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APPROXIMATION BY NONLINEAR HERMITE-FEJÉR INTERPOLATION OPERATORS OF MAX-PRODUCT KIND ON CHEBYSHEV NODES

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Abstract. The aim of this note is that by using the so-called max-product method, to associate to the Hermite-Fejér polynomials based on the Chebyshev knots of first kind, a new interpolation operator for which a Jackson-type approximation order in terms of $\omega_1(f; 1/n)$ is obtained.

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1. INTRODUCTION

Based on the Open Problem 5.5.4, pp. 324–326 in [8], in a series of recent papers [1, 2, 3, 4, 5], we have introduced and studied the so-called max-product operators attached to the Bernstein polynomials and to other linear Bernstein-type operators, like those of Favard-Szász-Mirakjan operators (truncated and nontruncated case), Baskakov operators (truncated and nontruncated case) and Bleimann-Butzer-Hahn operators.

This idea applied, for example, to the linear Bernstein operators $B_n(f)(x) = \sum_{k=0}^n p_{n,k}(x)f(k/n)$, where $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$, works as follows. Writing in the equivalent form $B_n(f)(x) = \frac{\sum_{k=0}^n p_{n,k}(x)f(k/n)}{\sum_{k=0}^n p_{n,k}(x)}$ and then replacing everywhere the sum operator Σ by the maximum operator $\bigvee$ (that is $\sum_{k=0}^n p_{n,k}(x)f(k/n)$ is replaced by the maximum $\max_{\{k=0,..,n\}}\{p_{n,k}(x)f(k/n)\}$

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and $\sum_{k=0}^{n} p_{n,k}(x)$ by $\max_{\{k=0,...,n\}}\{p_{n,k}(x)\}$), one obtains the nonlinear Bernstein operator of max-product kind

$$B_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{n} p_{n,k}(x)},$$

for which, surprisingly nice approximation and shape preserving properties were found.

For example, it is proved that for some classes of functions (like those of monotonous concave functions), the order of approximation given by the max-product Bernstein operators, are essentially better than the approximation order of their linear counterparts.

The aim of the present paper is to use the same idea to interpolation polynomials. In the case of the Hermite-Fejér kind polynomials based on the Chebyshev nodes of first kind, for example, we will obtain that in the class of Lipschitz functions with positive values, the new obtained interpolation operator has essentially better approximation property than the Hermite-Fejér polynomials.

Thus, let $f : [-1,1] \to \mathbb{R}$ and $x_{n,k} = \cos\left(\frac{2(n-k)+1}{2(n+1)}\pi\right) \in (-1,1)$, $k \in \{0,...,n\}$, $-1 < x_{n,0} < x_{n,1} < ... < x_{n,n} < 1$, be the roots of the first kind Chebyshev polynomial $T_{n+1}(x) = \cos[(n+1)\arccos(x)]$. Consider the Hermite-Fejér interpolation polynomial of degree $\leq 2n+1$ attached to f and to the nodes $(x_{n,k})_k$,

$$H_{2n+1}(f)(x) = \sum_{k=0}^{n} h_{n,k}(x) f(x_{n,k}),$$

with

$$h_{n,k}(x) = (1 - x x_{n,k}) \cdot \left(\frac{T_{n+1}(x)}{(n+1)(x-x_{n,k})}\right)^2.$$

It is well known that $\sum_{k=0}^{n} h_{n,k}(x) = 1$ for all $x \in \mathbb{R}$ (and that $h_{n,k}(x) \geq 0$ for all $x \in [-1,1]$ and $k = 0,...,n$), which allows us to write

$$H_{2n+1}(f)(x) = \frac{\sum_{k=0}^{n} h_{n,k}(x) f(x_{n,k})}{\sum_{k=0}^{n} h_{n,k}(x)}, \text{ for all } x \in \mathbb{R}.$$

Therefore, applying the max-product method as in the above case of Bernstein polynomials, the corresponding max-product Hermite-Fejér interpolation operator will be given by

$$H_{2n+1}^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} h_{n,k}(x) f(x_{n,k})}{\bigvee_{k=0}^{n} h_{n,k}(x)}.$$

REMARK 1.1. Firstly, it is clear that $H_{2n+1}^{(M)}(f)(x)$ is a nonlinear (more exactly sublinear on the space of positive functions) operator, well-defined for all $x \in \mathbb{R}$ and a continuous, piecewise rational function on $\mathbb{R}$. Indeed, by $\sum_{k=0}^{n} h_{n,k}(x) = 1$, for all $x \in \mathbb{R}$, for any x there exists an index $k \in \{0, ..., n\}$ such that $h_{n,k}(x) > 0$, which implies that $\bigvee_{k=0}^{n} h_{n,k}(x) > 0$. Indeed, contrariwise would follow that $h_{n,k}(x) \leq 0$ for all k and therefore we would obtain the contradiction $\sum_{k=0}^{n} h_{n,k}(x) \leq 0$. The continuity of the numerator and denominator of $H_{2n+1}^{(M)}(f)(x)$ as maximum of finite number of continuous functions is immediate, which implies the continuity of $H_{2n+1}^{(M)}(f)(x)$ on $\mathbb{R}$. The sublinearity follows from the property of the maximum operator $\bigvee$.

Also, by the property $h_{n,k}(x_{n,j}) = 1$ if $k = j$ and $h_{n,k}(x_{n,j}) = 0$ if $k \neq j$, we immediately obtain the interpolation property $H_{2n+1}^{(M)}(f)(x_{n,j}) = f(x_{n,j})$, for all $j \in \{0, ..., n\}$. $\square$

The plan of the paper goes as follows: in Section 2 we present some auxiliary results, in Section 3 we prove the main approximation result while in Section 4 we compare the approximation result in Section 3 with those for the linear Hermite-Fejér interpolation polynomials based on the Chebyshev knots of first kind.

2. AUXILIARY RESULTS

In all what follows, f will be considered continuous and with positive values, that is

$$f \in C_+[-1, 1] = \{f : [-1, 1] \to \mathbb{R}_+; f \text{ is continuous on } [-1, 1]\}.$$

Firstly, we present a general type approximation result, which in fact is valid for all the max-product type operators (including those of Bernstein type proved in [1]).

THEOREM 2.1. *For all $f \in C_+[-1, 1]$, $n \in \mathbb{N}$, $\delta > 0$ and $x \in [-1, 1]$ we have*

$$|f(x) - H_{2n+1}^{(M)}(f)(x)| \leq \left[1 + \tfrac{1}{\delta} H_{2n+1}^{(M)}(\varphi_x)(x)\right] \omega_1(f; \delta),$$

where $\varphi_x(t) = |t - x|$ for all $t, x \in [-1, 1]$, and $\omega_1(f; \delta) = \max\{|f(x) - f(y)|; x, y \in [-1, 1], |x - y| \leq \delta\}$.

Proof. First it is easy to check that as a consequence of the properties of the operator $\bigvee$, $f \leq g$ implies $H_{2n+1}^{(M)}(f) \leq H_{2n+1}^{(M)}(g)$ and also we have $H_{2n+1}^{(M)}(f + g) \leq H_{2n+1}^{(M)}(f) + H_{2n+1}(g)$, for all $f, g \in C_+[-1, 1]$.

Further, we have $f = f - g + g \leq |f - g| + g$, which by the above two properties successively implies $H_{2n+1}^{(M)}(f)(x) \leq H_{2n+1}^{(M)}(|f - g|)(x) + H_{2n+1}^{(M)}(g)(x)$, that is $H_{2n+1}^{(M)}(f)(x) - H_{2n+1}^{(M)}(g)(x) \leq H_{2n+1}^{(M)}(|f - g|)(x)$.

Writing now $g = g - f + f \leq |f - g| + f$ and applying the above reasonings, it follows $H_{2n+1}(g)^{(M)}(x) - H_{2n+1}^{(M)}(f)(x) \leq H_{2n+1}^{(M)}(|f - g|)(x)$, which combined with the above inequality gives

$$|H_{2n+1}^{(M)}(f)(x) - H_{2n+1}^{(M)}(g)(x)| \leq H_{2n+1}^{(M)}(|f-g|)(x).$$

Also, it is immediate that $H_{2n+1}^{(M)}(f)$ is positive homogenous, that is

$$H_{2n+1}^{(M)}(\lambda f) = \lambda H_{2n+1}^{(M)}(f) \text{ for all } \lambda \geq 0.$$

Now, since it is clear that $H_{2n+1}^{(M)}(e_0) = e_0$, where $e_0(x) = 1$ for all x, from the identity (for a fixed $x \in [-1,1]$)

$$H_{2n+1}^{(M)}(f)(x) - f(x) = H_{2n+1}^{(M)}(f(t))(x) - H_{2n+1}^{(M)}(f(x))(x)$$

and from the above proved properties of $H_{2n+1}^{(M)}(f)$, it easily follows

$$|f(x) - H_{2n+1}^{(M)}(f)(x)| \leq H_{2n+1}^{(M)}(|f(t) - f(x)|)(x).$$

Since for all $t, x \in [-1,1]$ we have

$$|f(t) - f(x)| \leq \omega_1(f;|t-x|) \leq \left[\tfrac{1}{\delta}|t-x| + 1\right]\omega_1(f;\delta),$$

replacing above we immediately obtain the estimate in the statement. □

As in case of the Bernstein type max-product operators, first it will be useful to exactly calculate $\bigvee_{k=0}^{n} h_{n,k}(x)$ for $x \in [-1,1]$. In this sense we have the following result.

LEMMA 2.2. *For each $j \in \{0,...,n-1\}$, there exists a unique point $y_{n,j} \in (x_{n,j}, x_{n,j+1})$, such that we have*

$$\bigvee_{k=0}^{n} h_{n,k}(x) = h_{n,j+1}(x), \text{ for all } x \in [y_{n,j}, y_{n,j+1}], \ j \in \{0,...,n-2\}.$$

In addition,

$$\bigvee_{k=0}^{n} h_{n,k}(x) = h_{n,0}(x), \text{ for all } x \in [-1, y_{n,0}],$$

and

$$\bigvee_{k=0}^{n} h_{n,k}(x) = h_{n,n}(x), \text{ for all } x \in [y_{n,n-1}, 1].$$

Proof. First we show that for fixed $n \in \mathbb{N}$ and $0 \leq k < k+1 \leq n$, there exists a unique point $y_{n,k} \in (x_{n,k}, x_{n,k+1})$ such that we have

$$0 \leq h_{n,k+1}(x) \leq h_{n,k}(x),$$

$$\text{if and only if } x \in [0, y_{n,k}] \bigcup \{x_{n,j}; j \in \{0,1,...,n\}, j \neq k+1\}. \tag{2.1}$$

Indeed, the inequality $h_{n,k+1}(x) \leq h_{n,k}(x), x \in [-1,1]$ is equivalent to

$$0 \leq \tfrac{T_{n+1}^2(x)}{(n+1)^2(x-x_{n,k})^2(x-x_{n,k+1})^2}(x_{n,k+1} - x_{n,k}) \cdot P_{n,k}(x), x \in [-1,1],$$

with $P_{n,k}(x) = x^3 - x[2 + x_{n,k} \cdot x_{n,k+1}] + (x_{n,k} + x_{n,k+1})$. Therefore, the inequality $h_{n,k+1}(x) \le h_{n,k}(x), x \in [-1,1]$ is equivalent to the condition that $x \in \{x \in [-1,1]; P_{n,k}(x) \ge 0\} \bigcup \{x_{n,j}; j \in \{0,1,...,n\}, j \ne k+1\}$.

But, since $P_{n,k}(-1) = (1 + x_{n,k})(1 + x_{n,k+1}) > 0$, $P_{n,k}(1) = (x_{n,k} - 1)(1 - x_{n,k+1}) < 0$ and $P'_{n,k}(x) = 0$ has the two solutions

$$z_1 = -\sqrt{\tfrac{2+x_{n,k}\cdot x_{n,k+1}}{3}}, z_2 = \sqrt{\tfrac{2+x_{n,k}\cdot x_{n,k+1}}{3}} \in (-1,1),$$

it easily follows that $P_{n,k}(x)$ has at z_1 a maximum point, at z_2 a minimum point, the equation $P_{n,k}(x) = 0$ has a unique solution $y_{n,k} \in (z_1, z_2)$ and that $P_{n,k}(x) \ge 0$ on $[-1,1]$ if and only if $x \in [0, y_{n,k}]$.

Now we will prove that in fact $y_{n,k} \in (x_{n,k}, x_{n,k+1})$. Indeed, this is immediate from the following simple calculation

$$P_{n,k}(x_{n,k}) \cdot P_{n,k+1}(x_{n,k+1})$$
$$= (x_{n,k}^3 - x_{n,k}^2 \cdot x_{n,k+1} - x_{n,k} + x_{n,k+1})(x_{n,k+1}^3 - x_{n,k} \cdot x_{n,k+1}^2 + x_{n,k} - x_{n,k+1})$$
$$= (x_{n,k} - x_{n,k+1})^2(x_{n,k}^2 - 1)(1 - x_{n,k+1}) < 0.$$

Therefore, as a first conclusion it follows (2.1).

By taking $k = 0,1,..,n-1$ in the inequality (2.1), we get

$$h_{n,1}(x) \le h_{n,0}(x), \text{ if and only if } x \in [-1, y_{n,0}] \bigcup \{x_{n,j}; j \ne 1\},$$

$$h_{n,2}(x) \le h_{n,1}(x), \text{ if and only if } x \in [-1, y_{n,1}] \bigcup \{x_{n,j}; j \ne 2\},$$

$$h_{n,3}(x) \le h_{n,2}(x), \text{ if and only if } x \in [-1, y_{n,2}] \bigcup \{x_{n,j}; j \ne 3\},$$

so on,

$$h_{n,k+1}(x) \le h_{n,k}(x), \text{ if and only if } x \in [-1, y_{n,k}] \bigcup \{x_{n,j}; j \ne k+1\},$$

so on,

$$h_{n,n-2}(x) \le h_{n,n-3}(x), \text{ if and only if } x \in [-1, y_{n,n-3}] \bigcup \{x_{n,j}; j \ne 1\},$$

$$h_{n,n-1}(x) \le h_{n,n-2}(x), \text{ if and only if } x \in [-1, y_{n,n-2}] \bigcup \{x_{n,j}; j \ne n-1\},$$

$$h_{n,n}(x) \le h_{n,n-1}(x), \text{ if and only if } x \in [-1, y_{n,n-1}] \bigcup \{x_{n,j}; j \ne n\}.$$

From all these inequalities, reasoning by recurrence we easily obtain:

$$\text{if } x \in [-1, y_{n,0}] \text{ then } h_{n,k}(x) \le h_{n,0}(x), \text{ for all } k = 0,1,...,n,$$

$$\text{if } x \in [y_{n,0}, y_{n,1}] \text{ then } h_{n,k}(x) \le h_{n,1}(x), \text{ for all } k = 0,1,...,n,$$

and so on finally

$$\text{if } x \in [y_{n,n-2}, y_{n,n-1}] \text{ then } h_{n,k}(x) \le h_{n,n-1}(x), \text{ for all } k = 0,1,...,n,$$

$$\text{if } x \in [y_{n,n-1}, 1] \text{ then } h_{n,k}(x) \le h_{n,n}(x), \text{ for all } k = 0,1,...,n,$$

which proves the lemma. □

For the proof of the main results we need some notations and auxiliary results, as follows.

Let us denote $y_{n,-1} = -1$ and $y_{n,n} = 1$. Then, for all $k, j \in \{0, 1, ..., n\}$, and for each $x \in [y_{n,j-1}, y_{n,j}]$, we denote

$$m_{k,n,j}(x) = \frac{h_{n,k}(x)}{h_{n,j}(x)},\ M_{k,n,j}(x) = m_{k,n,j}(x) \left| x_{n,k} - x \right| .$$

We observe that for $k \geq j+1$ we have $x_{n,k} - x \geq x_{n,j+1} - y_{n,j} \geq 0$ and it follows that $M_{k,n,j}(x) = m_{k,n,j}(x)(x_{n,k} - x)$. Also for $j \geq 1$ and $k \leq j-1$ we have $x - x_{n,k} \geq y_{n,j-1} - x_{n,j-1} \geq 0$ and it follows that $M_{k,n,j}(x) = m_{k,n,j}(x)(x - x_{n,k})$.

Lemma 2.3. *For all $k, j \in \{0, 1, ..., n\}$, and for each $x \in [y_{n,j-1}, y_{n,j}]$, we have*

$$m_{k,n,j}(x) \leq 1.$$

Proof. By Lemma 2.2 it immediately follows that

$$h_{n,0}(x) \leq h_{n,1}(x) \leq ... \leq h_{n,j}(x) \geq h_{n,j+1}(x) \geq ... h_{n,n}(x)$$

for all $x \in [y_{n,j-1}, y_{n,j}]$. Multiplying the above inequalities with $1/h_{n,j}(x)$ we get

$$m_{0,n,j}(x) \leq m_{1,n,j}(x) \leq ... \leq m_{j,n,j}(x) \geq m_{j+1,n,j}(x) \geq ... \geq m_{n,n,j}(x).$$

Since $m_{j,n,j}(x) = 1$ we immediately obtain the desired conclusion. □

Lemma 2.4. *Let $k,\ j \in \{0, 1, ..., n\}$ and let $x \in [y_{n,j-1}, y_{n,j}]$.*

(i) *If $k \in \{j+1, j+2, ..., n-1\}$, then $M_{k,n,j}(x) \geq M_{k+1,n,j}(x)$.*

(ii) *If $j \geq 1$ and $k \in \{0, 1, ..., j-1\}$, then $M_{k,n,j}(x) \geq M_{k-1,n,j}(x)$.*

Proof. (i) We observe that for all $k \geq j+1$ we get

$$\frac{M_{k,n,j}(x)}{M_{k+1,n,j}(x)} = \frac{h_{n,k}(x)}{h_{n,k+1}(x)} \cdot \frac{x_{n,k} - x}{x_{n,k+1} - x} = \frac{1 - x x_{n,k}}{1 - x x_{n,k+1}} \cdot \frac{x_{n,k} - x}{x_{n,k+1} - x} \cdot \left(\frac{x - x_{n,k+1}}{x - x_{n,k}}\right)^2 =$$
$$= \frac{1 - x x_{n,k}}{1 - x x_{n,k+1}} \cdot \frac{x_{n,k+1} - x}{x_{n,k} - x} \geq 1$$

which proves (i).

(ii) For all $k \leq j-1$ we get

$$\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} = \frac{h_{n,k}(x)}{h_{n,k-1}(x)} \cdot \frac{x - x_{n,k}}{x - x_{n,k-1}} = \frac{1 - x x_{n,k}}{1 - x x_{n,k-1}} \cdot \frac{x - x_{n,k}}{x - x_{n,k-1}} \cdot \left(\frac{x - x_{n,k-1}}{x - x_{n,k}}\right)^2 =$$
$$= \frac{1 - x x_{n,k}}{1 - x x_{n,k-1}} \cdot \frac{x - x_{n,k-1}}{x - x_{n,k}} \geq 1,$$

which proves (ii). □

Remark 2.5. It is of interest to find good estimates for each $y_{n,j}$. For this purpose we take into account that from $x_{n,j} < y_{n,j} < y_{n,j+1}$, we immediately obtain the following estimates for $y_{n,j}$:

$$\min\{|x_{n,j}|, |x_{n,j+1}|\} \leq |y_{n,j}| \leq \max\{|x_{n,j}|, |x_{n,j+1}|\},\ \text{if } x_{n,j} \cdot x_{n,j+1} \geq 0,$$

and $y_{n,j} = 0$, if $x_{n,j} \cdot x_{n,j+1} < 0$. Indeed, in this last case we necessarily have $x_{n,j} + x_{n,j+1} = 0$ (the roots of the Chebyshev polynomial $T_{n+1}(x)$ are symmetric with respect to the origin), which replaced in the proof of Lemma 2.2 immediately implies $y_{n,j} = 0$. But by the formula $\cos(\alpha) = \sin(\pi/2 - \alpha)$, we get

$$x_{n,j} = \cos\left(\frac{2(n-j)+1}{2(n+1)}\pi\right) = \sin\left(\frac{\pi}{2} \cdot \frac{2j-n}{n+1}\right), j = 0, 1, ..., n-1$$

and by the well-known double inequality (see e.g. [10], p. 57) $(2/\pi)u \leq \sin(u) \leq u$, for all $u \in [0, \pi/2]$, we immediately get

$$\frac{|2j-n|}{n+1} \leq |x_{n,j}| \leq \frac{|2j-n|}{n+1} \cdot \frac{\pi}{2}, \text{ for all } j = 0, 1, ..., n.$$

□

Remark 2.6. Note that due to the symmetry of the nodes $x_{n,j}$, the "intermediate" nodes $y_{n,j} \in (x_{n,j}, x_{n,j+1})$, $j \in \{0, ..., n-1\}$ in Lemma 2.2 also are symmetric with respect the origin. Indeed, since each $y_{n,j}$ satisfies the equation $y_{n,j}^3 - y_{n,j}[2 + x_{n,j} \cdot x_{n,j+1}] + (x_{n,j} + x_{n,j+1}) = 0$ and since $x_{n,j} = -x_{n,n-j}$, we get $y_{n,n-j}^3 - y_{n,n-j}[2 + x_{n,n-j} \cdot x_{n,n-(j+1)}] + (x_{n,n-j} + x_{n,n-(j+1)}) = 0$. Adding these two relationships we obtain $y_{n,j}^3 + y_{n,n-j}^3 - [2 + x_{n,j} \cdot x_{n,j+1}](y_{n,j} + y_{n,n-j}) = 0$, that is

$$(y_{n,j} + y_{n,n-j})(y_{n,j}^2 - y_{n,j} \cdot y_{n,j+1} + y_{n,j+1}^2 - 2 - x_{n,j} \cdot x_{n,j+1}) = 0.$$

Because it easily follows that the second term above is always < 0, we get $y_{n,j} + y_{n,n-j} = 0$, which proves the desired assertion. □

Remark 2.7. Since $H_n^{(M)}(f)(x_{n,j}) - f(x_{n,j}) = 0$ for all $n \in \mathbb{N}$ and $j = 0, 1, ..., n$, we note that in the next notations, proofs and statements of the all approximation results, in fact we always may suppose that $x \in [-1, 1]$ and $x \neq x_{n,j}$, for all $j = 0, 1, ..., n$. □

3. APPROXIMATION RESULTS

The main result is the following Jackson-type estimate.

Theorem 3.1. *Let $f : [-1, 1] \to \mathbb{R}_+$ be continuous on $[-1, 1]$. Then we have the estimate*

$$|H_{2n+1}^{(M)}(f)(x) - f(x)| \leq 14\omega_1\left(f, \frac{1}{n+1}\right), \textit{ for all } n \in \mathbb{N}, \ x \in [-1, 1].$$

Proof. By Theorem 2.1 we have

$$|H_{2n+1}^{(M)}(f)(x) - f(x)| \leq \left(1 + \frac{1}{\delta_n} H_{2n+1}^{(M)}(\varphi_x)(x)\right) \omega_1(f, \delta_n), \tag{3.1}$$

where $\varphi_x(t) = |t - x|$. So, it is enough to estimate

$$E_n(x) := H_{2n+1}^{(M)}(\varphi_x)(x) = \frac{\bigvee_{k=0}^{n} h_{n,k}(x)|x_{n,k} - x|}{\bigvee_{k=0}^{n} h_{n,k}(x)}, \ x \in [-1, 1].$$

Let $x \in [y_{n,j-1}, y_{n,j}]$, where $j \in \{0, 1, ..., n\}$ is fixed arbitrary. By Lemma 2.2 we easily obtain

$$E_n(x) = \max_{k=0,1,\ldots,n} \{M_{k,n,j}(x)\}, x \in [y_{n,j-1}, y_{n,j}].$$

It remains to obtain an upper estimate for each $M_{k,n,j}(x)$ when $j \in \{0, 1, ..., n\}$ is fixed, $x \in [y_{n,j-1}, y_{n,j}]$ and $k \in \{0, 1, ..., n\}$. In fact we will prove that

$$M_{k,n,j}(x) \le \tfrac{2\pi}{n+1}, \text{ for all } x \in [y_{n,j-1}, y_{n,j}], k = 0, 1, ..., n,$$

which immediately will imply that

$$E_n(x) \le \tfrac{2\pi}{n+1}, \text{ for all } x \in [-1, 1], n \in \mathbb{N}, \tag{3.2}$$

and taking $\delta_n = \frac{2\pi}{n+1}$ in (3.1), since $[2\pi] = 6$, from the property $\omega_1(f; \lambda\delta) \le ([\lambda] + 1)\omega_1(f; \delta)$ we immediately obtain the estimate in the statement.

In order to prove (3.2), we distinguish the following cases:

1) $j = 0$; 2) $j = n$ and 3) $j \in \{1, 2, ..., n - 1\}$.

Case 1) By Lemma 2.4, (i), it follows that $E_n(x) = \max_{k=0,1}\{M_{k,n,0}(x)\}$ for all $x \in [-1, y_{n,0}]$.

If $k = 0$ then $M_{0,n,0}(x) = |x_{n,0} - x|$. Since $x \in [-1, y_{n,0}] \subseteq [-1, x_{n,1}]$, we obtain

$$\begin{aligned} |x_{n,0} - x| &\le x_{n,1} + 1 = \cos\left(\tfrac{2(n-1)+1}{2(n+1)}\pi\right) + 1 \\ &= 2\cos^2\left(\tfrac{2(n-1)+1}{4(n+1)}\pi\right) = 2\sin^2\left(\tfrac{\pi}{2} - \tfrac{2(n-1)+1}{4(n+1)}\pi\right) \\ &= 2\sin^2\left(\tfrac{3\pi}{4(n+1)}\right) \le \tfrac{9\pi^2}{8(n+1)^2}. \end{aligned}$$

If $k = 1$ then $M_{1,n,0}(x) = m_{1,n,0}(x)\,|x_{n,1} - x|$. By Lemma 2.3, it follows that $m_{1,n,0} \le 1$ and we obtain

$$M_{1,n,0}(x) \le |x_{n,1} - x| = x_{n,1} - x \le x_{n,1} + 1 \le \tfrac{9\pi^2}{8(n+1)^2}.$$

In conclusion we obtain $E_n(x) \le \frac{9\pi^2}{8(n+1)^2}$ for all $x \in [-1, y_{n,0}]$.

Case 2) By Lemma 2.4, (ii), it follows that $E_n(x) = \max_{k=n-1,n}\{M_{k,n,n}(x)\}$ for all $x \in [y_{n,n-1}, 1]$.

If $k = n$ then $M_{n,n,n}(x) = |x_{n,n} - x|$. Since $x \in [y_{n,n-1}, 1] \subseteq [x_{n,n-1}, 1]$, we obtain

$$|x_{n,n} - x| \le 1 - x_{n,n-1} \le \tfrac{9\pi^2}{8(n+1)^2}$$

where we used the obvious equality $1 - x_{n,n-1} = x_{n,1} + 1$.

If $k = n - 1$ then $M_{n-1,n,n}(x) = m_{n-1,n,n}(x)\,|x_{n,n-1} - x| \le |x_{n,n-1} - x| = x - x_{n,n-1} \le 1 - x_{n,n-1} \le \frac{9\pi^2}{8(n+1)^2}$.

In conclusion, we obtain $E_n(x) \le \frac{9\pi^2}{8(n+1)^2}$ for all $x \in [y_{n,n-1}, 1]$.

Case 3) By Lemma 2.4 it follows that $E_n(x) = \max_{k=j-1,j,j+1}\{M_{k,n,j}(x)\}$ for all $x \in [y_{n,j-1}, y_{n,j}]$.

If $k=j$ then $M_{j,n,j}(x)=|x_{n,j}-x|$. For $x\in[y_{n,j-1},y_{n,j}]\subseteq[x_{n,j-1},x_{n,j+1}]$, we obtain

$$\begin{aligned}|x_{n,j}-x| &\le x_{n,j+1}-x_{n,j-1}=2\sin\left(\tfrac{\pi}{n+1}\right)\sin\left(\tfrac{2n-2j+1}{2(n+1)}\pi\right)\\ &\le 2\sin\left(\tfrac{\pi}{n+1}\right)\le\tfrac{2\pi}{n+1}.\end{aligned}$$

If $k=j+1$ then $M_{j+1,n,j}(x)=m_{j+1,n,j}(x)\,|x_{n,j+1}-x|\le x_{n,j+1}-x$. Since $x\in[y_{n,j-1},y_{n,j}]\subseteq[x_{n,j-1},x_{n,j+1}]$ it follows that

$$x_{n,j+1}-x\le x_{n,j+1}-x_{n,j-1}\le\tfrac{2\pi}{n+1}.$$

If $k=j-1$ then $M_{j-1,n,j}(x)=m_{j-1,n,j}(x)\,|x_{n,j-1}-x|\le x-x_{n,j-1}\le x_{n,j+1}-x_{n,j-1}\le\frac{2\pi}{n+1}$.

Collecting all the estimates obtained above and taking into account that $9\pi^2/[8(n+1)^2]\le 2\pi/(n+1)$ for all $n\in\mathbb{N}$, we easily get (3.2), which completes the proof. □

Remark 3.2. The order of approximation in terms of $\omega_1(f;1/(n+1))$ obtained by the proof of Theorem 3.1 cannot be improved, in the sense that the order of $\max_{x\in[0,1]}\{E_n(x)\}$ is exactly $\frac{1}{n+1}$ (here $E_n(x)$ is defined in the proof of Theorem 3.1). Indeed, for each $n\in\mathbb{N}$ we have $x_{2n+1,n}+x_{2n+1,n+1}=0$ which by the Remark 2.5 after the proof of Lemma 2.4 immediately implies $y_{2n+1,n}=0$ and (since $0\in[y_{2n+1,n},y_{2n+1,n+1}]$)

$$M_{n+1,2n+1,n+1}(0)=x_{2n+1,n+1}\ge\tfrac{2(n+1)-(2n+1)}{2n+2}=\tfrac{1}{2(n+1)}.\quad\square$$

4. COMPARISON WITH THE HERMITE-FEJÉR POLYNOMIALS

Firstly we present a brief history on the order in approximation by the Hermite-Fejér polynomials, $H_{2n+1}(f)(x)$. Denoting $A_{n+1}(f)=\|H_{2n+1}-f\|$, where $\|\cdot\|$ is the uniform norm on $C[-1,1]$, a famous result of Fejér [7] states that $lim_{n\to\infty}A_{n+1}(f)=0$, for all $f\in C[-1,1]$. The first estimate of the rate of convergence, $A_{n+1}(f)=O\left(\omega_1\left(f;\frac{1}{\sqrt{n+1}}\right)\right)$, obtained by T. Popoviciu [12], was improved by E. Moldovan to $A_{n+1}(f)=O\left(\omega_1\left(f;\frac{ln(n+1)}{n+1}\right)\right)$ in [11], where $ln(n)$ denotes the logarithm of n. In Xie Hua Sun and Dechang Jiang [17], it was proved that above, ω_1 can be replaced by the Diztzian-Totik modulus ω_1^{φ}.

In a sense, the two previous results are the best possible, because for $g(x)=|x|$ we have $|H_{2n+1}(g)(0)-g(0)|\ge c_1\frac{ln(n+1)}{n+1}, n\in\mathbb{N}$, with $c_1>0$ independent of n. In fact, by R. Bojanic [6], if $f\in Lip_1$ then the order $O(ln(n+1)/(n+1))$ cannot be improved.

On the other hand, as it was remarked in the book of J. Szabados and P. Vértesi [14], p. 168, Theorem 5.1, the above order is not the best possible for $g(x)=|x|^{\delta}$, $0<\delta<1$, the correct estimate being of order $1/n^{\delta}$. This remark also follows from the equivalence proved by Theorem 2.3 in [17],

$$\|H_{2n+1}(f)-f\|=O(1/n^{\delta})\text{ iff }E_{n+1}(f)=O(1/n^{\delta})$$

Other good estimates were obtained, for example, in R. Bojanic [6] for the uniform approximation, and in J. Prasad [13], which generalizes the estimate of P. Vértesi in [16] for the pointwise approximation. Also, the saturation order $\frac{1}{n}$, was proved by J. Szabados in [15].

Now, from Theorem 3.1, we easily get that the order of approximation obtained by the max-product interpolation operator $H_{2n+1}^{(M)}(f)(x)$ for the positive function $f \in Lip_1[-1,1]$, is essentially better than that given by the Hermite-Fejér interpolation polynomials, $H_{2n+1}(f)(x)$. Indeed, in this case by Theorem 3.1 we get that $\|H_{2n+1}^{(M)}(f) - f\| \leq \frac{c}{n+1}$, while by [6] we have $\|H_{2n+1}(f) - f\| \sim \frac{ln(n+1)}{n+1}$. Here $a_n \sim b_n$ means that there exists $c_1, c_2 > 0$ independent of n, such that $c_1 b_n \leq a_n \leq c_2 b_n$ for all $n \in \mathbb{N}$.

Finally, let us mention that in Hermann-Vértesi [9], some linear interpolatory rational operators are constructed, for which a Jackson-type order of approximation is obtained and, in addition, a saturation result is obtained. It remains an open question to prove a saturation result for the nonlinear max-product Hermite-Fejér operator in the present paper, possibly by using some ideas in [9].

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