0 \) is arbitrarily chosen, one obtains:
\[ G(u) - G(v) \leq L(f, Y) d^a(u, v), \]
for \( u, v \in X \), i.e. \( G \in \Lambda_\alpha(X) \). Moreover \( L(G, X) \leq L(f, Y) \). Because \( G|_Y = f \) one obtains also
\[ L(f, Y) = L(G|_Y, Y) \leq L(G, X) \]
and consequently, \( L(f, Y) = L(G, X) \). \( \square \)

**Remark 3.** Observe that the function
\[
F(x) = \inf_{y \in Y} \{ f(y) + L(f, Y) d^a(x, y) \}, \ x \in X.
\]
is another extension of \( f \in \Lambda_\alpha(f, Y) \).
Moreover, if \( H \) is any extension of \( f \), i.e. \( H|_Y = f \) and \( L(H, X) = L(f, Y) \) then
\[
G(x) \leq H(x) \leq F(x), \ x \in X
\]
where \( G \) is defined by (2.2) and \( F \) is defined by (2.3).
From (2.4) it follows that \( G \) is the minimal extension of \( f \), and \( F \) is the maximal extension of \( f \). \( \square \)

Indeed let \( H \) an extension of \( f \in \Lambda_\alpha(Y) \). Then, for arbitrary \( x \in X \) and \( y \in Y \) we have
\[ H(x) - H(y) \leq L(f, Y) d^a(x, y) \]
implying
\[ H(x) \leq H(y) + L(f, Y) d^a(x, y) \]
\[ = f(y) + L(f, y) d^a(x, y) \]
Taking the infimum with respect to \( y \in Y \) one obtain
\[ H(x) \leq F(x), \ x \in X \]
Analogously,
\[ H(y) - H(x) \leq L(f, Y) d^a(y, x) \]
implies
\[ f(y) - L(f, Y) d^a(y, x) \geq H(x), \]
and taking the supremum with respect to \( y \in Y \) one obtain
\[ G(x) \geq H(x), \ x \in X. \]
It follows (2.4).
Remark 4. For $f \in \Lambda_\alpha(Y)$ denote by $\mathcal{E}(f)$ the (non-empty) set of all extensions of $f$ i.e.

$$\mathcal{E}(f) := \{H \in \Lambda_\alpha(X) : H|_Y = f \text{ and } L(H, X) = L(f, Y)\}$$

Obviously, $\mathcal{E}(f)$ is a convex subset of the cone $\Lambda_\alpha(X)$.

Remark 5. Let $f \in \Lambda_\alpha(Y)$ and let $\|f\|_{\alpha,Y}$ be the smallest $d$-semi-Hölder constant for $f$ on $Y$ (see (2.5)). Then the functions $G$ and $F$ defined by (2.2) and (2.3), where $L(f, Y) = \|f\|_{\alpha,Y}$ are extensions for $f$, preserving the constant $\|f\|_{\alpha,Y}$.

Consider a fixed element $y_0 \in Y$, and let

$$\Lambda_{\alpha,0}(Y) := \{f \in \Lambda_\alpha(Y) : f(y_0) = 0\}.$$  \hspace{1cm} (2.6)

Then the functional $\|\cdot\|_{\alpha,Y} : \Lambda_{\alpha,0}(Y) \to [0, \infty)$ defined by

$$\|f\|_{\alpha,Y} = \sup\{\frac{|f(x)-f(y)|}{d(x, y)} : d(x, y) > 0; x, y \in Y\}$$  \hspace{1cm} (2.7)

is a quasi-norm on $\Lambda_{\alpha,0}(Y)$, i.e. the following properties hold:

a) $\|f\|_{\alpha,Y} \geq 0$; $f = 0$ iff $f \in \Lambda_{\alpha,0}(Y)$ and $\|f\|_{\alpha,Y} = \|\!-\!f\|_{\alpha,Y} = 0$, $f \in \Lambda_{\alpha,0}(Y)$;

b) $\|af\|_{\alpha,Y} = a \|f\|_{\alpha,Y}$ for every $f \in \Lambda_{\alpha,0}(Y)$ and $a \geq 0$;

c) $\|f + g\|_{\alpha,Y} \leq \|f\|_{\alpha,Y} + \|g\|_{\alpha,Y}$ for all $f, g \in \Lambda_{\alpha,0}(Y)$.

Remark 6. Let $f \in \Lambda_\alpha(X)$ and $Y_1 \subset Y_2 \subset X$. Suppose that $q \geq \|f\|_{\alpha,X}$ and let

$$G_1(x) = \sup\{f(y) - q d^\alpha(y, x)\}, \quad x \in X,$$

$$G_2(x) = \sup\{f(y) - q d^\alpha(x, y)\}, \quad x \in X$$

Then $G_1(x) \leq G_2(x) \leq f(x)$, for all $x \in X$ and $G_1|_{Y_1} = G_2|_{Y_1} = f|_{Y_1}$.

Also, if

$$F_1(x) = \inf\{f(y) + q d^\alpha(y, x)\}, \quad x \in X$$

and

$$F_2(x) = \inf\{f(y) + q d^\alpha(x, y)\}, \quad x \in X$$

then

$$F_1(x) \geq F_2(x) \geq f(x), \text{ for all } x \in X$$

and

$$F_1|_{Y_1} = F_2|_{Y_1} = f|_{Y_1}.$$  \hspace{1cm} (2.8)

Consequently, if $(Y_n)_{n \geq 1}$ is a sequence in $2^X$ such that $Y_1 \subset Y_2 \subset \ldots \subset Y_n \subset \ldots$, $f \in \Lambda_{\alpha,0}(X)$ and $q \geq \|f\|_{\alpha,X}$ then the sequences $(G_n)_{n \geq 1}$ and $(F_n)_{n \geq 1}$, where

$$G_n(x) = \sup\{f(y) - q d^\alpha(y, x)\}, \quad x \in X$$

and

$$F_n(x) = \inf\{f(y) + q d^\alpha(y, x)\}, \quad x \in X$$

for all $n \geq 1$.  \hspace{1cm} (2.8)
and
\[ F_n(x) = \inf_{y \in Y_n} \{ f(y) + qd^\alpha(x, y) \}, \quad x \in X \]
are monotonically increasing, respectively decreasing, \( G_n, F_n \in \Lambda_\alpha(X) \),
\( n = 1, 2, \ldots, \) and \( G_n(x) \leq f(x) \leq F_n(x) \), for all \( x \in X \).

Because, for every \( y \in Y_n \)
\[ f(y) - qd^\alpha(y, x) \leq G_n(x) \leq F_n(x) \leq f(y) + qd^\alpha(x, y) \]
it follows that,
\[ F_n(x) - G_n(x) \leq q \inf_{y \in Y_n} [d^\alpha(x, y) + d^\alpha(y, x)]. \]
Taking the infimum with respect to \( y \in Y_n \) one obtains:
\begin{equation}
(2.8) \quad F_n(x) - G_n(x) \leq q \inf_{y \in Y_n} [d^\alpha(x, y) + d^\alpha(y, x)].
\end{equation}
for every \( x \in X \).

If \( Y_n \) is \( d^s \)-dense in \( X \), where \( d^s(x, y) = d(x, y) \lor d(y, x) \) for every \( x, y \in X \) then, by (2.8) it follows that \( F_n(x) = G_n(x), \quad x \in X \). Consequently, a function \( f \in \Lambda_\alpha(Y) \) where \( Y \) is \( d^s \)-dense in \( X \) has an unique extension \( F \in \Lambda_\alpha(X) \).

REFERENCES