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KOROVKIN-TYPE CONVERGENCE RESULTS FOR MULTIVARIATE SHEPARD FORMULAE

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Abstract. We present a new convergence proof for classic multivariate Shepard formulae within the context of Korovkin-type convergence results for positive operators on spaces of continuous real valued functions.

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1. INTRODUCTION

In his work on 'A two-dimensional interpolation function for irregularly-spaced data' [10], Shepard introduced a new two-dimensional interpolation technique for scattered data. In the last decades, there were introduced many multivariate generalizations by various authors, as for instance *moving least squares interpolation* due to Lancaster and Šalkauskas [6]. Farwig provides in [4] a general treatment of the 'Rate of Convergence of Shepard's Global Interpolation Formula', even in a multivariate setting. In his work he avoids the use of the fact that interpolation operators of Shepard-type are positive. Operators of that kind are liable to the famous convergence results as introduced by Bohman [2] and Korovkin [5] in the early 1950s. In this work we prove the convergence of the multivariate Shepard interpolants within the positive operator context. However, we do not intend to improve existing error estimates for multivariate Shepard formulae as given by Farwig and other authors.

2. MULTIVARIATE SHEPARD FORMULAE

One of the keynotes for multivariate Shepard formulae is the construction of arbitrarily smooth interpolating functions for a given set of multivariate scattered data, where it is assumed that the data depend on certain function values. Within this paper we consider the following setting: let $f \in C(X; \mathbb{R})$

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be a continuous function mapping the *compact* subset $X \subset \mathbb{R}^d$ into the set of real numbers and consider a set

$$\mathscr{B}_n = \{x_1, x_2, \dots, x_{c(n)-1}, x_{c(n)}\} \subset X,$$

where $c(n) = \text{card}\mathscr{B}_n$, which is said to be an *interpolation node distribution.* For even $\alpha > 1$ let us denote $w_\alpha : \mathbb{R}^d \backslash \{0\} \to \mathbb{R}_{>0}$ the function defined by

$$w_\alpha(x) = ||x||_2^{-\alpha},$$

and for an arbitrary index $i \in \{1, \dots, c(n)\}$ let $\varphi_\alpha^{(i)} : \mathbb{R}^d \to \mathbb{R}$ be given by

$$\varphi_\alpha^{(i)}(x) = \frac{w_\alpha(x - x_i)}{\sum\limits_{j=1}^{c(n)} w_\alpha(x - x_j)}. \tag{1}$$

This definition implies the following properties:

Proposition 1. *The (cardinal basis) functions* $\varphi_\alpha^{(1)}, \dots, \varphi_\alpha^{(c(n))}$ *as introduced in equation* (1) *are arbitrarily smooth, i.e.* $\varphi_\alpha^{(i)} \in C^\infty(X; \mathbb{R})$, *and enjoy the properties*

(a) $\varphi_\alpha^{(i)}(x_j) = \delta_{ij}$, $i, j = 1, \dots, c(n)$,

(b) $0 \le \varphi_\alpha^{(i)}(x) \le 1$ *for every* $x \in X$,

(c) $\varphi_\alpha^{(i)}(x) = 0$ *if and only if* $x = x_j$, $i \neq j$,

(d) $\sum_{i=1}^{c(n)} \varphi_\alpha^{(i)}(x) = 1$ *for every* $x \in X$,

(e) $D^\mu \varphi_\alpha^{(i)}(x)\big|_{x=x_j} = 0$ *for* $1 \le j \le c(n)$ *and multi-indices* $|\mu| = 1$.

Proof. Consider Lancaster and Šalkauskas [6], as well as Sonar [11] for (e) within a univariate treatment, respectively. □

The functions introduced in (1) allow the construction of an interpolant to f with cardinal basis representation.

Definition 2. *Let* $\alpha > 1$ *be fixed and let* $\mathscr{B}_n \subset X$ *be an interpolation node distribution. The function* $\mathbf{S}_{\mathscr{B}_n}^\alpha f : C(X; \mathbb{R}) \to C^\infty(X; \mathbb{R})$ *defined by*

$$\mathbf{S}_{\mathscr{B}_n}^\alpha f(x) = \sum_{i=1}^{c(n)} f(x_i) \varphi_\alpha^{(i)}(x), \quad x \in X,$$

is called multivariate Shepard interpolant to f. *The functions* $\varphi_\alpha^{(i)}$ *are given by equation* (1).

Note that for fixed $x \in X$ the value $\mathbf{S}_{\mathscr{B}_n}^\alpha f(x)$ can be interpreted as the minimizer of the weighted least squares problem

$$\sum_{j=1}^{c(n)} \big(a(x) - f(x_j)\big)^2 w_\alpha(x - x_j) \to \min!,$$

consider [9] for details. Furthermore, with regard to (a) in Proposition 1, $\mathbf{S}^{\alpha}_{\mathscr{B}_n} f$ is equipped with the desired interpolation property

$$\mathbf{S}^{\alpha}_{\mathscr{B}_n} f(x_i) = f(x_i), \quad 1 \le i \le c(n),$$

and it is stable in the sense

$$\min_{1 \le i \le c(n)} f(x_i) \le \mathbf{S}^{\alpha}_{\mathscr{B}_n} f(x) \le \max_{1 \le i \le c(n)} f(x_i), \quad x \in X.$$

The smoothness of $\mathbf{S}^{\alpha}_{\mathscr{B}_n} f$ is inherited from the functions $\varphi^{(i)}_{\alpha}$.

At this point, we have to bring up the question under which conditions $\mathbf{S}^{\alpha}_{\mathscr{B}_n} f$ converges uniformly to f as $n \to \infty$, where this limit has to be interpreted in a sense corresponding to the following definition.

Definition 3. *Let $n_0 \in \mathbb{N}$ and let $\mathscr{B}_{n_0} = \{x_1, \ldots, x_{c(n_0)}\} \subset K \subset \mathbb{R}^d$ be an interpolation node distribution. Then, the sequence $(\mathscr{B}_n)_{n \ge n_0}$ is said to be a* monotone sequence of interpolation node distributions in X, *provided that there exists $\eta : \mathbb{N} \to \mathbb{R}$ such that $\lim_{n \to \infty} \eta(n) = 0$ and*

(a) *$\mathscr{B}_n \subset \mathscr{B}_{n+1} \subset K$ for all $n \ge n_0$,*

(b) *$c(n)\, r(\mathscr{B}_n)^{d+1} \le \eta(n)$ for $n \ge n_0$, where*

$$r(\mathscr{B}_n) = \inf\left\{\delta > 0 \,\middle|\, \forall x \in X : \, card(B_\delta(x) \cap \mathscr{B}_n) \ge 1\right\},$$

holds true with $B_\delta(x) = \{y \in X \mid ||x - y||_2 < \delta\}$.

Below we give an example for a monotone sequence of interpolation node distributions according to the last definition.

Example 4. Let $X = \left\{(x_1, \ldots, x_d)^{\mathrm{T}} \in \mathbb{R}^d \,\middle|\, 0 \le x_i \le 1 \text{ for } 1 \le i \le d\right\}$ and assume that $\mathscr{B}_n$ is defined by

$$\mathscr{B}_n = X \cap \left\{\frac{z}{n} \in \mathbb{Q}^d \,\middle|\, z \in \mathbb{Z}^d\right\}, \quad n = 2^k, k \ge 0.$$

Then, we have $c(n) = \mathrm{card}\mathscr{B}_n = (n+1)^d$ and

$$r(\mathscr{B}_n) = c_d \tfrac{1}{n}, \tag{2}$$

where $c_d := \sqrt{d}/2$, that means

$$r(\mathscr{B}_n)^{d+1} c(n) = (c_d)^{d+1} \tfrac{1}{n^{d+1}} (n+1)^d = (c_d)^{d+1} \tfrac{1}{n} \left(1 + \tfrac{1}{n}\right)^d \le \underbrace{(c_d)^{d+1} \tfrac{2^d}{n}}_{=:\eta(n)},$$

i.e. $(\mathscr{B}_n)_{n \ge 1}$ is a monotone sequence of interpolation node distributions in X. The identity (2) follows from some reflections on basic geometry since the Euclidean norm ν of X_n's center of mass, where

$$X_n = \left\{(x_1, \ldots, x_d)^{\mathrm{T}} \in \mathbb{R}^d \,\middle|\, 0 \le x_i \le 1/n \text{ for } 1 \le i \le d\right\},$$

is $\nu = \frac{\sqrt{d}}{2n}$. □

In the subsequent considerations we will show that the convergence question can be answered with the concept of monotone sequences of interpolation node distributions and, moreover, it suffices to prove the uniform convergence of $\mathbf{S}^{\alpha}_{\mathscr{B}_n}$ for the $2d+1$ monomial functions

$$e_i^j : x \mapsto x(i)^j, \quad 1 \le i \le d, \quad 0 \le j \le 2, \tag{3}$$

where we use the notation $x = \big(x(1), \dots, x(d)\big)^{\mathrm{T}} \in \mathbb{R}^d$. Note that $e_k^0 = e_l^0$ for $1 \le k, l \le d$.

This fact can be concluded directly from a multivariate extension of some famous results due to Bohman and Korovkin.

Theorem 5 (Bohman, Korovkin). *Let* $(P_n)_{n \ge n_0}$ *be a sequence of positive and linear operators mapping* $C(X;\mathbb{R})$ *into itself, such that*

$$\lim_{n\to\infty} ||P_n e_i^j - e_i^j||_\infty = 0, \quad 1 \le i \le d, \quad 0 \le j \le 2, \tag{4}$$

where e_i^j *denotes the monomial function as given in equation* (3)*. Then,*

$$\lim_{n\to\infty} ||P_n f - f||_\infty = 0$$

holds true for arbitrary $f \in C(X;\mathbb{R})$*. Moreover, for fixed* $n \in \mathbb{N}$*, one has*

$$\begin{aligned} |P_n f(x) - f(x)| \quad &\le \quad |f(x)|\,|e_0(x) - P_n e_0(x)| \\ &\quad + \Big(P_n e_0(x) + \sqrt{P_n e_0(x)}\Big)\,\omega\big(f, \gamma_n(x)\big), \end{aligned}$$

where $\gamma_n^2(x) = P_n\big(\sum_{i=1}^d (\cdot - x(i))^2\big)(x)$ *and* $\omega(f,\cdot)$ *denotes the usual smoothness module*

$$\omega(f,h) = \sup_{0 \le ||t||_2 \le h} \ \sup_{x \in X} ||f(x+t) - f(x)||_2 .$$

Proof. Korovkin [5], Lorentz [7], DeVore [3], Altomare and Campiti [1]. □

In this setting the positivity of an operator $P : C(X;\mathbb{R}) \to C(X;\mathbb{R})$ means that $f \ge 0$ implies $Pf \ge 0$, while monotonicity means that $f \le g$ implies $Pf \le Pg$. Positivity and monotonicity are equivalent notions in the case of linear operators. Moreover, it is quite easy to realize that the Shepard interpolation operator $\mathbf{S}^{\alpha}_{\mathscr{B}_n}$ is positive and linear. Therefore it suffices to prove the uniform convergence of $\mathbf{S}^{\alpha}_{\mathscr{B}_n}$ for the $2d+1$ monomial functions e_i^j, $1 \le i \le d$, $0 \le j \le 2$, in order to prove the convergence for arbitrary continuous f, under consideration of Theorem 5.

3. PROOF OF CONVERGENCE

As in the foregoing section, let e_i^j denote the monomial function as given by (3) and consider $f \in C(X;\mathbb{R})$. As from now, we will use the abbreviation $|x| = ||x||_2$ for $x \in \mathbb{R}^d$. The proof of convergence is divided into three steps.

(I) The Shepard interpolation operator reproduces the functions $e_i^0 \equiv 1$ exactly, that means

$$\mathbf{S}^\alpha_{\mathscr{B}_n} e_i^0 = e_i^0$$

holds true for $1 \le i \le d$ and arbitrary underlying interpolation node distribution $\mathscr{B}_n$ under consideration of Proposition 1.

(II) Suppose an interpolation node distribution $\mathscr{B}_n = \{x_1, \dots, x_{c(n)}\}$ and consider the case $j = 1$. Then, for arbitrary $1 \le i \le d$, we have

$$\begin{aligned}
\left|\mathbf{S}^\alpha_{\mathscr{B}_n} e_i^1(x) - e_i^1(x)\right| \le & \\
\le & \sum_{j=1}^{c(n)} |x_j(i) - x(i)| \tfrac{|x-x_j|^{-\alpha}}{\sum_{l=1}^{c(n)} |x-x_l|^{-\alpha}} \\
\le & \sum_{j=1}^{c(n)} |x_j - x| \tfrac{|x-x_j|^{-\alpha}}{\sum_{l=1}^{c(n)} |x-x_l|^{-\alpha}} \\
= & \sum_{\substack{j \\ |x-x_j| < r(\mathscr{B}_n)^{d+1}}} |x_j - x| \tfrac{|x-x_j|^{-\alpha}}{\sum_{l=1}^{c(n)} |x-x_l|^{-\alpha}} \\
& + \sum_{\substack{j \\ |x-x_j| \ge r(\mathscr{B}_n)^{d+1}}} |x_j - x| \tfrac{|x-x_j|^{-\alpha}}{\sum_{l=1}^{c(n)} |x-x_l|^{-\alpha}} \\
\le & \; r(\mathscr{B}_n)^{d+1} + \sum_{\substack{j \\ |x-x_j| \ge r(\mathscr{B}_n)^{d+1}}} |x_j - x| \tfrac{|x-x_j|^{-\alpha}}{\sum_{l=1}^{c(n)} |x-x_l|^{-\alpha}} \\
= & \; r(\mathscr{B}_n)^{d+1} + c(n)\mathcal{O}\big(r(\mathscr{B}_n)^{d+1}\big) = \mathcal{O}\big(r(\mathscr{B}_n)^{d+1}\big) + \mathcal{O}\big(\eta(n)\big),
\end{aligned}$$

in view of

$$|x - x_j|^{-\alpha+1} = \mathcal{O}\big(r(\mathscr{B}_n)^{(-\alpha+1)(d+1)}\big)$$

and

$$\sum_{l=1}^{c(n)} |x - x_l|^{-\alpha} \ge \sum_{\substack{l \\ |x-x_l| < r(\mathscr{B}_n)^{d+1}}} |x - x_l|^{-\alpha} \ge r(\mathscr{B}_n)^{-\alpha(d+1)}.$$

(III) Finally, let denote $M = \max_{x \in K} |x|$ and consider the case $j = 2$. Then, for arbitrary $1 \le i \le d$, we conclude

$$\begin{aligned}
\left|\mathbf{S}^\alpha_{\mathscr{B}_n} e_i^2(x) - e_i^2(x)\right| \le & \sum_{j=1}^{c(n)} \left|\big(x_j(i)\big)^2 - x(i)^2\right| \tfrac{|x-x_j|^{-\alpha}}{\sum_{l=1}^{c(n)} |x-x_l|^{-\alpha}} \\
\le & \; 2\,M \sum_{j=1}^{c(n)} |x_j - x| \tfrac{|x-x_j|^{-\alpha}}{\sum_{l=1}^{c(n)} |x-x_l|^{-\alpha}} \\
= & \; \mathcal{O}\big(r(\mathscr{B}_n)^{d+1}\big) + \mathcal{O}\big(\eta(n)\big),
\end{aligned}$$

in agreement with (II).

Consequently, the uniform convergence of $\mathbf{S}^{\alpha}_{\mathscr{B}_n} f$ towards f as $n \to \infty$ follows from Theorem 5 by setting $P_n = \mathbf{S}^{\alpha}_{\mathscr{B}_n}$. The required convergence (4) is established by virtue of

$$\sup_{x\in X} \left| e_i^j(x) - \mathbf{S}^{\alpha}_{\mathscr{B}_n} e_i^j(x) \right| = \mathcal{O}\big(r(\mathscr{B}_n)^{d+1}\big) + \mathcal{O}\big(\eta(n)\big) \longrightarrow 0,$$

as $n \to \infty$.

We conclude this section with a brief summary of the results given above.

Theorem 6. *Let $f \in C(X;\mathbb{R})$ and denote $\mathbf{S}^{\alpha}_{\mathscr{B}_n} f$, $n \geq n_0$, the appropriate Shepard interpolant as given in Definition 2 corresponding to a monotone sequence of interpolation node distributions $(\mathscr{B}_n)_{n \geq n_0}$ as introduced in Definition 3. Then, $\mathbf{S}^{\alpha}_{\mathscr{B}_n} f$ converges uniformly to f as $n \to \infty$,*

$$\lim_{n\to\infty} ||\mathbf{S}^{\alpha}_{\mathscr{B}_n} - f||_{\infty} = 0,$$

and

$$|\mathbf{S}^{\alpha}_{\mathscr{B}_n} f(x) - f(x)| \leq 2\, \omega(f, \gamma_n(x))$$

for arbitrary $x \in X$, where $\gamma_n^2(x) = \mathbf{S}^{\alpha}_{\mathscr{B}_n}\big(\sum_{i=1}^{d} (\cdot - x(i))^2\big)(x)$. □

4. CONCLUDING REMARKS

This work presents a new convergence proof for multivariate Shepard formulae which rests upon a classical convergence statement for positive operators on spaces of continuous functions due to Bohman and Korovkin. The achieved results extend quantitative and qualitative convergence statements due to Farwig [4] for multivariate Shepard formulae within the case $\alpha < d$, which couldn't be handled with the methods of Farwig's approach. To the authors knowledge there is no comparable convergence proof for multivariate Shepard formulae within this positive operator context.

Shepard interpolation suffers from the *flat spot phenomenon*, that means $D^{\mu}\mathbf{S}^{\alpha}_{\mathscr{B}_n} f(x_i) = 0$ for $1 \leq i \leq c(n)$ and multi-indices $|\mu| = 1$. That is why it is not the right choice in a context of derivative approximations. However, generalized multivariate interpolation concepts, such as *moving least squares interpolation*, which is applicable also for derivative approximations, can be analyzed theoretically, even within the context of simultaneous approximation, with methods in the style of this paper. We encourage to consider [8].

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