REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 38 (2009) no. 2, pp. 177–181
ictp.acad.ro/jnaat

EXTENDING THE COLLAGE THEOREM TO CONTRACTIVE LIKE OPERATORS

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Abstract. We generalize the classical "collage" theorem, due to Barnsley, to contractive like operators.

MSC 2000. 65J22, 47N40, 47H10.

Keywords. contractive like operators.

1. INTRODUCTION

Let X be a real Banach space, $T : X \to X$ be an operator. The following result of Barnsley, see [1], becomes "a classic".

THEOREM 1. (Collage Theorem) *Let $x \in X$ be given and $T : X \to X$ a contraction with contraction factor $L \in (0,1)$, (i.e. $\|Tx - Ty\| \leq L\|x - y\|$, $\forall\ x,\ y \in X$), and fixed point x^*. Then*

$$\|x - x^*\| \leq \tfrac{1}{1-L}\,\|x - Tx\|.$$

In fractal-based applications, T produces a union of shrunken copies of x, i.e. "a collage" of itself. The term $\|x - Tx\|$ is referred as "collage distance". Such "collages" are sufficient to be taken in finite number, in order to have a good approximation of an denoised image; fact which is very useful in Image Compression (both Analysis and Synthesis of an image). Kunze et all., see [5], [6], [7], were able to apply the Collage Theorem to inverse problems in ODE , that is to reconstruct the field of an ODE, from a given "target" (trajectory). Our aim is to generalize the above Collage result for a larger operatorial class than contractions. Recently, similar results were introduced for other operatorial classes, see [11] and [12].

The following operatorial class, satisfying (1), was introduced in [4]. Since they failed to named it, we shall do so here. It is just a convention.

DEFINITION 2. *The operator T is contractive-like, or CL for short, if there exist a constant $q \in (0,1)$ and a monotone increasing and continuous function*

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$\psi : [0,\infty) \to [0,\infty)$ *with* $\psi(0) = 0$ *such that for each* $x, y \in X$,

$$\|Tx - Ty\| \leq q\,\|x - y\| + \psi(\|x - Tx\|). \tag{1}$$

Let $F(T)$ denote the fixed point set with respect to X for the map T. Suppose that $x^* \in F(T)$. The following operators are called Zamfirescu operators, see [13] or [10].

Definition 3. [13] *The operator* $T : X \to X$ *satisfies condition Z (or is a quasi-contraction) if and only if there exist the real numbers* a, b, c *satisfying* $0 < a < 1$, $0 < b < 1/2$, $0 < c < 1/2$ *such that for each pair* x, y *in* X, *at least one condition is true*
(z_1) $\|Tx - Ty\| \leq a\ \|x - y\|$,
(z_2) $\|Tx - Ty\| \leq b\ (\|x - Tx\| + \|y - Ty\|)$,
(z_3) $\|Tx - Ty\| \leq c\ (\|x - Ty\| + \|y - Tx\|)$.

It has been shown in [2] (see also [3]) that conditions $(z_1) - (z_3)$ lead to

$$\|Tx - Ty\| \leq \delta\,\|x - y\| + 2\delta\,\|x - Tx\|, \forall x, y \in D, \tag{2}$$

where

$$\delta := \max\left\{a, \tfrac{b}{1-b}, \tfrac{c}{1-c}\right\}.$$

Set

$$q := \delta, \psi(a) := 2\delta a$$

to obtain (1). Thus, relation (1) generalizes (2). In [4] was introduced this more general class of operators satisfying (1).

Remark 4. Set $\psi(t) = 2t$ to see that a CL operators need not have a fixed point, as pointed in [8] or [9]. Therefore, *we shall suppose implicitly throughout this paper that all CL operators involved have a fixed point.* □

A typical inverse problem is the following:

Problem 5. For given $\varepsilon > 0$ and a "target" $\bar{x}$, find $T_\varepsilon \in CL$ such that $\left\|\bar{x} - x^*_{T_\varepsilon}\right\| < \varepsilon$, where $x^*_{T_\varepsilon} = T_\varepsilon\left(x^*_{T_\varepsilon}\right)$ is the unique fixed point of the CL mapping T_ε.

Randomly selecting various maps in CL, finding their fixed points and computing the distance from our target is an extremely tedious procedure. Consider now the following problem which we shall fit in our framework and which is very useful for practitioners, see [5].

Problem 6. Let $\bar{x} \in X$ be a target and let $\delta > 0$ be given. Find $T_\delta \in CL$, such that $\|\bar{x} - T_\delta \bar{x}\| < \delta$.

In other words, instead of searching for CL maps whose fixed points lie close to target $\bar{x}$, we search for CL maps that send $\bar{x}$ close to itself.

2. MAIN RESULTS

We shall give a lower bound to approximation error in terms of the collage error.

PROPOSITION 7. *Let X be a real Banach space and T a CL map with contraction factor $q \in (0,1)$ and fixed point $x^* \in X$. Then for any $x \in X$,*

$$\tfrac{1}{1+q} \|x - Tx\| \leq \|x^* - x\| .$$

Proof. For any $x \in X$ satisfying $x = x^*$, the above inequality holds. If $x \neq x^*, \forall x \in X$, then one obtains

$$\begin{aligned} \|Tx - x\| &\leq \|x^* - x\| + \|x^* - Tx\| \\ &= \|x^* - x\| + \|Tx^* - Tx\| \\ &\leq \|x^* - x\| + q \|x^* - x\| + \psi \left(\|x^* - Tx^*\|\right) \\ &\leq (1+q) \|x^* - x\| . \end{aligned}$$

From which one gets the conclusion. □

THEOREM 8. (Collage theorem for contractive-like maps) *Let X be a real Banach space and T a CL map with contraction factor $q \in (0,1)$ and fixed point $x^* \in X$. Then for any $x \in X$,*

$$\|x^* - x\| \leq \tfrac{1}{1-q} \|x - Tx\| .$$

Proof. The CL condition assures that the fixed point x^* is unique. If $x = x^*$, the above inequality holds. If $x \neq x^*, \forall x \in X$, then one obtains

$$\begin{aligned} \|x^* - x\| &\leq \|Tx^* - Tx\| + \|Tx - x\| \\ &\leq q \|x^* - x\| + \psi \left(\|x^* - Tx^*\|\right) + \|Tx - x\| \\ &= q \|x^* - x\| + \|Tx - x\| . \end{aligned}$$

From which one gets the conclusion. □

REMARK 9. To summarize, we have the following bounds

$$\tfrac{1}{1+q} \|x - Tx\| \leq \|x^* - x\| \leq \tfrac{1}{1-q} \|x - Tx\| .$$

□

The above "Collage Theorem" allows us to reformulate the inverse Problem 5 in the particular and more convenient Problem 6.

THEOREM 10. *If Problem 6 has a solution, then Problem 5 has a solution too.*

Proof. Let $\varepsilon > 0$ and $\bar{x} \in X$ be given. For $\delta := (1-q)\,\varepsilon$, let $T_\delta \in CL$ be such that $\|\bar{x} - T_\delta \bar{x}\| < \delta$. If $x^*_{T_\delta}$ is the unique fixed point of the CL mapping T_δ, then, by Theorem 8,

$$\left\|\bar{x} - x^*_{T_\delta}\right\| \leq \tfrac{1}{1-q} \|\bar{x} - T_\delta \bar{x}\| \leq \tfrac{1}{1-q} \delta = \varepsilon.$$

□

Note that shrinking the distance between two operators, one of them from CL, reduces the distance between their fixed points.

PROPOSITION 11. *Let X be a real Banach space and $T_1 \in CL$ with contraction factor $q_1 \in (0,1)$ and $T_2 : X \to X$ a map such that $x_1^*, x_2^* \in X$ are distinct fixed points for T_1 and T_2. Then,*

$$\|x_1^* - x_2^*\| \leq \tfrac{1}{1-q_1} \sup_{x\in X} \|T_1x - T_2x\| .$$

Proof. Using (1) one obtains

$$\begin{aligned}\|x_1^* - x_2^*\| &= \|T_1x_1^* - T_2x_2^*\| \leq \|T_1x_1^* - T_1x_2^*\| + \|T_1x_2^* - T_2x_2^*\| \\ &\leq q_1 \|x_1^* - x_2^*\| + \psi\left(\|x_1^* - T_1x_1^*\|\right) + \sup_{x\in X} \|T_1x - T_2x\| ,\end{aligned}$$

from which we get the conclusion. □

THEOREM 12. *Let X be a real Banach space, $T : X \to X$, $\bar{x} = T\bar{x}$ and suppose there exists $T_1 \in CL$ with contraction factor q, such that*

$$\sup_{x\in X} \|T_1x - Tx\| \leq \varepsilon.$$

Then

$$\|\bar{x} - T_1\bar{x}\| \leq \tfrac{1+q}{1-q}\varepsilon.$$

Proof. Let $x^* = T_1x^*$, and by use of Proposition 11 we obtain

$$\|\bar{x} - x^*\| \leq \tfrac{1}{1-q} \left(\sup_{x\in X} \|T_1x - Tx\|\right).$$

Thus,

$$\begin{aligned}\|\bar{x} - T_1\bar{x}\| &\leq \|\bar{x} - x^*\| + \|x^* - T_1\bar{x}\| \\ &\leq \|\bar{x} - x^*\| + \|T_1x^* - T_1\bar{x}\| \\ &\leq \|\bar{x} - x^*\| + q\|\bar{x} - x^*\| + \psi\left(\|x^* - T_1x^*\|\right) \\ &= (1+q)\|\bar{x} - x^*\| \\ &\leq \tfrac{1+q}{1-q}\left(\sup_{x\in X}\|T_1x - Tx\|\right) \leq \tfrac{1+q}{1-q}\varepsilon.\end{aligned}$$

□

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Received by the editors: September 17, 2008.