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THE KANTOROVICH FORM OF SOME EXTENSIONS FOR SZÁSZ-MIRAKJAN OPERATORS

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Abstract. Recently, C. Mortici defined a class of linear and positive operators depending on a certain function φ. These operators generalize the well known Szász-Mirakjan operators. A convergence theorem for the defined sequence by the mentioned operators was given. Other interesting approximation properties of these generalized Szász-Mirakjan operators and also their bivariate form were obtained by D. Bărbosu, O. T. Pop and D. Miclăuş. In the present paper we are dealing with the Kantorovich form of the generalized Szász-Mirakjan operators. We construct the Kantorovich associated operators and then we establish a convergence theorem for the defined operators. The degree of approximation is expressed in terms of the modulus of continuity. Next, we construct the bivariate and respectively the GBS corresponding operators and we establish some of their approximation properties.

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1. INTRODUCTION

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Considering an analytic function $\varphi : \mathbb{R} \to]0, +\infty[$, C. Mortici [19] defined the operators

$$\varphi S_n : C^2([0, +\infty[) \to C^\infty([0, +\infty[),$$

given by

$$(\varphi S_n f)(x) = \tfrac{1}{\varphi(nx)} \sum_{k=0}^{\infty} \tfrac{\varphi^{(k)}(0)}{k!} (nx)^k f\left(\tfrac{k}{n}\right), \tag{1.1}$$

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for any $x \in [0, +\infty[$ and any $n \in \mathbb{N}$.

The operators (1.1) are called the φ-Szász-Mirakjan operators, because in the case when $\varphi(y) = \mathrm{e}^y$, they become the classical Mirakjan-Favard-Szász operators [2], [14], [18], [23], [25]. Some nice and interesting approximation properties of operators (1.1) were obtained by C. Mortici [19] and by D. Bărbosu, O. T. Pop and D. Miclăuş [8], [20].

Remark 1.1. Similar generalization of this type are the operators defined and studied by Jakimovski and Leviatan [15] or the operators defined by Baskakov in 1957 (see, e.g., the book [2], subsection 5.3.11, p. 344, where they are attributed to Mastroianni).

Remark 1.2. The classical Mirakjan-Favard-Szász operators $S_n : C_2([0, +\infty[) \to C([0, +\infty[)$ are defined by

$$(S_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \tfrac{(nx)^k}{k!} f\left(\tfrac{k}{n}\right),$$

where

$$C_2([0, +\infty[) := \left\{ f \in C([0, +\infty[) : \lim_{x \to \infty} \tfrac{f(x)}{1+x^2} \text{ exists and is finite} \right\}.$$

In what follows, we shall use the classical definition of Mirakjan-Favard-Szász operators, i.e. $f \in C_2([0, +\infty[)$.

The main goal of the present paper is to construct the Kantorovich type operators, associated to the φ-Szász-Mirakjan operators (1.1).

Using the method of parametric extensions [7], [12], the bivariate $\varphi_1\varphi_2$-Szász-Mirakjan-Katorovich operators are constructed and some of their approximation properties are established. The last section is devoted to the construction of the associated GBS $\varphi_1\varphi_2$-Szász-Mirakjan-Kantorovich operators and to study some of their approximation properties.

2. φ-SZÁSZ-MIRAKJAN-KATOROVICH OPERATORS

Let $\varphi : \mathbb{R} \to]0, +\infty[$ be an analytic function. Following the idea of L. V. Kantorovich [16] we define the operators

$$\varphi K_n : C_2([0, +\infty[) \to C([0, +\infty[),$$

given by

$$(\varphi K_n f)(x) = \tfrac{n}{\varphi(nx)} \sum_{k=0}^{\infty} \tfrac{\varphi^{(k)}(0)}{k!} (nx)^k \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) \mathrm{d}t, \tag{2.1}$$

for any $x \in [0, +\infty[$ and $n \in \mathbb{N}$.

The operators (2.1) will be called the φ-Szász-Mirakjan-Kantorovich operators, because in the case when $\varphi(y) = \mathrm{e}^y$, they reduce to the classical Szász-Mirakajan-Kantorovich operators [1], [17].

REMARK 2.1. The operators (2.1) are linear and positive. □

In order to obtain the convergence of the sequence $(\varphi K_n)_{n\in\mathbb{N}}$ we need the following:

LEMMA 2.2. *Let $e_j(x) = x^j$, $j = 0, 1, 2$ be the test functions. The φ-Szász-Mirakjan-Kantorovich operators satisfy the following relations:*

i) $(\varphi K_n e_0)(x) = 1$,

ii) $(\varphi K_n e_1)(x) = \frac{\varphi^{(1)}(nx)}{\varphi(nx)} x + \frac{1}{2n}$,

iii) $(\varphi K_n e_2)(x) = \frac{\varphi^{(2)}(nx)}{\varphi(nx)} x^2 + \frac{2}{n}\frac{\varphi^{(1)}(nx)}{\varphi(nx)} x + \frac{1}{3n^2}$.

Proof. Taking that the function φ is analytic into account, it follows

$$\sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} y^k = \varphi(y)$$

and next, by differentiation

$$\sum_{k=1}^{\infty} \frac{\varphi^{(k)}(0)}{(k-1)!} y^{k-1} = \varphi^{(1)}(y), \qquad \sum_{k=2}^{\infty} \frac{\varphi^{(k)}(0)}{(k-2)!} y^{k-2} = \varphi^{(2)}(y).$$

For the test functions e_0, e_1, e_2, the following identities

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} e_0(t)\mathrm{d}t = \frac{1}{n}, \qquad \int_{\frac{k}{n}}^{\frac{k+1}{n}} e_1(t)\mathrm{d}t = \frac{2k+1}{2n^2}, \qquad \int_{\frac{k}{n}}^{\frac{k+1}{n}} e_2(t)\mathrm{d}t = \frac{3k^2+3k+1}{3n^3}$$

hold.

Recall that, the images of test functions by the operators φ-Szász-Mirakjan [19] are

$$\begin{aligned} &(\varphi S_n e_0)(x) = 1, \\ &(\varphi S_n e_1)(x) = \frac{\varphi^{(1)}(nx)}{\varphi(nx)} x, \\ &(\varphi S_n e_2)(x) = \frac{\varphi^{(2)}(nx)}{\varphi(nx)} x^2 + \frac{1}{n}\frac{\varphi^{(1)}(nx)}{\varphi(nx)} x, \end{aligned} \tag{2.2}$$

then

$$\begin{aligned} (\varphi K_n e_0)(x) &= \frac{n}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \int_{\frac{k}{n}}^{\frac{k+1}{n}} e_0(t)\mathrm{d}t \\ &= \frac{n}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \frac{1}{n} \\ &= \frac{1}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \\ &= (\varphi S_n e_0)(x) = 1. \end{aligned}$$

$$\begin{aligned}
(\varphi K_n e_1)(x) &= \frac{n}{\varphi(nx)}\sum_{k=0}^{\infty}\frac{\varphi^{(k)}(0)}{k!}(nx)^k\int_{\frac{k}{n}}^{\frac{k+1}{n}} e_1(t)\mathrm{d}t\\
&= \frac{n}{\varphi(nx)}\sum_{k=0}^{\infty}\frac{\varphi^{(k)}(0)}{k!}(nx)^k\frac{2k+1}{2n^2}\\
&= \frac{1}{\varphi(nx)}\sum_{k=0}^{\infty}\frac{\varphi^{(k)}(0)}{k!}(nx)^k\frac{k}{n}+\frac{1}{2n}\frac{1}{\varphi(nx)}\sum_{k=0}^{\infty}\frac{\varphi^{(k)}(0)}{k!}(nx)^k\\
&= (\varphi S_n e_1)(x)+\frac{1}{2n}(\varphi S_n e_0)(x)\\
&= \frac{\varphi^{(1)}(nx)}{\varphi(nx)}x+\frac{1}{2n}.
\end{aligned}$$

$$\begin{aligned}
(\varphi K_n e_2)(x) &= \frac{n}{\varphi(nx)}\sum_{k=0}^{\infty}\frac{\varphi^{(k)}(0)}{k!}(nx)^k\int_{\frac{k}{n}}^{\frac{k+1}{n}} e_2(t)\mathrm{d}t\\
&= \frac{1}{n^2}\frac{1}{\varphi(nx)}\sum_{k=0}^{\infty}\frac{\varphi^{(k)}(0)}{k!}(nx)^k\left(k^2+k+\frac{1}{3}\right)\\
&= \frac{1}{n^2}\Bigg(\frac{n^2}{\varphi(nx)}\sum_{k=0}^{\infty}\frac{\varphi^{(k)}(0)}{k!}(nx)^k\frac{k^2}{n^2}\\
&\qquad+\frac{n}{\varphi(nx)}\sum_{k=0}^{\infty}\frac{\varphi^{(k)}(0)}{k!}(nx)^k\frac{k}{n}+\frac{1}{3\varphi(nx)}\sum_{k=0}^{\infty}\frac{\varphi^{(k)}(0)}{k!}(nx)^k\Bigg)\\
&= (\varphi S_n e_2)(x)+\frac{(\varphi S_n e_1)(x)}{n}+\frac{(\varphi S_n e_0)(x)}{3n^2}\\
&= \frac{\varphi^{(2)}(nx)}{\varphi(nx)}x^2+\frac{2}{n}\frac{\varphi^{(1)}(nx)}{\varphi(nx)}x+\frac{1}{3n^2}. \qquad \square
\end{aligned}$$

Applying Lemma 2.2. we shall prove the following:

THEOREM 2.3. *Let $\varphi : \mathbb{R} \to]0,+\infty[$ be an analytic function having the properties*

$$\lim_{y\to\infty}\frac{\varphi^{(1)}(y)}{\varphi(y)}=1,\quad \lim_{y\to\infty}\frac{\varphi^{(2)}(y)}{\varphi(y)}=1. \tag{2.3}$$

Then, for any function $f \in C_2([0,+\infty[)$ and any $x \in [a,b]$, it holds

$$\lim_{n\to\infty}(\varphi K_n f)(x)=f(x)$$

uniformly on $[a,b]$, where $[a,b] \subset [0,+\infty[$ is a compact interval.

Proof. Applying the Lemma 2.2. and the hypothesis (2.3), it follows

$$\lim_{n\to\infty}(\varphi K_n e_1)(x)=\lim_{n\to\infty}\left(\frac{\varphi^{(1)}(nx)}{\varphi(nx)}x+\frac{1}{2n}\right)=x$$

and

$$\lim_{n\to\infty}(\varphi K_n e_2)(x)=\lim_{n\to\infty}\left(\frac{\varphi^{(2)}(nx)}{\varphi(nx)}x^2+\frac{2}{n}\frac{\varphi^{(1)}(nx)}{\varphi(nx)}x+\frac{1}{3n^2}\right)=x^2$$

uniformly on any compact interval $[a,b] \subset [0,+\infty[$.

Next, from the well known Bohman-Korovkin Theorem one arrives to the desired result. $\square$

In order to obtain the degree of approximation of $f \in C_2([0,+\infty[)$, by means of the φ-Szász-Mirakjan-Kantorovich operators, let us to recalling some known results, concerning the modulus of continuity.

Let $I \subset \mathbb{R}$ be an interval, $C(I)$ be the set of real-valued functions continuous on I, $B(I)$ be the set of real-valued functions bounded on I and $C_B(I)$ be the set of real-valued functions continuous, bounded on I.

Definition 2.4. *Let $f \in C_B(I)$ be given. The function $\omega_1 : [0,+\infty[\to \mathbb{R}$ defined by*

$$\omega_1(f;\delta) := \sup\{|f(x+h) - f(x)| \;:\; x,\ x+h \in I,\ 0 < h \leq \delta\} \tag{2.4}$$

is called the modulus of continuity (the first modulus of smoothness) of the function f.

Remark 2.5. *Its properties can be found in the monograph* [1]. $\square$

In 1968, O. Shisha and B. Mond [21] established the following:

Theorem 2.6. [1] *Let $L : C(I) \to B(I)$ be a linear positive operator and let the function φ be defined by*

$$\varphi_x(t) = |t-x|,\ (x,t) \in I \times I.$$

If $f \in C_B(I)$, then for any $x \in I$ and $\delta > 0$ the following

$$\begin{aligned} |(Lf)(x) - f(x)| &\leq |f(x)||(Le_0)(x) - 1| \\ &\quad + \left((Le_0)(x) + \delta^{-1}\sqrt{(Le_0)(x)(L\varphi_x^2)(x)}\right)\omega_1(f;\delta) \end{aligned} \tag{2.5}$$

holds.

For obtaining the degree of approximation of $f \in C_2([0,+\infty[)$, on any compact interval $[a,b] \subset [0,+\infty[$, by means of the φ-Szász-Mirakjan-Kantorovich operators we need the following:

Lemma 2.7. *Let the function φ_x be defined by $\varphi_x = |t-x|$, $(x,t) \in [a,b] \times [a,b]$. The operators* (2.1) *verify the relation*

$$(\varphi K_n \varphi_x^2)(x) = \left(\frac{\varphi^{(2)}(nx)}{\varphi(nx)} - 2\frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 1\right)x^2 + \frac{1}{n}\left(2\frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 1\right)x + \frac{1}{3n^2}. \tag{2.6}$$

Proof. Because the operators φ-Szász-Mirakjan-Kantorovich are linear, then taking the definition of φ_x into account, we get

$$(\varphi K_n \varphi_x)(x) = (\varphi K_n e_2)(x) - 2x(\varphi K_n e_1)(x) + x^2(\varphi K_n e_0)(x).$$

Next, one applies Lemma 2.2. $\square$

In the following, we suppose that the analytic function $\varphi:\mathbb{R}\to]0,+\infty[$ satisfy the conditions (2.3) and taking these conditions into account, it results

$$\lim_{n\to\infty}\left(\frac{\varphi^{(2)}(nx)}{\varphi(nx)}-2\frac{\varphi^{(1)}(nx)}{\varphi(nx)}+1\right)=0.$$

Then we suppose that, there exists $0<\gamma\leq 1$, so that

$$\lim_{n\to\infty}n^{\gamma}\left(\frac{\varphi^{(2)}(nx)}{\varphi(nx)}-2\frac{\varphi^{(1)}(nx)}{\varphi(nx)}+1\right)=\beta_2(x), \tag{2.7}$$

for any $x\in[0,+\infty[$ and where β_2 is a function, $\beta_2:[0,+\infty[\to\mathbb{R}$.

THEOREM 2.8. *For any function $f\in C_2([0,+\infty[)$, any compact interval $[a,b]\subset[0,+\infty[$, any $x\in[a,b]$ and any $\delta>0$, the φ-Szász-Mirakjan-Kantorovich operators verify the inequality*

$$|(\varphi K_n f)(x)-f(x)|\leq K\omega_1\left(f;\tfrac{b-a}{\sqrt{n^{\gamma}}}\right), \tag{2.8}$$

where

$$m_2([a,b]):=\sup_{x\in[a,b]}|\beta_2(x)|$$

and

$$K=1+\tfrac{1}{b-a}\sqrt{(m_2([a,b])+1)b^2+2b+\tfrac{1}{3}}.$$

Proof. The relation (2.8) yields from (2.5), if we choose $\delta=\frac{b-a}{\sqrt{n^{\gamma}}}$ and if we take the definition of limit and relation (2.7) into account. □

3. THE BIVARIATE $\varphi_1\varphi_2$-SZÁSZ-MIRAKJAN-KANTOROVICH OPERATORS

Suppose that $\varphi_1,\varphi_2:\mathbb{R}\to]0,+\infty[$ are analytic functions. The operators

$$\varphi_1K_m^x : C_{2,2}([0,+\infty[\times[0,+\infty[)\to C([0,+\infty[\times[0,+\infty[),$$

$$\varphi_1K_n^y : C_{2,2}([0,+\infty[\times[0,+\infty[)\to C([0,+\infty[\times[0,+\infty[),$$

given by

$$(\varphi_1K_m^x f)(x,y)=\tfrac{m}{\varphi_1(mx)}\sum_{k=0}^{\infty}\tfrac{\varphi_1^{(k)}(0)}{k!}(mx)^k\int_{\frac{k}{m}}^{\frac{k+1}{m}}f(t,y)\mathrm{d}t, \tag{3.1}$$

$$(\varphi_1K_n^y f)(x,y)=\tfrac{n}{\varphi_2(ny)}\sum_{j=0}^{\infty}\tfrac{\varphi_2^{(j)}(0)}{j!}(ny)^j\int_{\frac{j}{n}}^{\frac{j+1}{n}}f(x,s)\mathrm{d}s \tag{3.2}$$

are called the parametric extensions of the φ-Szász-Mirakjan-Kantorovich operators; (for the notion of "parametric extensions" see [7], [12]).

It is immediately the result contained in the following:

LEMMA 3.1. *The parametric extension of the φ-Szász-Mirakjan-Kantorovich operators defined at* (3.1) *and* (3.2) *are linear and positive. They commute on* $C_{2,2}([0,+\infty[\times[0,+\infty[)$ *and their product are the bivariate* $\varphi_1\varphi_2$*-Szász-Mirakjan-Kantorovich operators*

$$\varphi_1\varphi_2 K_{m,n} : C_{2,2}([0,+\infty[\times[0,+\infty[) \to C([0,+\infty[\times[0,+\infty[)$$

defined by

$$(3.3)\quad (\varphi_1\varphi_2 K_{m,n} f)(x,y) = \\ = \frac{m}{\varphi_1(mx)}\frac{n}{\varphi_2(ny)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{\varphi_1^{(k)}(0)}{k!}\frac{\varphi_2^{(j)}(0)}{j!}(mx)^k(ny)^j\int_{\frac{k}{m}}^{\frac{k+1}{m}}\int_{\frac{j}{n}}^{\frac{j+1}{n}} f(t,s)\mathrm{d}t\mathrm{d}s,$$

for any $x,y\in[0,+\infty[$ *and* $m,n\in\mathbb{N}$.

LEMMA 3.2. *Let* $e_{ij}(x,y)=x^iy^j$, $i,j\in\mathbb{N}_0$, $i+j\le 2$ *be the test functions. The operators* (3.3) *verify the following identities:*

i) $(\varphi_1\varphi_2K_{m,n}e_{0,0})(x,y)=1$,

ii) $(\varphi_1\varphi_2K_{m,n}e_{1,0})(x,y)=\frac{\varphi_1^{(1)}(mx)}{\varphi_1(mx)}x+\frac{1}{2m}$,

iii) $(\varphi_1\varphi_2K_{m,n}e_{0,1})(x,y)=\frac{\varphi_2^{(1)}(ny)}{\varphi_2(ny)}y+\frac{1}{2n}$,

iv) $(\varphi_1\varphi_2K_{m,n}e_{2,0})(x,y)=\frac{\varphi_1^{(2)}(mx)}{\varphi_1(mx)}x^2+\frac{2}{m}\frac{\varphi_1^{(1)}(mx)}{\varphi_1(mx)}x+\frac{1}{3m^2}$,

v) $(\varphi_1\varphi_2K_{m,n}e_{0,2})(x,y)=\frac{\varphi_2^{(2)}(ny)}{\varphi_2(ny)}y^2+\frac{2}{n}\frac{\varphi_2^{(1)}(ny)}{\varphi_2(ny)}y+\frac{1}{3n^2}$.

Proof. One applies relation (3.3) and takes Lemma 2.2. into account. □

LEMMA 3.3. *Let* $\varphi_x,\varphi_y:[0,+\infty[\times[0,+\infty[\to\mathbb{R}$ *be defined by*

$$\varphi_x(s,t)=|s-x|,\quad \varphi_y(s,t)=|t-y|.$$

The operators (3.3) *satisfy*

$$(3.4)\qquad (\varphi_1\varphi_2K_{m,n}\varphi_x^2)(x,y)= \\ =\left(\frac{\varphi_1^{(2)}(mx)}{\varphi_1(mx)}-2\frac{\varphi_1^{(1)}(mx)}{\varphi_1(mx)}+1\right)x^2+\frac{1}{m}\left(2\frac{\varphi_1^{(1)}(mx)}{\varphi_1(mx)}-1\right)x+\frac{1}{3m^2}$$

and

$$(3.5)\qquad (\varphi_1\varphi_2K_{m,n}\varphi_y^2)(x,y)= \\ =\left(\frac{\varphi_2^{(2)}(ny)}{\varphi_2(ny)}-2\frac{\varphi_2^{(1)}(ny)}{\varphi_2(ny)}+1\right)y^2+\frac{1}{n}\left(2\frac{\varphi_2^{(1)}(ny)}{\varphi_2(ny)}-1\right)y+\frac{1}{3n^2}.$$

Proof. Taking the linearity of operator (3.3) and the definitions of the functions φ_x, φ_y into account, one obtains

$$(\varphi_1\varphi_2K_{m,n}\varphi_x^2)(x,y)=(\varphi_1\varphi_2K_{m,n}e_{2,0})(x,y)-2x(\varphi_1\varphi_2K_{m,n}e_{1,0})(x,y) \\ +x^2(\varphi_1\varphi_2K_{m,n}e_{0,0})(x,y),$$

$$(\varphi_1\varphi_2 K_{m,n}\varphi_y^2)(x,y) = (\varphi_1\varphi_2 K_{m,n}e_{0,2})(x,y) - 2y(\varphi_1\varphi_2 K_{m,n}e_{0,1})(x,y) + y^2(\varphi_1\varphi_2 K_{m,n}e_{0,0})(x,y).$$

Next, applying Lemma 3.2. one arrives to (3.4) and (3.5). □

Suppose now that $a_1, b_1, a_2, b_2 \in \mathbb{R}$ such that $a_1 < b_1,\ a_2 < b_2$ and let $f : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ be a bounded function.

The function $\omega_f : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ defined by

$$\omega_f(\delta_1, \delta_2) = \sup\left\{|f(x', y') - f(x'', y'')| \ : (x', y'), (x'', y'') \in [a_1, b_1] \times [a_2, b_2],\ |x' - x''| \le \delta_1,\ |y' - y''| \le \delta_2\right\} \tag{3.6}$$

is called modulus of continuity of the bivariate function f.

Its properties are similar with the properties of the modulus of continuity for univariate functions [3], [7].

It is known from [7], [24] the following analogous of Shisha-Mond Theorem for the bivariate case:

THEOREM 3.4. *Let $L : C([a_1, b_1] \times [a_2, b_2]) \to B([a_1, b_1] \times [a_2, b_2])$ be a linear positive operator. For any $f \in C([a_1, b_1] \times [a_2, b_2])$, any $(x, y) \in [a_1, b_1] \times [a_2, b_2]$ and $\delta_1 > 0,\ \delta_2 > 0$ the following inequality*

(3.7)

$$|(Lf)(x,y) - f(x,y)| \le |f(x,y)||(Le_{00})(x,y) - 1| + \Big((Le_{00})(x,y) + \delta_1^{-1}\sqrt{(Le_{00})(x,y)(L\varphi_x^2)(x,y)} + \delta_2^{-1}\sqrt{(Le_{00})(x,y)(L\varphi_y^2)(x,y)} + \delta_1^{-1}\delta_2^{-1}\sqrt{(Le_{00})(x,y)(L\varphi_x^2)(x,y)(L\varphi_y^2)(x,y)}\Big)\ \omega_f(\delta_1, \delta_2)$$

holds.

Suppose that the analytic functions $\varphi_1, \varphi_2 : \mathbb{R} \to]0, +\infty[$ satisfy the conditions

$$\lim_{y\to\infty} \frac{\varphi_k^{(1)}(y)}{\varphi_k(y)} = \lim_{y\to\infty} \frac{\varphi_k^{(2)}(y)}{\varphi_k(y)} = 1 \tag{3.8}$$

and taking relation (3.8) into account, it follows

$$\lim_{n\to\infty} \left(\frac{\varphi_k^{(2)}(nx)}{\varphi_k(nx)} - 2\frac{\varphi_k^{(1)}(nx)}{\varphi_k(nx)} + 1 \right) = 0,$$

for $k \in \{1, 2\}$.

Then we suppose that, there exists $0 < \gamma_k \le 1$, so that

$$\lim_{n\to\infty} n^{\gamma_k} \left(\frac{\varphi_k^{(2)}(nx)}{\varphi_k(nx)} - 2\frac{\varphi_k^{(1)}(nx)}{\varphi_k(nx)} + 1 \right) = \beta_{2,k}, \tag{3.9}$$

for any $x \in [0, +\infty[$ and where $\beta_{2,k}$ are functions, $\beta_{2,k} : [0, +\infty[\to \mathbb{R}$, with $k \in \{1, 2\}$.

THEOREM 3.5. *For any function* $f \in C_{2,2}([0,+\infty[\times[0,+\infty[)$, *any bivariate compact interval* $[a_1,b_1]\times[a_2,b_2]\subset[0,+\infty[\times[0,+\infty[$, *any* $(x,y)\in[a_1,b_1]\times[a_2,b_2]$ *and* $\delta_1>0$, $\delta_2>0$, *the operators* (3.3) *satisfy the following inequality*

$$|(\varphi_1\varphi_2K_{m,n}f)(x,y)-f(x,y)|\le K\omega_f\left(\tfrac{b_1-a_1}{\sqrt{m^{\gamma_1}}},\tfrac{b_2-a_2}{\sqrt{n^{\gamma_2}}}\right), \tag{3.10}$$

where

$$m_{2,k}([a_k,b_k]):=\sup_{x\in[a_k,b_k]}|\beta_{2,k}(x)|, \tag{3.11}$$

for $k\in\{1,2\}$ *and*

$$K=\left(1+\tfrac{1}{b_1-a_1}\sqrt{(m_{2,1}([a_1,b_1])+1)\,b_1^2+2b_1+\tfrac{1}{3}}\right) \tag{3.12}$$
$$\cdot\left(1+\tfrac{1}{b_2-a_2}\sqrt{(m_{2,2}([a_2,b_2])+1)\,b_2^2+2b_2+\tfrac{1}{3}}\right).$$

Proof. In the Theorem 3.4. one replaces L by $\varphi_1\varphi_2K_{m,n}$ and next, one takes the proof of Theorem 2.8. into account. □

It is known from [7], [24] the following analogous of Bohman-Korovkin Theorem for the bivariate case:

THEOREM 3.6. *Let* $(L_{m,n})_{m,n\in\mathbb{N}}$ *be a sequence of linear positive operators,* $L_{m,n}:C([a_1,b_1[\times[a_2,b_2[)\to C([a_1,b_1[\times[a_2,b_2[)$ *and let* $e_{ij}(x,y)=x^iy^j$, $i,j\in\mathbb{N}_0$, $i+j\le 2$ *be the test functions.*

Suppose that the following relations

i) $(L_{m,n}e_{00})(x,y)=1$,
ii) $(L_{m,n}e_{10})(x,y)=x+u_{m,n}(x,y)$,
iii) $(L_{m,n}e_{01})(x,y)=y+v_{m,n}(x,y)$,
iv) $(L_{m,n}(e_{20}+e_{02}))(x,y)=x^2+y^2+w_{m,n}(x,y)$

hold, for any $(x,y)\in[a_1,b_1]\times[a_2,b_2]$.

If the sequence $(u_{m,n})$, $(v_{m,n})$ *and* $(w_{m,n})$, $m,n\in\mathbb{N}$ *converge to zero uniformly on* $[a_1,b_1]\times[a_2,b_2]$, *then* $L_{m,n}f$ *converges to* f *uniformly on* $[a_1,b_1]\times[a_2,b_2]$, *for any* $f\in C([a_1,b_1]\times[a_2,b_2])$.

THEOREM 3.7. *Suppose that the analytic functions* $\varphi_1,\varphi_2:\mathbb{R}\to]0,+\infty[$ *satisfy the conditions* (3.8).

Then, for every function $f\in C_{2,2}([0,+\infty[\times[0,+\infty[)$ *it holds*

$$\lim_{m,n\to\infty}(\varphi_1\varphi_2K_{m,n}f)(x,y)=f(x,y)$$

uniformly on any bivariate compact interval $[a_1,b_1]\times[a_2,b_2]\subset[0,+\infty[\times[0,+\infty[$.

Proof. Applying Lemma 3.2. and the hypothesis (3.8), it follows

$$\lim_{m,n\to\infty}(\varphi_1\varphi_2K_{m,n}e_{10})(x,y)=x,$$

$$\lim_{m,n\to\infty}(\varphi_1\varphi_2K_{m,n}e_{01})(x,y)=y$$

and

$$\lim_{m,n\to\infty}\big(\varphi_1\varphi_2 K_{m,n}(e_{20}+e_{02})\big)(x,y)=x^2+y^2$$

uniformly on any bivariate compact interval $[a_1,b_1]\times[a_2,b_2]\subset[0,+\infty[\times[0,+\infty[$.

Next, using Theorem 3.6. one arrives to the desired result. $\square$

4. THE GBS $\varphi_1\varphi_2$-SZÁSZ-MIRAKJAN-KANTOROVICH OPERATOR

In this section we shall construct the $\varphi_1\varphi_2$-Szász-Mirakjan-Kantorovich operator associated to a B-continuous function.

A function $f: I\times J\to\mathbb{R}$ is called B-continuous function in $(x_0,y_0)\in I\times J$, if

$$\lim_{(x,y)\to(x_0,y_0)}\Delta f[(x,y),(x_0,y_0)]=0,$$

where

$$\Delta f[(x,y),(x_0,y_0)]=f(x,y)-f(x_0,y)-f(x,y_0)+f(x_0,y_0) \tag{4.1}$$

denotes a so-called mixed difference of function f. A function $f: I\times J\to\mathbb{R}$ is called B-continuous function on $I\times J$, if it is B-continuous at any point of $I\times J$. The definition of B-continuity was introduced by K. Bögel in [10]. The function $f: I\times J\to\mathbb{R}$ is B-bounded on $I\times J$, if there exists $k>0$ so that

$$|\Delta f[(x,y),(s,t)]|\le k,\ \text{for any } ((x,y),(s,t))\in I\times J.$$

We shall use the function sets:
$B(I\times J)=\{f|\ f: I\times J\to\mathbb{R},\ f \text{ is bounded on } I\times J\}$,
with the usual sup-norm $\|\cdot\|_\infty$,
$B_b(I\times J)=\{f|\ f: I\times J\to\mathbb{R},\ f \text{ is B-bounded on } I\times J\}$,
$C_b(I\times J)=\{f|\ f: I\times J\to\mathbb{R},\ f \text{ is B-continuous on } I\times J\}$.
Let $f\in B_b(I\times J)$. The function $\omega_{mixed}(f;*,*):[0,+\infty[\times[0,+\infty[\to\mathbb{R}$ defined by

$$\omega_{mixed}(f;\delta_1,\delta_2):=\sup\{|\Delta f[(x,y),(s,t)]|:|x-s|\le\delta_1,\ |y-t|\le\delta_2\}, \tag{4.2}$$

for any $(\delta_1,\delta_2)\in[0,+\infty[\times[0,+\infty[$ is called the mixed modulus of smoothness. The notion of mixed modulus of smoothness was introduced by I. Badea in [3] and then studied by I. Badea, C. Badea, C. Cottin and H. H. Gonska [4], [6].

DEFINITION 4.1. [4] *Let $I,J\subset\mathbb{R}$ be compact intervals and let $L: C_b(I\times J)\to B(I\times J)$ be a linear positive operator. Suppose that $f(\cdot,*)\in C_b(I\times J)$. The operator $U: C_b(I\times J)\to B(I\times J)$ defined for any $f\in C_b(I\times J)$ and $(x,y)\in I\times J$ by*

$$(Uf)(x,y)=L\left(f(\cdot,y)+f(x,*)-f(\cdot,*)\right)(x,y) \tag{4.3}$$

is called the GBS (Generalized Boolean Sum) operator associated to L.

REMARK 4.2. The notion of GBS operator was introduced by C. Badea and C. Cottin [4]. $\square$

Remark 4.3. The most natural way to construct the GBS operator U is the following:

- one considers the univariate operators
$$L_1^x : C_b(I) \to B(I), \quad L_2^y : C_b(J) \to B(J);$$
- if

$L_1^x, L_2^y : C_b(I \times J) \to B(I \times J)$ are their parametric extensions [7], [12], then

$$U = L_1^x + L_2^y - L_1^x L_2^y. \tag{4.4}$$

□

Lemma 4.4. *The GBS operator associated to the bivariate $\varphi_1\varphi_2$-Szász-Mirakjan-Kantorovich operators is defined for any $f : [0,+\infty[\times[0,+\infty[\to \mathbb{R}$ and $(x,y) \in [0,+\infty[\times[0,+\infty[$ by*

$$(\varphi_1\varphi_2 U_{m,n} f)(x,y) = \tfrac{m}{\varphi_1(mx)} \tfrac{n}{\varphi_2(ny)} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \tfrac{\varphi_1^{(k)}(0)}{k!} \tfrac{\varphi_2^{(j)}(0)}{j!} \tag{4.5}$$
$$\cdot (mx)^k (ny)^j \int_{\frac{k}{m}}^{\frac{k+1}{m}} \int_{\frac{j}{n}}^{\frac{j+1}{n}} \left(f\left(\tfrac{k}{m}, t\right) + f\left(s, \tfrac{j}{n}\right) - f\left(\tfrac{k}{m}, \tfrac{j}{n}\right)\right) \mathrm{d}s\mathrm{d}t.$$

Proof. One applies relation (4.4) with $L_1^x := \varphi_1 S_m^x$, $L_2^y := \varphi_2 S_n^y$ and one takes Lemma 3.1. into account. □

The analogous of Shisha-Mond Theorem in terms of mixed modulus of smoothness is the following:

Theorem 4.5. [4] *Let $L : C_b(I \times J) \to B(I \times J)$ be an linear and positive operator reproducing constants and let $U : C_b(I \times J) \to B(I \times J)$ be the GBS associated operator.*

For any $(x,y) \in I \times J$ and $(\delta_1, \delta_2) \in]0,+\infty[\times]0,+\infty[$ the following inequality:

(4.6)
$$|f(x,y) - (Uf)(x,y)| \le \Big(1 + \delta_1^{-1}\sqrt{(L\varphi_x^2)(x,y)} + \delta_2^{-1}\sqrt{(L\varphi_y^2)(x,y)}$$
$$+\delta_1^{-1}\delta_2^{-1}\sqrt{(L\varphi_x^2)(x,y)(L\varphi_y^2)(x,y)}\Big)\, \omega_{mixed}(f;\delta_1,\delta_2)$$

holds.

For the GBS operator associated to the $\varphi_1\varphi_2$-Szász-Mirakjan-Kantorovich operator we have the following:

Theorem 4.6. *For any function $f \in C_b([0,+\infty[\times[0,+\infty[)$, any bivariate interval $[a_1,b_1] \times [a_2,b_2] \subset [0,+\infty[\times[0,+\infty[$ and $(x,y) \in [a_1,b_1] \times [a_2,b_2]$, the GBS operator (4.5) satisfies the following inequality*

$$|f(x,y) - (\varphi_1\varphi_2 U_{m,n} f)(x,y)| \le K\omega_{mixed}\left(f; \tfrac{b_1-a_1}{\sqrt{m^{\gamma_1}}}, \tfrac{b_2-a_2}{\sqrt{n^{\gamma_2}}}\right), \tag{4.7}$$

where K is defined at (3.11) and (3.12).

Proof. One applies the Theorem 4.5. with $L := \varphi_1\varphi_2 K_{m,n}$, respectively with $U := \varphi_1\varphi_2 U_{m,n}$. $\square$

Next we recall the Korovkin type theorem for B-continuous functions due to C. Badea, I. Badea and H. H. Gonska in [5].

THEOREM 4.7. [5] *Let $(L_{m,n})_{m,n\in\mathbb{N}}$ be a sequence of linear positive operators, $L_{m,n} : C_b(I \times J) \to B(I \times J)$. If*

i) $(L_{m,n}e_{00})(x, y) = 1$,
ii) $(L_{m,n}e_{10})(x, y) = x + u_{m,n}(x, y)$,
iii) $(L_{m,n}e_{01})(x, y) = y + v_{m,n}(x, y)$,
iv) $(L_{m,n}(e_{20} + e_{02}))(x, y) = x^2 + y^2 + w_{m,n}(x, y)$,
v) $\lim_{m,n\to\infty} u_{m,n}(x, y) = \lim_{m,n\to\infty} v_{m,n}(x, y) = \lim_{m,n\to\infty} w_{m,n}(x, y) = 0$ *uniformly on $I \times J$,*

then for any $f \in C_b(I \times J)$, any $(x, y) \in I \times J$, the sequence $(U_{m,n})_{m,n\in\mathbb{N}}$ converges to f uniformly on $I \times J$, where the operator $U_{m,n}$, $m, n \in \mathbb{N}$ is defined by

$$(U_{m,n}f)(x, y) = L_{m,n}(f(\cdot, y) + f(x, *) - f(\cdot, *))(x, y). \tag{4.8}$$

For the GBS operator associated to the $\varphi_1\varphi_2$-Szász-Mirakjan-Kantorovich operator we have the following:

THEOREM 4.8. *Suppose that the analytic functions $\varphi_1, \varphi_2 : \mathbb{R} \to]0, +\infty[$ satisfy the conditions* (3.8)*. Then, for any function $f \in C_b([0, +\infty[\times[0, +\infty[)$ and any $(x, y) \in [a_1, b_1] \times [a_2, b_2]$, it holds*

$$\lim_{m,n\to\infty} (\varphi_1\varphi_2 U_{m,n}f)(x, y) = f(x, y)$$

uniformly on any bivariate compact interval $[a_1, b_1]\times[a_2,b_2] \subset [0,+\infty[\times[0,+\infty[$.

Proof. One applies Theorem 4.7. with $L_{m,n} := \varphi_1\varphi_2 K_{m,n}$, respectively with $U_{m,n} := \varphi_1\varphi_2 U_{m,n}$. $\square$

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