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ON THE COMPOSITE BERNSTEIN TYPE QUADRATURE FORMULA

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Abstract. Considering a given function $f \in C[0,1]$, the interval $[0,1]$ is divided in m equally spaced subintervals $\left[\frac{k-1}{m}, \frac{k}{m}\right]$, $k = \overline{1,m}$. On each of such type of interval the Bernstein approximation formula is applied and a corresponding Bernstein type quadrature formula is obtained. Making the sum of mentioned quadrature formulas, the composite Bernstein type quadrature formula is obtained.

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1. PRELIMINARIES

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The operators $B_n : C([0,1]) \to C([0,1])$ given by

$$(B_n f)(x) = \sum_{j=0}^{n} p_{n,j}(x) f\left(\tfrac{j}{n}\right), \tag{1.1}$$

where $p_{n,j}$ are the fundamental Bernstein's polynomials defined by

$$p_{n,j}(x) = \tbinom{n}{j} x^j (1-x)^{n-j}, \tag{1.2}$$

for any $x \in [0,1]$, any $j \in \{0, 1, \dots, n\}$ and any $n \in \mathbb{N}$, are called Bernstein operators and were first introduced in [3]. The approximation properties of the Bernstein operator were intensively studied in [1], [4], [5].

For any $f \in C[0,1]$, any $x \in [0,1]$ and any $n \in \mathbb{N}$, the following equality

$$f(x) = (B_n f)(x) + (R_n f)(x) \tag{1.3}$$

is called the Bernstein approximation formula, where R_n is the remainder operator associated to the Bernstein operator B_n, i.e. $R_n f$ is the remainder term of the approximation formula (1.3). Regarding the remainder term of (1.3), Tiberiu Popoviciu [4] established the following:

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THEOREM 1.1. *For any* $f \in C[0,1]$ *there exist the distinct points* $\xi_1, \xi_2, \xi_3 \in [0,1]$ *such that, for any* $x \in [0,1]$*, the remainder term of* (1.3) *can be represented under the form*

$$(R_n f)(x) = -\tfrac{x(1-x)}{n}[\xi_1, \xi_2, \xi_3; f]. \tag{1.4}$$

In (1.4) the brackets denote the divided difference of function f with respect the distinct knots ξ_1, ξ_2, ξ_3. It is well known the following estimation of the remainder term of (1.3), (see [7]).

THEOREM 1.2. *Suppose that* $f \in C^2[0,1]$*. The following inequality*

$$|(R_n f)(x)| \leq \tfrac{x(1-x)}{2n} M_2[f] \tag{1.5}$$

holds, where

$$M_2[f] = \max_{x \in [0,1]} |f''(x)|. \tag{1.6}$$

The inequality (1.5) follows directly from (1.4), applying the mean value theorem for divided differences and it is attributed to D. D. Stancu.

In the following we suppose that $f \in C^2[0,1]$. Starting with the Bernstein approximation formula (1.3), in [7] the following Bernstein quadrature formula

$$\int_0^1 f(x)\mathrm{d}x = \sum_{j=0}^{n} A_j f\left(\tfrac{j}{n}\right) + R_n[f] \tag{1.7}$$

is obtained, where

$$A_j = \tfrac{1}{n+1},\ (\forall)\ j = \overline{0,n} \tag{1.8}$$

and

$$|R_n[f]| \leq \tfrac{1}{12n} M_2[f]. \tag{1.9}$$

The focus of the present paper is to construct the composite Bernstein type quadrature formula. For this aim, the interval $[0,1]$ will be divided in m equally spaced subintervals $\left[\frac{k-1}{m}, \frac{k}{m}\right]$, $k = \overline{1,m}$. On each of such type of interval, the Bernstein quadrature formula (1.7) will be applied. Next, adding the mentioned quadrature formulas, the desired Bernstein type quadrature formula on $[0,1]$ will be obtained.

2. MAIN RESULTS

We start with two auxiliary results.

LEMMA 2.1. *Suppose that* $a, b \in \mathbb{R}$*,* $a < b$ *and* $f \in C[a,b]$*. Then, the Bernstein polynomial associated to the function* f *is defined by*

$$(B_n f)(x) = \tfrac{1}{(b-a)^n} \sum_{j=0}^{n} \tbinom{n}{j} (x-a)^j (b-x)^{n-j} f\left(a + j\tfrac{b-a}{n}\right). \tag{2.1}$$

Proof. It is easy to observe that the correspondence $t \to \frac{x-a}{b-a}$ transform the interval $[a, b]$ in the interval $[0, 1]$. Taking (1.1), (1.2) and the above remark into account, yields (2.1). □

LEMMA 2.2. *Suppose that $a, b \in \mathbb{R}$, $a < b$ and $f \in C^2[a, b]$. Then, the remainder term of the Bernstein approximation formula on $[a, b]$ verifies the inequality*

$$|(R_n f)(x)| \leq \tfrac{(x-a)(b-x)}{2n(b-a)^2} M_2[f], \tag{2.2}$$

where $M_2[f]$ is defined at (1.6).

Proof. One applies in a way similar to the case of relation (1.5), taking the transformation $t \to \frac{x-a}{b-a}$ into account. □

In what follows, let us to consider the interval $[0, 1]$ divided in the equally spaced subintervals $\left[\frac{k-1}{m}, \frac{k}{m}\right]$, $k = \overline{1, m}$. In each interval $\left[\frac{k-1}{m}, \frac{k}{m}\right]$, $k = \overline{1, m}$, one considers the distinct knots $x_i = \frac{kn-n+i}{mn}$, $i = \overline{0, n}$. Applying Lemma 2.1., yields the following Bernstein type polynomial

$$(B_{n,k} f)(x) = m^n \sum_{i=0}^{n} \tbinom{n}{i} \left(x - \tfrac{k-1}{m}\right)^i \left(\tfrac{k}{m} - x\right)^{n-i} f\left(\tfrac{kn-n+i}{mn}\right). \tag{2.3}$$

The corresponding Bernstein type approximation formula on the interval $\left[\frac{k-1}{m}, \frac{k}{m}\right]$, $k = \overline{1, m}$, becomes

$$f(x) = (B_{n,k} f)(x) + (R_{n,k} f)(x). \tag{2.4}$$

If $f \in C^2[0, 1]$, the remainder term of (2.4) verifies the inequality

$$|(R_{n,k} f)(x)| \leq \frac{\left(x - \frac{k-1}{m}\right)\left(\frac{k}{m} - x\right)}{2n} m^2 M_2[f]. \tag{2.5}$$

THEOREM 2.3. *If $f \in C^2[0, 1]$, the following Bernstein type quadrature formula*

$$\int_{\frac{k-1}{m}}^{\frac{k}{m}} f(x)\mathrm{d}x = \sum_{i=0}^{n} A_{i,k} f\left(\tfrac{kn-n+i}{mn}\right) + R_k[f] \tag{2.6}$$

holds, for any $k = \overline{1, m}$, where

$$A_{i,k} = \tfrac{1}{m(n+1)}, \ (\forall)\ i = \overline{0, n} \tag{2.7}$$

and

$$|R_k[f]| \leq \tfrac{1}{12mn} M_2[f]. \tag{2.8}$$

Proof. Integrating (2.4) on $\left[\frac{k-1}{m}, \frac{k}{m}\right]$, $k = \overline{1,m}$, and taking (2.3) into account, yields

$$\begin{aligned} A_{i,k} &= m^n \binom{n}{i} \int_{\frac{k-1}{m}}^{\frac{k}{m}} \left(x - \tfrac{k-1}{m}\right)^i \left(\tfrac{k}{m} - x\right)^{n-i} \mathrm{d}x \\ &= m^n \binom{n}{i} \int_0^1 \left(\tfrac{t}{m}\right)^i \left[\tfrac{1}{m}(1-t)\right]^{n-i} \tfrac{1}{m}\mathrm{d}t \\ &= \tfrac{1}{m}\binom{n}{i} \int_0^1 t^i (1-t)^{n-i} \mathrm{d}t. \end{aligned}$$

The last integral is the Euler function of first kind $B(i+1, n-i+1)$. Using the well known properties of Euler function of first kind, it follows

$$A_{i,k} = \tfrac{1}{m}\binom{n}{i} B(i+1, n-i+1) = \tfrac{1}{m} \tfrac{n!}{i!(n-i)!} \tfrac{i!(n-i)!}{(n+1)!} = \tfrac{1}{m(n+1)}.$$

For the remainder term, taking (2.5) into account, we get

$$|R_k[f]| \le M_2[f] \tfrac{m^2}{2n} \int_{\frac{k-1}{m}}^{\frac{k}{m}} \left(x - \tfrac{k-1}{m}\right)\left(\tfrac{k}{m} - x\right) \mathrm{d}x. \tag{2.9}$$

Because

$$\int_{\frac{k-1}{m}}^{\frac{k}{m}} \left(x - \tfrac{k-1}{m}\right)\left(\tfrac{k}{m} - x\right) \mathrm{d}x = \tfrac{1}{6m^3}$$

and from (2.9) one arrives to the desired inequality (2.8). □

THEOREM 2.4. *For any $f \in C^2[0,1]$, the following composite Bernstein type quadrature formula*

$$\int_0^1 f(x)\mathrm{d}x = \tfrac{1}{m(n+1)} \sum_{k=1}^{m} \sum_{i=0}^{n} f\left(\tfrac{kn-n+i}{mn}\right) + R_n[f] \tag{2.10}$$

holds, where

$$|R_n[f]| \le \tfrac{1}{2n} M_2[f]. \tag{2.11}$$

Proof. Adding the Bernstein type quadrature formulas (2.6) for $k = \overline{1,m}$, we get the following composite Bernstein type quadrature formula

$$\int_0^1 f(x)\mathrm{d}x = \tfrac{1}{m(n+1)} \sum_{k=1}^{m} \sum_{i=0}^{n} f\left(\tfrac{kn-n+i}{mn}\right) + \sum_{k=1}^{m} R_k[f]. \tag{2.12}$$

Denoting $R_n[f] = \sum\limits_{k=1}^{m} R_k[f]$ and taking (2.8) into account, yields

$$|R_n[f]| \le \sum_{k=1}^{m} |R_k[f]| \le \tfrac{1}{12mn} M_2[f] \cdot m = \tfrac{1}{12n} M_2[f]. \tag{2.13}$$

□

REMARK 2.5. It is easy to see that we get the same result for the remainder term of the composite Bernstein type quadrature formula as the result obtained by D. D. Stancu in [7], for the Bernstein quadrature formula. □

COROLLARY 2.6. *For any $f \in C^2[0,1]$ and any $m \in \mathbb{N}$, the following equality*

$$\lim_{n\to\infty} \tfrac{1}{m(n+1)} \sum_{k=1}^{m} \sum_{i=0}^{n} f\left(\tfrac{kn-n+i}{mn}\right) = \int_0^1 f(x)\mathrm{d}x \tag{2.14}$$

holds.

Proof. From (2.11) follows $\lim\limits_{n\to\infty} R_n[f] = 0$ and then, taking (2.10) into account one arrives to (2.14). □

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REFERENCES

[1] AGRATINI, O., *Approximation by linear operators*, Presa Universitară Clujeană, 2000, (in Romanian).
[2] BĂRBOSU, D., *A Schurer-Stancu type quadrature formula*, Carpathian J. Math., **23**, nos. 1–2, pp. 27–31, 2007.
[3] BERNSTEIN, S.N., *Démonstration du théorème de Weierstrass fondée sur le calcul de probabilités*, Commun. Soc. Math. Kharkow, **13**, no. 2, pp. 1–2, 1912–1913.
[4] POPOVICIU, T., *Sur le rest dans certains formules lineaires d'approximation de l'analyse*, Mathematica **I**, **24**, pp. 95–142, 1959.
[5] STANCU, D. D., *Quadrature formulas constructed by using certain linear positive operators*, Numerical Integration (Proc. Conf. Math. Res. Inst. Oberwolffach) (Basel) (G. Hammerlin, ed.), Birkhäuser, pp. 241–251, 1982.
[6] STANCU, D. D. and VERNESCU, A., *On some remarkable positive polynomial operators of approximation*, Rev. Anal. Numér. Théor. Approx., **28**, pp. 85–95, 1999.
[7] STANCU, D. D., COMAN, GH. and BLAGA, P., *Numerical Analysis and Approximation Theory*, **II**, Presa Universitară Clujeană, 2002 (in Romanian).

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