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INVERSE THEOREM FOR THE SZÁSZ-DURRMEYER OPERATORS

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Abstract. In the present paper we establish direct and inverse local properties for the Szász-Durrmeyer operators. These operators are introduced in [1] and independently considered in [4] as the generalized integral operators proposed by S.M. Mazhar and V. Totik in [2].

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1. INTRODUCTION

S.M. Mazhar and V. Totik in [2] introduced the modified Szász-Mirakyan operators defined on $[0,\infty)$:

$$L_n(f,x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k}(t)f(t)\mathrm{d}t,$$

where $p_{n,k}(x) = \mathrm{e}^{-nx}\frac{(nx)^k}{k!}$ and $n \in \mathbb{N}$.

The Szász-Durrmeyer operators are defined as

$$M_n^{\nu}(f,x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k}(t)f(t)\mathrm{d}t,$$

where $\nu \in (-1,\infty)$. Let us notice, that $M_n^0(f) = L_n(f)$.

Some asymptotic properties of the operators M_n^{ν} in polynomial weight spaces, global approximation theorem and Voronovskaya type theorem have been given in [4]. Moreover, the connection between these operators and boundary problem was shown, using the relationship discussed in [5].

The Szász-Durrmeyer operators were introduced by A. Ciupa and I. Gavrea [1]. In the same paper the commutativity these operators, use in the proof of theorem 1 was shown.

In this paper we give relation between local smoothness of function f and local convergence of Szász-Durrmeyer operators M_n^{ν}. Results of this type for Berenstain-Durrmeyer operators have been given by Ding-Xuan Zhou [6] and

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Song Li [3]. By $K_i(a,b)$ or $K_i(a,b,c)$, $i \in \mathbb{N}^*$ we denote suitable positive constants depending only on indicated parameters.

2. AUXILIARY RESALTS

Let B_r denote set of measurable function $f : [0,\infty) \to R$ such that

$$\exists M > 0\ \forall\ t \in [0,\infty)\ |f(t)p_r(t)| \le M$$

where $p_0(t) = 1$ and $p_r(t) = \frac{1}{1+t^r}$ for $r \in \mathbb{N}^*$. The norm on B_r is defined by formula

$$\| f \|_{B_r} = \sup_{t\in[0,\infty)} |f(t)p_r(t)|.$$

In [4], it has been given that M_n^ν exist for $f \in B_r$ and $M_n^\nu \in B_r$.

Later on we use the following formulas [4]:

$$\sum_{k=0}^{\infty} p_{n,k}(x) = 1, \tag{1}$$

$$\sum_{k=0}^{\infty} kp_{n,k}(x) = nx, \tag{2}$$

$$\sum_{k=0}^{\infty} k^2 p_{n,k}(x) = (nx)^2 + nx, \tag{3}$$

$$\int_{k=0}^{\infty} t^r p_{n,k+\nu}(t)\mathrm{d}t = n^{-(r+1)}\frac{\Gamma(k+\nu+r+1)}{\Gamma(k+\nu+1)}, \tag{4}$$

$$M_n^\nu(1,x) = 1, \tag{5}$$

$$M_n^\nu((t-x)^2,x) = \frac{x}{n} + \frac{(\nu+1)(\nu+2)}{n^2}, \tag{6}$$

where $r \in \mathbb{N}$, $n \in \mathbb{N}^*$ and $x \in [0,\infty)$.

Let $E \subset [0,\infty)$, $t \in [0,\infty)$ and $\alpha \in (0,1]$. We denote

$$\phi_{\alpha,n}(t) = \left(\frac{t}{n}\right)^{\frac{\alpha}{2}} + n^{-\alpha} + (d(t,E))^\alpha,$$

where $d(t,E)$ is the distance between t i E defined as

$$d(t,E) = \inf_{y\in E}\{|t-y|\}.$$

We shall prove the following:

Lemma 1. *Let $\alpha \in (0,1]$ and $E \subset [0,\infty)$. Suppose, that $f \in B_0$ satisfies*

$$|f(t)| \le \phi_{\alpha,n}(t), \quad t \in [0,\infty). \tag{7}$$

Then, we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}x} M_{n,\nu}(f,x)\right| \le K_1(\alpha,n)\sqrt{\frac{n}{x}}\phi_{\alpha,n}(x), \quad x \in (0,\infty). \tag{8}$$

Proof. We start with observations

$$(x)^{\frac{\alpha}{2}} \leq (t)^{\frac{\alpha}{2}} + (|t-x|)^{\frac{\alpha}{2}}$$

and

$$(d(t,E))^{\alpha} \leq (d(x,E))^{\alpha} + |t-x|^{\alpha} \quad for\ t,x \in [0,\infty).$$

By (7) we get

$$\begin{aligned}
\mid \tfrac{d}{dx} M_n^{\nu}(f,x) \mid &\leq \mid n \sum_{k=0}^{\infty} \tfrac{k-nx}{x} p_{n,k}(x) \int_0^{\infty} f(t) p_{n,k+\nu}(t) \mathrm{d}t \mid \\
&\leq \tfrac{n}{x} \sum_{k=0}^{\infty} |k-nx| p_{n,k}(x) \int_0^{\infty} [(\tfrac{t}{n})^{\frac{\alpha}{2}} + n^{-\alpha} + d(t,E)^{\alpha}] p_{n,k+\nu}(t) \mathrm{d}t \\
&\leq \tfrac{1}{x} \sum_{k=0}^{\infty} |k-nx| p_{n,k}(x) \phi_{\alpha,n}(x) \\
&\quad + \tfrac{n}{x} \sum_{k=0}^{\infty} |k-nx| p_{n,k}(x) \int_0^{\infty} \left(\tfrac{|t-x|}{n}\right)^{\frac{\alpha}{2}} p_{n,k+\nu}(t) \mathrm{d}t \\
&\quad + \tfrac{n}{x} \sum_{k=0}^{\infty} |k-nx| p_{n,k}(x) \int_0^{\infty} |t-x|^{\alpha} p_{n,k+\nu}(t) \mathrm{d}t \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

By Cauchy inequality and (1–3) we have

$$\begin{aligned}
I_1 &= \tfrac{1}{x} \phi_{\alpha,n}(x) \sum_{k=0}^{\infty} [(k-nx)^2 p_{n,k}(x)]^{\frac{1}{2}} [p_{n,k}(x)]^{\frac{1}{2}} \\
&\leq \tfrac{1}{x} \phi_{\alpha,n}(x) [\sum_{k=0}^{\infty} (k-nx)^2 p_{n,k}(x)]^{\frac{1}{2}} [\sum_{k=0}^{\infty} p_{n,k}(x)]^{\frac{1}{2}} \\
&= \sqrt{\tfrac{n}{x}} \phi_{\alpha,n}(x).
\end{aligned}$$

Similarly, by the Hölder inequality and (4), we obtain

$$\begin{aligned}
I_3 &\leq \tfrac{1}{x} \sum_{k=0}^{\infty} |k-nx| p_{n,k}(x) [n \int_0^{\infty} (t-x)^2 p_{n,k+\nu}(t) \mathrm{d}t]^{\frac{\alpha}{2}} [n \int_0^{\infty} p_{n,k+\nu}(t) \mathrm{d}t]^{1-\frac{\alpha}{2}} \\
&= \tfrac{1}{x} \sum_{k=0}^{\infty} |k-nx| p_{n,k}(x) n^{-\alpha} [(k+\nu+1)(k+\nu+2) - 2(k+\nu+1)nx + (nx)^2]^{\frac{\alpha}{2}} \\
&= \tfrac{1}{x} \sum_{k=0}^{\infty} |k-nx| (p_{n,k}(x))^{1-\frac{\alpha}{2}} \\
&\quad \times \left(n^{-\alpha} [(k+\nu+1)(k+\nu+2) - 2(k+\nu+1)nx + (nx)^2] p_{n,k}(x)\right)^{\frac{\alpha}{2}}
\end{aligned}$$

$$\leq \tfrac{1}{x}[\sum_{k=0}^{\infty}|k-nx|^{\frac{2}{2-\alpha}}p_{n,k}(x)]^{1-\frac{\alpha}{2}}$$

$$\times\, n^{-\alpha}\left(\sum_{k=0}^{\infty}[(k+\nu+1)(k+\nu+2)-2(k+\nu+1)nx+(nx)^2]p_{n,k}(x)\right)^{\frac{\alpha}{2}}.$$

Applying (1–3), we get

$$\begin{aligned} n^{-\alpha}&\left(\sum_{k=0}^{\infty}[(k+\nu+1)(k+\nu+2)-2(k+\nu+1)nx+(nx)^2]p_{n,k}(x)\right)^{\frac{\alpha}{2}} \\ &= n^{-\alpha}[2nx+(\nu+1)(\nu+2)]^{\frac{\alpha}{2}} \\ &\leq K_2(\alpha,\nu)[(\tfrac{x}{n})^{\frac{\alpha}{2}}+n^{-\alpha}]. \end{aligned}$$

Using the Hölder inequality and (1–3), we have

$$\begin{aligned} &\tfrac{1}{x}\left[\sum_{k=0}^{\infty}|k-nx|^{\frac{2}{2-\alpha}}p_{n,k}(x)\right]^{1-\frac{\alpha}{2}} = \\ &= \tfrac{1}{x}\left[\big(\sum_{k=0}^{\infty}(k-nx)^2p_{n,k}(x)\big)^{\frac{1}{2-\alpha}}\big(\sum_{k=0}^{\infty}p_{n,k}(x)\big)^{\frac{1-\alpha}{2-\alpha}}\right]^{1-\frac{\alpha}{2}} \\ &= \tfrac{1}{x}(nx)^{\frac{1}{2}} = \sqrt{\tfrac{n}{x}}. \end{aligned}$$

Hence,

$$I_3 \quad\leq\quad K_2(\alpha,\nu)\sqrt{\tfrac{n}{x}}[(\tfrac{x}{n})^{\frac{\alpha}{2}}+n^{-\alpha}].$$

Now, we estimate I_2. First, we remark that

$$\left(\tfrac{|t-x|}{n}\right)^{\frac{\alpha}{2}} \leq \tfrac{1}{2}(n^{-\alpha}+|t-x|^{\alpha}),\quad t,x\in[0,\infty),\ n\in\mathbb{N}^*.$$

Hence,

$$\begin{aligned} I_2 &\leq \tfrac{1}{2x}n^{-\alpha}\sum_{k=0}^{\infty}|k-nx|p_{n,k}(x)n\int_0^{\infty}p_{n,k+\nu}(t)\mathrm{d}t \\ &\quad+\tfrac{1}{2x}\sum_{k=0}^{\infty}|k-nx|p_{n,k}(x)n\int_0^{\infty}|t-x|^{\alpha}p_{n,k+\nu}(t)\mathrm{d}t = J_{2,1}+J_{2,2}. \end{aligned}$$

By Cauchy inequality and (1–4) we get

$$\begin{aligned} J_{2,1} &= \tfrac{1}{2x}n^{-\alpha}[\sum_{k=0}^{\infty}(k-nx)^2p_{n,k}(x)]^{\frac{1}{2}}[\sum_{k=0}^{\infty}p_{n,k}(x)]^{\frac{1}{2}} \\ &\leq \tfrac{1}{2x}n^{-\alpha}\sqrt{nx} \leq \tfrac{1}{2}\sqrt{\tfrac{n}{x}}n^{-\alpha}. \end{aligned}$$

By Hölder inequality and (1–4) we obtain

$$J_{2,2} = \tfrac{1}{2x}\sum_{k=0}^{\infty}|k-nx|p_{n,k}(x)[n\int_0^{\infty}(t-x)^2 p_{n,k+\nu}(t)\mathrm{d}t]^{\frac{\alpha}{2}}$$

$$\times [n\int_0^{\infty} p_{n,k+\nu}(t)\mathrm{d}t]^{1-\frac{\alpha}{2}}$$

$$\le \tfrac{1}{2x}\sum_{k=0}^{\infty}|k-nx|p_{n,k}(x)(n^{-2}[(k+\nu+1)(k+\nu+2)-2(k+\nu+1)nx+(nx)^2])^{\frac{\alpha}{2}}$$

$$\le \tfrac{1}{2x}[\sum_{k=0}^{\infty}|k-nx|^{\frac{2}{2-\alpha}}p_{n,k}(x)]^{\frac{2-\alpha}{2}}$$

$$\times [\sum_{k=0}^{\infty}(n^{-2}[(k+\nu+1)(k+\nu+2)-2(k+\nu+1)nx+(nx)^2])p_{n,k}(x)]^{\frac{\alpha}{2}}$$

$$\le K_3(\alpha,\nu)\sqrt{\tfrac{n}{x}}[(\tfrac{x}{n})^{\frac{\alpha}{2}}+n^{-\alpha}].$$

Hence,

$$I_2 \le (K_3(\alpha,\nu)+\tfrac{1}{2})\sqrt{\tfrac{n}{x}}[(\tfrac{x}{n})^{\frac{\alpha}{2}}+n^{-\alpha}]$$

and finally we get (8). □

Lemma 2. *Let $\alpha\in(0,1]$, $E\subset[0,\infty)$. Suppose that $f\in B_0$ satisfies* (7). *Then*

(9) $$|\tfrac{\mathrm{d}}{\mathrm{d}x}M_n(\nu)(f,x)| \le K_4(\alpha,\nu)n\phi_{\alpha,n}(x), \quad x\in[0,\infty).$$

Proof. Observe that

$$|\tfrac{\mathrm{d}}{\mathrm{d}t}M_{n,\nu}(f,x)| = n^2|\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}f(t)[p_{n,k+1+\nu}(t)-p_{n,k+\nu}(t)]\mathrm{d}t|$$

$$\le n^2\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}\phi_{\alpha,n}(t)|p_{n,k+1+\nu}(t)-p_{n,k+\nu}(t)|\mathrm{d}t$$

$$\le n^2\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}[(\tfrac{t-x}{n})^{\frac{\alpha}{2}}+(\tfrac{x}{n})^{\frac{\alpha}{2}}$$

$$+n^{-\alpha}+d(x,E)^{\alpha}+|t-x|^{\alpha}][p_{n,k+1+\nu}(t)+p_{n,k+\nu}(t)]\mathrm{d}t.$$

By the estimation similar to the proof of lemma 1, we get (9). □

Theorem 1. *Let $f\in B_0$, $\alpha\in(0,1), x\in[0,\infty)$, i $E\subset[0,\infty)$. Then*

(10) $$|f(x)-f(y)| \le K_5(f,\alpha,\nu)|x-y|^{\alpha} \text{ for } y\in E$$

if and only if

(11) $$|M_{n,\nu}(f,x)-f(x)| \le K_6(f,\alpha,\nu)\phi_{\alpha,n}(x).$$

Proof. Suppose that (10) holds. Let $x_E \in clE$ such that $|x - x_E| = d(x, E)$. So we have

$$|f(t) - f(x_E) \leq K_5(f, \alpha, \nu)|t - x|^\alpha \quad for\ t \in [0, \infty)$$

and

$$\begin{aligned}
&|M_{n,\nu}(f, x) - f(x)| \leq |M_{n,\nu}(f(t) - f(x_E), x)| + |f(x_E) - f(x)| \\
&\leq K_5(f, \alpha, \nu)(|M_{n,\nu}(|t - x_E|^\alpha, x) + |x - x_E|^\alpha) \\
&\leq K_5(f, \alpha, \nu)[M_{n,\nu}(|t - x|^\alpha, x) + M_{n,\nu}(|x - x_E|^\alpha, x) + |x - x_E|^\alpha].
\end{aligned}$$

Thus, by (5) we obtain

$$\begin{aligned}
|M_{n,\nu}(f, x) - f(x)| &\leq K_5(f, \alpha, \nu)[\left(\tfrac{x}{n} + \tfrac{(\nu+1)(\nu+2)}{n^2}\right)^{\frac{\alpha}{2}} + 2|x - x_E|^\alpha] \\
&\leq K_6(f, \alpha, \nu)\phi_{\alpha,n}(x).
\end{aligned}$$

And the proof sufficiency is complete. We note that this part holds also for $\alpha = 1$.

Suppose that (11) is valid. Let $x \in [0, \infty)$, $y \in E$. If $x = y$ the (10) is obvious. If $|x - y| \in (\frac{1}{4}, \infty)$ we have

$$|f(x) - f(y)| \leq 2 \parallel f \parallel \leq K_7(f, \alpha)|x - y|^\alpha.$$

If $|x - y| \in (0, \frac{1}{4})$, we take $n \in \mathbb{N}$ and $n > 4$, such that

$$\frac{|x-y|}{2} < \delta_n(x, y) \leq |x - y|, \tag{12}$$

where $\delta_n(x, y) = \max\left\{\frac{1}{2^{n-2}}, \sqrt{\frac{x}{2^{n-2}}}, \sqrt{\frac{y}{2^{n-2}}}\right\}$.

This implies that

$$\phi_{\alpha,2^n}(x) \leq K_8(\alpha)|x - y|^\alpha. \tag{13}$$

We have

$$\begin{aligned}
&|f(x) - f(y)| \leq \\
&\leq |M^\nu_{2^n}(f, x) - f(x)| + |M^\nu_{2^n}(f, y) - f(y)| + |M^\nu_{2^n}(M^\nu_{2^{n-1}}(f, .) - f, x)| \\
&\quad + |M^\nu_{2^n}(M^\nu_{2^{n-1}}(f, .) - f, y)| + |M^\nu_{2^n}(M^\nu_{2^{n-1}}(f, .), x) - M^\nu_{2^n}(M^\nu_{2^{n-1}}(f, .), y)| \\
&= A_1 + A_2 + A_3 + A_4 + A_5.
\end{aligned}$$

By assumption, (12) and (13) we get

$$A_1 \leq K_6(f, \alpha, \nu)\phi_{\alpha,2^n}(x) \leq K_9(f, \alpha, \nu)|x - y|^\alpha, \tag{14}$$

$$A_2 \leq K_6(f, \alpha, \nu)\phi_{\alpha,2^n}(y) \leq K_{10}(f, \alpha, \nu)|x - y|^\alpha, \tag{15}$$

By (5), (6) and (12) we have

$$\begin{aligned} M_{2^n}^{\nu}(\phi_{\alpha,2^{n-1}},x) &\le \phi_{\alpha,2^{n-1}}(x) + M_{2^n}^{\nu}\Big(\Big(\tfrac{|t-x|}{2^{n-1}}\Big)^{\frac{\alpha}{2}},x\Big) + M_{2^n}^{\nu}(|t-x|^{\alpha},x) \\ &\le \phi_{\alpha,2^{n-1}}(x) + \tfrac{1}{2}M_{2^n}^{\nu}\Big(\Big(\tfrac{1}{2^{n-1}}\Big)^{\alpha},x\Big) + \tfrac{3}{2}M_{2^n}^{\nu}(|t-x|^{\alpha},x) \\ &\le \phi_{\alpha,2^{n-1}}(x) + \tfrac{1}{2}\left(\tfrac{1}{2^{n-1}}\right)^{\alpha} + \tfrac{3}{2}\left(M_{2^n}^{\nu}((t-x)^2,x)\right)^{\frac{\alpha}{2}} \\ &\le K_9(f,\alpha,\nu)|x-y|^{\alpha} \end{aligned}$$

and by (11) we get

$$A_3 \le K_6(\alpha,\nu)K_9(f,\alpha,\nu)|x-y|^{\alpha}. \tag{16}$$

Similarly

$$A_4 \le K_6(\alpha,\nu)K_{10}(f,\alpha,\nu)|x-y|^{\alpha}. \tag{17}$$

In the next step we estimate A_5. By commutativity of M_n^{ν} (see [1]) we obtain

$$\begin{aligned} M_{2^n}^{\nu}(M_{2^{n-1}}^{\nu}(f,.),t) = \sum_{i=3}^{n} & M_{2^{i-1}}^{\nu}(M_{2^i}^{\nu}(f,.) - M_{2^{i-2}}^{\nu}(f,.),t) \\ & + M_4^{\nu}(M_2(f,.),t). \end{aligned}$$

Furthermore we have

$$\begin{aligned} &|M_{2^n}^{\nu}(M_{2^{n-1}}^{\nu}(f,.),x) - M_{2^n}^{\nu}(M_{2^{n-1}}^{\nu}(f,.),y)| \le \\ &\qquad \le \sum_{i=3}^{n}\int_x^y |\tfrac{\mathrm{d}}{\mathrm{d}x}M_{2^{i-1}}^{\nu}(M_{2^i}^{\nu}(f,.) - M_{2^{i-2}}^{\nu}(f,.),t)|\mathrm{d}t \\ &\qquad\quad + \int_x^y |\tfrac{\mathrm{d}}{\mathrm{d}x}M_4^{\nu}(M_2(f,.),t)|\mathrm{d}t = A_{5,1} + A_{5,2}. \end{aligned}$$

By simple calculation we get

$$A_{5,2} \le K_{11}(f,\alpha)|x-y| \le K_{11}(f,\alpha)|x-y|^{\alpha}.$$

Moreover,

$$\begin{aligned} |M_{2^i}^{\nu}(f,t) - M_{2^{i-2}}^{\nu}(f,t)| &\le |M_{2^i}^{\nu}(f,t) - f(t)| + |M_{2^{i-2}}^{\nu}(f,t) - f(t)| \\ &\le K_6(f,\alpha,\nu)(\phi_{\alpha,2^i}(t) + \phi_{\alpha,2^{i-2}}(t)) \\ &\le K_{12}(f,\alpha,\nu)\phi_{\alpha,2^{i-1}}(t). \end{aligned}$$

Now, we assume that $x < y$. Thus we have

$$\delta_n(x,y) = \max\left\{\tfrac{1}{2^{n-2}}, \sqrt{\tfrac{y}{2^{n-2}}}\right\}$$

and

$$d(t,E) \le |x-y| \quad \text{for } t \in [x,y].$$

We estimate $A_{5,1}$ in two cases. First, let $\delta_n(x,y)=\sqrt{\frac{y}{2^{n-2}}}$. Then $2^{-n+2}\le y$ and

$$\tfrac{1}{2}|x-y|\le\sqrt{\tfrac{y}{2^{n-2}}}\le|x-y|. \tag{18}$$

Let $i\in\{3,4,...,n\}$. If $2^{-i+2}\le y$, then by (12), lemma 1 and inequality

$$\int_x^y t^{-\beta}\mathrm{d}t=\tfrac{|x-y|}{(1-\beta)y^{\beta}},\quad \beta\in[0,1),$$

we get

$$\begin{aligned}
I_i&=\int_x^y|\tfrac{\mathrm{d}}{\mathrm{d}t}M^{\nu}_{2^{i-1}}(M^{\nu}_{2^i}(f,.)-M^{\nu}_{2^{i-2}}(f,.),t)|\mathrm{d}t\\
&\le K_1(\alpha,\nu)K_{12}(f,\alpha,\nu)\int_x^y\sqrt{\tfrac{2^{i-1}}{t}}\phi_{\alpha,n}(t)\mathrm{d}t\\
&\le K_{13}(f,\alpha,\nu)(2^{i-1})^{(1-\frac{\alpha}{2})}\int_x^y t^{-\frac{1-\alpha}{2}}\mathrm{d}t+(2^{i-1})^{(\alpha-\frac{1}{2})}\int_x^y t^{\frac{1}{2}}\mathrm{d}t\\
&\quad+(2^{i-1})^{-\frac{1}{2}}|x-y|^{\alpha}\int_x^y t^{-\frac{1}{2}}\mathrm{d}t\\
&\le K_{14}(f,\alpha,\nu)\left[\left(\sqrt{\tfrac{y}{2^{i-1}}}\right)^{\alpha-1}|x-y|+\left(\sqrt{\tfrac{y}{2^{i-1}}}\right)^{-1}|x-y|^{1+\alpha}\right].
\end{aligned}$$

If $2^{-i+2}>y$, then by lemma 2 and (12) we obtain

$$\begin{aligned}
I_i&\le K_4(\alpha,\nu)K_{11}(f,\alpha,\nu)\int_x^y 2^{i-1}\phi_{\alpha,2^{i-1}}(t)\mathrm{d}t\\
&\le K_{15}(f,\alpha,\nu)\left((2^{i-1})^{1-\frac{\alpha}{2}}y^{\frac{\alpha}{2}}|x-y|+(2^{i-1})^{1-\alpha}|x-y|+2^{i-1}|x-y|^{1+\alpha}\right)\\
&\le K_{15}(f,\alpha,\nu)\left[\left(\sqrt{\tfrac{y}{2^{i-1}}}\right)^{\alpha-1}|x-y|+\left(\sqrt{\tfrac{y}{2^{i-1}}}\right)^{-1}|x-y|^{1+\alpha}\right].
\end{aligned}$$

Above inequality and (12) imply

$$\begin{aligned}
A_{5,1}&\le\sum_{i=3}^{n}I_i\le K_{17}(f,\alpha,\nu)\left[\left(\sqrt{\tfrac{y}{2^{n-2}}}\right)^{\alpha-1}|x-y|+\left(\sqrt{\tfrac{y}{2^{n-2}}}\right)^{-1}|x-y|^{1+\alpha}\right]\\
&=2K_{17}(f,\alpha,\nu)|x-y|^{\alpha}.
\end{aligned}$$

In the next step we suppose that $\delta_n(x,y)=\frac{1}{2^{n-2}}$. In this case $y<2^{-n+2}$ thus by lemma 2 and (12) we have

$$I_i\le K_{14}(f,\alpha,\nu)\left((2^{i-1})^{1-\frac{\alpha}{2}}y^{\frac{\alpha}{2}}|x-y|+(2^{i-1})^{1-\alpha}|x-y|+2^{i-1}|x-y|^{1+\alpha}\right)$$

and

$$\begin{aligned}
A_{5,1}&\le\sum_{i=3}^{n}I_iK_{18}(f,\alpha,\nu)\left(2^{n-2}|x-y|^{\alpha+1}+2\cdot(2^{n-2})^{1-\alpha}|x-y|\right)\\
&\le 3K_{18}(f,\alpha,\nu)|x-y|^{\alpha}.
\end{aligned}$$

Hence,

$$A_5 \leq K_{19}(f,\alpha,\nu)|x-y|^{\alpha} \tag{19}$$

Finally, (14), (15), (16), (17) and (19) imply that

$$|f(x)-f(y)| \leq K(f,\alpha,\nu)|x-y|^{\alpha}$$

for all $x \in [0,\infty)$ and $y \in E$. □

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