REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 38 (2009) no. 2, pp. 164–169
ictp.acad.ro/jnaat

EXTENSIONS OF SEMI-HÖLDER REAL VALUED FUNCTIONS ON A QUASI–METRIC SPACE

COSTICĂ MUSTĂŢA*

Abstract. In this note the semi-Hölder real valued functions on a quasi–metric (asymmetric metric) space are defined. An extension theorem for such functions and some consequences are presented.

MSC 2000. 26A16, 46A22.

Keywords. Semi-Hölder functions, extensions.

1. PRELIMINARIES

Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is called a quasi–metric on X [9] (see also [1]) if the following conditions hold:

AM1) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$
AM2) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

When X is non-empty set and d a quasi–metric on X, the pair (X, d) is called a quasi–metric space.

The function $\overline{d} : X \times X \to [0, \infty)$ defined by $\overline{d}(x, y) = d(y, x)$ for all $x, y \in X$ is also a quasi–metric on X, called the conjugate quasi–metric of d.

Obviously, the function $d^s(x, y) = \max\{d(x, y), \overline{d}(x, y)\}$ is a metric on X. Each quasi–metric d on X induces a topology $\tau(d)$, which has as a base the family of balls (forward open balls [4]).

$$B^+(x, \varepsilon) : \{y \in X : d(x, y) < \varepsilon\},\ x \in X,\ \varepsilon > 0. \tag{1.1}$$

This topology is called the *forward topology* of X ([4]), [1]) and is denoted also by τ_+.

The topology induced by the quasi–metric $\overline{d}$ is called the *backward topology* and is denoted by τ_-.

The topology τ_+ is a T_0 topology. If the condition AM1) is replaced by the condition

AM0) $d(x, y) \geq 0$ and $d(x, y) = 0$, for all $x, y \in X$
then the topology τ_+ is a T_1 topology.

*Institutul de Calcul "T. Popoviciu", Cluj-Napoca, Romania, e-mail: cmustata2001@yahoo.com, cmustata@ictp.acad.ro.

Let (X,d) be quasi–metric space. A sequence $(x_k)_{k\geq 1}$ d-converge to $x_0 \in X$, respectively $\overline{d}$-converge to $x_0 \in X$, iff

$$\lim_{k\to\infty} d(x_0,x_k)=0, \text{ respectively} \tag{1.2}$$
$$\lim_{k\to\infty} \overline{d}(x_0,x_k)=\lim_{k\to\infty} d(x_k,x_0)=0.$$

A set $K\subset X$ is called d- compact if every open cover of K with respect to the topology τ_+ has a finite subcover. We say that K is d-sequentially compact if every sequence in K has a d-converget subsequence with limit in K (Definition 4.1 in [1]).

Finally, the set Y in (X,d) is called $(d,\overline{d})$-sequentially compact if every sequence $(y_n)_{n\geq 1}$ in Y has a subsequence $(y_{n_k})_{k\geq 1}$ d-convergent to $u\in Y$ and d-convergent to $v\in Y$. By Lemma 3.1 in [1] if the topology of X is T_1, i.e. d verifies the axioms AM0) and AM2), it follows that $u=v$.

The following definition of a *d-semi-Hölder* function (of exponent $\alpha\in(0,1)$) is inspired by the definition of semi-Lipschitz function in [9].

Definition 1. *Let Y be a non-empty subset of a quasi–metric space (X,d), and $\alpha\in(0,1)$ arbitrarily chosen, but fixed. A function $f:Y\to\mathbb{R}$ is called d-semi-Hölder (of exponent α) if there exists $L=L(f,Y)\geq 0$ (named a d-semi-Hölder constant for f) such that*

$$f(x)-f(y)\leq Ld^{\alpha}(x,y), \tag{1.3}$$

for all $x,y\in Y$.

The smallest d-semi-Hölder constant for f, verifying (1.3), is denoted by $\|f|_{\alpha,Y}$ and

$$\|f|_{\alpha,Y}=\sup\left\{\tfrac{(f(x)-f(y))\vee 0}{d^{\alpha}(x,y)}: d(x,y)>0,\quad x,y\in Y\right\}. \tag{1.4}$$

This means that $\|f|_{\alpha,Y}=\inf\{L\geq 0: L \text{ verifying (1.3)}\}$.

The set of all d-semi-Hölder function $f:Y\to\mathbb{R}$ is denoted by $\Lambda_\alpha(Y)$, i.e.

$$\Lambda_\alpha(Y):=\{f:Y\to\mathbb{R}, f \text{ is } d\text{-semi-Hölder of exponent } \alpha\}. \tag{1.5}$$

This set is a cone in the linear space $\mathbb{R}^Y$ of all functions $f:Y\to\mathbb{R}$, i.e. $\Lambda_\alpha(Y)$ is closed with respect to pointwise operations of multiplication with nonnegative real numbers of a function in $\Lambda_\alpha(Y)$, and of addition of two functions in $\Lambda_\alpha(Y)$.

The functional $\|\cdot|_{\alpha,Y}:\Lambda_\alpha(Y)\to[0,\infty)$ is nonnegative and sublinear, and the pair $(\Lambda_\alpha(Y),\|\cdot|_{\alpha,Y})$ is called the asymmetric normed cone of d-semi Hölder functions on Y (compare with [10]).

The cone $(\Lambda_\alpha(Y),\|\cdot|_{\alpha,Y})$ is different from the cone of d-semi-Lipschitz function ($\alpha=1$) considered in [9]. For example, if one considers $Y=[0,1]$ $d(x,y)=|x-y|$ and $f:[0,1]\to\mathbb{R}$, $f(x)=x\sin\frac{1}{x}$, $x\in(0,1]$; $f(0)=0$, then it is known that $f\in\Lambda_\alpha(X,d)$ if and only if $\alpha\in(0,1/2]$ (see[14], Problem 153) and in this case $\|f|_{\alpha,Y}\leq\{1+2\ln(1+2\pi)+2\pi\}^{1/2}$.

2. EXTENSIONS OF d-SEMI-HÖLDER FUNCTIONS

Let (X,d) be a quasi–metric space, $Y \subset X$ and $f \in \Lambda_\alpha(Y)$. A function $F \in \Lambda_\alpha(X)$ is called an extension of f (preserving the semi-Hölder constant $L(f,Y)$ if

$$F|_Y = f \text{ and } L(F,X) = L(f,Y) \tag{2.1}$$

The existence of extension in $\Lambda_\alpha(X)$ for each $f \in \Lambda_\alpha(X)$ is assured by the following theorem.

THEOREM 2. *Let (X,d) be a quasi–metric space, $Y \subset X$ and $f \in \Lambda_\alpha(Y)$ with d-semi-Hölder constant $L(f,Y)$. Then there exist $F \in \Lambda_\alpha(X)$ such that*

$$F|_Y = f \text{ and } L(F,X) = L(f,Y).$$

Proof. Let $G : X \to \mathbb{R}$ defined by

$$G(x) = \sup_{y\in Y}\{f(y) - L(f,Y)\cdot d^\alpha(y,x)\},\ x \in X. \tag{2.2}$$

Let $y_0 \in Y$ be a fixed element, and $x \in X$. For every $y \in Y$,

$$\begin{aligned} f(y) - L(f,Y)\cdot d^\alpha(y,x) &= f(y) - f(y_0) - L(f,Y)\ d^\alpha(y,x) + f(y_0) \\ &\le L(f,Y)d^\alpha(y,y_0) - L(f,Y)d^\alpha(y,x) + f(y_0) \\ &= f(y_0) + L(f,Y)[d^\alpha(y,y_0) - d^\alpha(y,x)] \\ &\le f(y_0) + L(f,Y)\ d^\alpha(x,y_0). \end{aligned}$$

Then it follows that the set

$$\{f(y) - L(f,Y)d^\alpha(y,x) : y \in Y\}$$

is bounded from above, and $G(x)$ exists for every $x \in X$. By the definition of $G(x)$, for every $y \in Y$

$$G(x) \ge f(y) - L(f,Y)d^\alpha(y,x),\ x \in X,$$

and for $x = y$ one obtains

$$G(y) \ge f(y).$$

On the other hand, for $y \in Y$ and every $y' \in Y$,

$$f(y') - f(y) \le L(f,Y)\cdot d^\alpha(y',y).$$

It follows

$$f(y') - L(f,Y)\cdot d^\alpha(y',y) \le f(y),$$

and taking the supremum with respect to $y' \in Y$ one obtains

$$G(y) \le f(y),\ y \in Y.$$

Consequently $G|_Y = f$.

Now, let $u, v \in X$ and $\varepsilon > 0$. Choosing $y \in Y$ such that

$$G(u) \le f(y) - L(f,Y)\ d^\alpha(y,u) + \varepsilon,$$

it follows

$$\begin{aligned} G(u) - G(v) &\leq f(y) - L(f,Y)d^{\alpha}(y,u) + \varepsilon - f(y) + L(f,Y)d^{\alpha}(y,v) \\ &= L(f,Y)[d^{\alpha}(y,v) - d^{\alpha}(y,u)] + \varepsilon \\ &\leq L(f,Y)d^{\alpha}(u,v) + \varepsilon. \end{aligned}$$

Because $\varepsilon > 0$ is arbitrarily chosen, one obtains:

$$G(u) - G(v) \leq L(f,Y)\ d^{\alpha}(u,v),$$

for $u, v \in X$, i.e. $G \in \Lambda_{\alpha}(X)$. Moreover $L(G,X) \leq L(f,Y)$. Because $G|_Y = f$ one obtains also

$$L(f,Y) = L(G|_{Y,} Y) \leq L(G,X)$$

and consequently, $L(f,Y) = L(G,X)$. □

REMARK 3. Observe that the function $F : X \to \mathbb{R}$,

$$F(x) = \inf_{y \in Y} \{f(y) + L(f,Y)\ d^{\alpha}(x,y)\},\ x \in X. \tag{2.3}$$

is another extension of $f \in \Lambda_{\alpha}(f,Y)$.

Moreover, if H is any extension of f, i.e. $H|_Y = f$ and $L(H,X) = L(f,Y)$ then

$$G(x) \leq H(x) \leq F(x),\ x \in X \tag{2.4}$$

where G is defined by (2.2) and F is defined by (2.3).

From (2.4) it follows that G is the minimal extension of f, and F is the maximal extension of f. □

Indeed let H an extension of $f \in \Lambda_{\alpha}(Y)$. Then, for arbitrary $x \in X$ and $y \in Y$ we have

$$H(x) - H(y) \leq L(f,Y)d^{\alpha}(x,y)$$

implying

$$\begin{aligned} H(x) &\leq H(y) + L(f,Y)d^{\alpha}(x,y) \\ &= f(y) + L(f,y)d^{\alpha}(x,y) \end{aligned}$$

Taking the infimum with respect to $y \in Y$ one obtain

$$H(x) \leq F(x),\ x \in X$$

Analogously,

$$H(y) - H(x) \leq L(f,Y) \cdot d^{\alpha}(y,x)$$

implies

$$f(y) - L(f,Y)d^{\alpha}(y,x) \geq H(x),$$

and taking the supremum with respect to $y \in Y$ one obtain

$$G(x) \geq H(x),\ x \in X.$$

It follows (2.4).

REMARK 4. For $f \in \Lambda_\alpha(Y)$ denote by $\mathcal{E}(f)$ the (non-empty) set of all extensions of f i.e.

(2.5) $$\mathcal{E}(f) := \{H \in \Lambda_\alpha(X) : H|_Y = f \text{ and } L(H,X) = L(f,Y)\}$$

Obviously, $\mathcal{E}(f)$ is a convex subset of the cone $\Lambda_\alpha(X)$. □

REMARK 5. Let $f \in \Lambda_\alpha(Y)$ and let $\|f|_{\alpha\cdot Y}$ be the smallest d-semi-Hölder constant for f on Y (see (1.4)). Then the functions G and F defined by (2.2) and (2.3), where $L(f,Y) = \|f|_{\alpha,Y}$ are extensions for f, preserving the constant $\|f|_{\alpha,Y}$. □

Consider a fixed element $y_0 \in Y$, and let

(2.6) $$\Lambda_{\alpha,0}(Y) := \{f \in \Lambda_\alpha(Y) : f(y_0) = 0\}.$$

Then the functional $\||_{\alpha,Y} : \Lambda_{\alpha,0}(Y) \to [0,\infty)$ defined by

(2.7) $$\|f|_{\alpha,Y} = \sup\{\tfrac{(f(x)-f(y))\vee 0}{d^\alpha(x,y)} : d(x,y) > 0; x,y \in Y\}$$

is a quasi-norm on $\Lambda_{\alpha,0}(Y)$, i.e. the following properties hold:

a) $\|f|_{\alpha,Y} \geq 0$; $f = 0$ iff $-f \in \Lambda_{\alpha,0}(Y)$ and $\|f|_{\alpha,Y} = \|-f|_{\alpha,Y} = 0$, $f \in \Lambda_{\alpha,0}(Y)$;
b) $\|af|_{\alpha,Y} = a\|f|_{\alpha,Y}$ for every $f \in \Lambda_{\alpha,0}(Y)$ and $a \geq 0$;
c) $\|f+g|_{\alpha,Y} \leq \|f|_{\alpha,Y} + \|g|_{\alpha,Y}$ for all $f,g \in \Lambda_{\alpha,0}(Y)$.

REMARK 6. Let $f \in \Lambda_\alpha(X)$ and $Y_1 \subset Y_2 \subset X$. Suppose that $q \geq \|f|_{\alpha,X}$ and let

$$G_1(x) = \sup_{y\in Y_1}\{f(y) - qd^\alpha(y,x)\},\ x \in X,$$

$$G_2(x) = \sup_{y\in Y_2}\{f(y) - qd^\alpha(y,x)\},\ x \in X$$

Then $G_1(x) \leq G_2(x) \leq f(x)$, for all $x \in X$ and $G_1|_{Y_1} = G_2|_{Y_1} = f|_{Y_1}$.

Also, if

$$F_1(x) = \inf_{y\in Y_1}\{f(y) + qd^\alpha(x,y)\},\ x \in X$$

and

$$F_2(x) = \inf_{y\in Y_2}\{f(y) + qd^\alpha(x,y)\},\ x \in X$$

then

$$F_1(x) \geq F_2(x) \geq f(x),\ \text{for all } x \in X$$

and

$$F_1|_{Y_1} = F_2|_{Y_1} = f|_{Y_1}.$$

Consequently, if $(Y_n)_{n\geq 1}$ is a sequence in 2^X such that $Y_1 \subset Y_2 \subset ... \subset Y_n \subset ...$, $f \in \Lambda_{\alpha,0}(X)$ and $q \geq \|f|_{\alpha,X}$ then the sequences $(G_n)_{n\geq 1}$ and $(F_n)_{n\geq 1}$, where

$$G_n(x) = \sup_{y\in Y_n}\{f(y) - qd^\alpha(y,x)\},\ x \in X$$

and

$$F_n(x) = \inf_{y\in Y_n} \{f(y) + qd^\alpha(x,y)\},\ x \in X$$

are monotonically increasing, respectively decreasing, $G_n, F_n \in \Lambda_\alpha(X)$, $n = 1,2,...$, and $G_n(x) \le f(x) \le F_n(x)$, for all $x \in X$. □

Because, for every $y \in Y_n$

$$f(y) - qd^\alpha(y,x) \le G_n(x) \le F_n(x) \le f(y) + qd^\alpha(x,y)$$

it follows that,

$$F_n(x) - G_n(x) \le q[d^\alpha(x,y) + d^\alpha(y,x)].$$

Taking the infimum with respect to $y \in Y_n$ one obtains:

$$F_n(x) - G_n(x) \le q \inf_{y\in Y_n} [d^\alpha(x,y) + d^\alpha(y,x)]. \tag{2.8}$$

for every $x \in X$.

If Y_n is d^s-dense in X, where $d^s(x,y) = d(x,y) \vee d(y,x)$ for every $x, y \in X$ then, by (2.8) it follows that $F_n(x) = G_n(x),\ x \in X$. Consequently, a function $f \in \Lambda_\alpha(Y)$ where Y is d^s-dense in X has an unique extension $F \in \Lambda_\alpha(X)$.

REFERENCES

[1] Collins, J. and Zimmer, J., *An asymmetric Arzelà-Ascoli theorem*, Topology and its Applications, **154** , pp. 2312–2322, 2007.

[2] Matouskova, E., *Extensions of continuous and Lipschitz functions*, Canad. Math. Bull, **43**, no. 2, pp. 208–217, 2000.

[3] McShane, E.T., *Extension of range of functions*, Bull. Amer. Math. Soc., **40**, pp. 837–842, 1934.

[4] Menucci, A., *On asymmetric distances.* Technical Report, Scuola Normale Superiore, Pisa, 2007, `http://cvgmt.sns.it/people/menucci`.

[5] Miculescu, R., *Some observations on generalized Lipschitz functions*, Rocky Mountain J. Math., **37**, no.3, pp. 893–903, 2007.

[6] Mustăţa, C., *Extension of semi-Lipschitz functions on quasi-metric spaces*, Rev. Anal. Number. Théor. Approx., **30**, no. 1, pp. 61–67, 2001.

[7] Mustăţa, C., *Best approximation and unique extension of Lipschitz functions*, J. Approx. Theory, **19**, no. 3, pp. 222–230, 1977.

[8] Mustăţa,C., *A Phelps type theorem for spaces with asymmetric norms*, Bul. Stiinţ. Univ. Baia Mare, Ser. B. Matematică-Informatică **18**, pp. 275–280, 2002.

[9] Romaguera, S. and Sanchis, M., *Semi-Lipschitz functions and best approximation in quasi–metric spaces*, J. Approx. Theory, **103**, pp. 292–301, 2000.

[10] Romaguera, S. and Sanchis, M., *Properties of the normed cone of semi-Lipschitz functions*, Acta Math Hungar, **108** no. 1–2, pp. 55–70, 2005.

[11] Sánchez-Álvarez, J.M., *On semi-Lipschitz functions with values in a quasi-normed linear space*, Applied General Topology, **6**, no. 2, pp. 216–228, 2005.

[12] Weaver, N., *Lattices of Lipschitz functions*, Pacific Journal of Math., **164**, pp. 179–193, 1994.

[13] Wells, J.H. and Williams, L.R. *Embeddings and Extensions in Analysis,* Springer-Verlag, Berlin, 1975.

[14] ** * *, The Otto Dunkel Memorial Problem Book,* New York, 1957.