REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 39 (2010) no. 1, pp. 32–49
ictp.acad.ro/jnaat

ON THE ACCELERATION OF THE CONVERGENCE OF CERTAIN ITERATIVE PROCEEDINGS (II)

ADRIAN DIACONU*

Abstract. The research reflected in this paper has its origin in the study of the convergence of the sequences generated through the use of certain methods derived from the well-known Newton-Kantorovich method for the approximation of the solution of an equation in a linear normed space, together with the inverse of the Fréchet differential on this solution. An important place is given in the paper to the notion of convergence speed order of an approximant sequence of the solution of an equation. Considering given an approximant sequence which verifies certain conditions expressed through the inequalities (25), we will build another approximant sequence through the relations (22), sequence which finds its convergence speed order ameliorated. We will analyze certain special cases and, in the same time, we will determine optimal methods from the point of view of the convergence speed order.

MSC 2000. 65J15.
Keywords. Convergence of the approximant sequences for operatorial equations in Banach spaces.

1. INTRODUCTION

Let us consider the linear normed spaces X and Y that are θ_X and θ_Y as null elements respectively, while $\|\cdot\|_X : X \to \mathbb{R}$ and $\|\cdot\|_Y : Y \to \mathbb{R}$ respectively are norms.

We will note by $(X, Y)^*$ the set of the linear and continuous mappings defined on X with values in Y; we know that this set is a linear normed space with the norm:

$$\|\cdot\| : (X,Y)^* \to \mathbb{R};\ \ \|U\| = \sup\left\{ \|U(x)\|_Y / \, x \in X, \|x\|_X = 1 \right\},$$

the supremum of the definition being finite of course.

It is also known that if $(Y, \|\cdot\|_Y)$ is a Banach space, then $((X,Y)^*, \|\cdot\|)$ is a Banach space as well.

We will consider now a set $D \subseteq X$, a nonlinear function $f : D \to Y$ and, using this function, the nonlinear equation:

$$f(x) = \theta_Y. \tag{1}$$

* "Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, Str. M. Kogălniceanu Nr. 1, 3400 Cluj-Napoca, Romania, e-mail: adiaconu@math.ubbcluj.ro.

The aim is to prove the existence of a solution of this equation, namely an element $\overline{x}$ from D so that for $x = \overline{x}$ the equality (1) is verified and the approximation of this solution through a sequence $(x_n)_{n\in\mathbb{N}\cup\{0\}} \subseteq D$, the elements of which will be obtained after a recurrence formula.

If we suppose that the function $f : D \to Y$ admits the Fréchet differential $f'(x) \in (X, Y)^*$ on every point $x \in D$, the sequence $(x_n)_{n\in\mathbb{N}\cup\{0\}}$ can be generated from the well-known iterative method known as the Newton-Kantorovich iterative method. In this case for any $n \in \mathbb{N}\cup\{0\}$ the equality:

$$f'(x_n)(x_{n+1} - x_n) + f(x_n) = \theta_Y \tag{2}$$

is verified.

If the function $f : D \to Y$ and the initial element $x_0 \in D$ verify certain conditions, we can prove that for any $n \in \mathbb{N}\cup\{0\}$ there exists the mapping $[f'(x_n)]^{-1} \in (Y, X)^*$ and in this way the recurrence relation (2) will be written for any $n \in \mathbb{N}\cup\{0\}$ as:

$$x_{n+1} = x_n - \left[f'(x_n)\right]^{-1} f(x_n), \tag{3}$$

see [10] and [11].

The mapping $[f'(x_n)]^{-1} \in (Y, X)^*$ will be found by solving of the linear equation (2) and this operation must be accomplished for every $n \in \mathbb{N}\cup\{0\}$, that is for every iteration step needed in order to obtain an element $x_n \in D$ corresponding to the criterion of error imposed to the approximant of the solution of equation (1).

We can surpass this difficulty using an additional sequence. Thus besides the solution $\overline{x} \in D$ that is approximed by the sequence $(x_n)_{n\in\mathbb{N}\cup\{0\}} \subseteq D$, the linear mapping $[f'(x)]^{-1} \in (Y, X)^*$ will be approximed in the same time through an additional sequence $(A_n)_{n\in\mathbb{N}\cup\{0\}} \subseteq (Y, X)^*$.

The manner of obtaining the additional sequence is that of the approximation of a certain linear mapping's inverse. So if we consider a mapping $U \in (X, Y)^*$ and another mapping $A_0 \in (Y, X)^*$ so that $\|\mathbf{I}_Y - UA_0\| < 1$ ($\mathbf{I}_Y$ represents the identical mapping from Y) then there exists the inverse to the right of the mapping U, denoted by $U_d^{-1} \in (Y, X)^*$ and this mapping will be obtained through the limit of the sequence $(A_n)_{n\in\mathbb{N}\cup\{0\}} \subseteq (Y, X)^*$ generated by the recurrence sequence:

$$A_{n+1} = A_n \sum_{k=0}^{r} (\mathbf{I}_Y - UA_n)^k \tag{4}$$

starting from the mapping A_0.

On account of the fact that for any $n \in \mathbb{N} \cup \{0\}$ the relations:

$$(5) \qquad \begin{cases} \mathbf{I}_Y - UA_{n+1} = (\mathbf{I}_Y - UA_n)^{r+1}, \\ \mathbf{I}_Y - UA_n = (\mathbf{I}_Y - UA_0)^{(r+1)^n}, \\ \|A_{n+1} - A_n\| \leq \|A_0\| \frac{d^{(r+1)^n}}{1-d^{(r+1)^n}} \exp \sum\limits_{k=1}^{r} \frac{d^k}{1-d^{kr}}, \end{cases}$$

are true. In the above $d = \|\mathbf{I}_Y - UA_0\|$ and for any $u \in \mathbb{R}$ we have $\exp(u) = \mathbf{e}^u$. Again for any $n \in \mathbb{N} \cup \{0\}$ and $p \in \mathbb{N}$ the following inequality is true:

$$(6) \qquad \|A_{n+p} - A_n\| \leq \|A_0\| \frac{d^{(r+1)^n}}{\left[1-d^{(r+1)^n}\right]^2} \exp \sum_{k=1}^{r} \frac{d^k}{1-d^{kr}}.$$

If $(X, \|\cdot\|_X)$ is a Banach space, we deduce the convergence of the sequence $(A_n)_{n \in \mathbb{N} \cup \{0\}}$ to the mapping $U_d^{-1} \in (Y, X)^*$ for which we have the estimates:

$$(7) \qquad \begin{cases} \left\|U_d^{-1}\right\| \leq \frac{\|A_0\|}{1-d^{(r+1)^n}} \exp \sum\limits_{k=1}^{r} \frac{d^k}{1-d^{kr}}, \\ \left\|U_d^{-1} - A_n\right\| \leq \|A_0\| \frac{d^{(r+1)^n}}{\left[1-d^{(r+1)^n}\right]^2} \exp \sum\limits_{k=1}^{r} \frac{d^k}{1-d^{kr}}. \end{cases}$$

On account of the aforementioned results the following definition is justified:

Definition 1. *For a given mapping $U \in (X, Y)^*$ and a number $r \in \mathbb{N}$, the mapping:*

$$(8) \qquad \mathbf{S}_U^{(r+1)} : (Y, X)^* \to (Y, X)^*, \quad \mathbf{S}_U^{(r+1)}(A) = A \sum_{k=0}^{r} (\mathbf{I}_Y - UA)^k$$

is called a mapping of approximation with the order $r+1$ of the inverse to right of U.

On account of the aforementioned elements we have:

Remark 2. If $A_0 \in (Y, X)^*$ is given, using a certain approximation mapping we can build a sequence $(A_n)_{n \in \mathbb{N} \cup \{0\}}$ that will be in fact the sequence of the successive approximations generated through the mapping (8), namely for any $n \in \mathbb{N} \cup \{0\}$ we have the equality $A_{n+1} = \mathbf{S}_U^{(r+1)}(A_n)$, a relation of recurrence that goes back to the relation (4) and if $\|\mathbf{I}_Y - UA_0\| < 1$ there exists $U_d^{-1} = \lim\limits_{n \to \infty} A_n \in (Y, X)^*$. □

We return to the iterative method of Newton-Kantorovich for the equation (1) and linked to this method we have the following:

Definition 3. *If the function $f : D \to Y$ admits the Fréchet differential on the every point $x \in D$ and this differential is invertible, then the function:*

$$(9) \qquad Q : D \to X; \; Q(x) = x - \left[f'(x)\right]^{-1} f(x)$$

is called the iterative operator of Newton attached to the function $f : D \to Y$.

It is clear that for the existence of a certain iterative operator of Newton in addition to the differentiability on every point $x \in D$ of the mapping $f : D \to Y$, it is necessary to suppose the invertibility of the mapping $f'(x) \in (X,Y)^*$ on every point $x \in D$ as well.

So the iterative method of Newton-Kantorovich is in fact the method of successive approximations generated by the mapping Q, the recurrence relation (3) will be written under the form $x_{n+1} = Q(x_n)$.

In order to avoid this last difficulty we will use the operator:

$$\mathbf{R} : D \times (Y,X)^* \to X,\ \mathbf{R}(x,A) = x - \mathbf{S}^{(r+1)}_{f'(x)}(A)\, f(x) \tag{10}$$

with an $r \in \mathbb{N} \cup \{0\}$.

In order to approximate the solution of the equation (1) we will use a sequence $(x_n)_{n \in \mathbb{N}\cup\{0\}} \subseteq D$ where for every $n \in \mathbb{N} \cup \{0\}$ we have:

$$\begin{cases} x_{n+1} = \mathbf{R}(x_n, A_n), \\ A_{n+1} = \mathbf{S}^{(q+1)}_{f'(x_{n+1})}(A_n), \end{cases} \tag{11}$$

namely more into detail:

$$\begin{cases} x_{n+1} = x_n - \mathbf{S}^{(r+1)}_{f'(x_n)}(A_n)\, f(x_n), \\ A_{n+1} = \mathbf{S}^{(q+1)}_{f'(x_{n+1})}(A_n), \end{cases} \tag{12}$$

or even more into detail:

$$\begin{cases} x_{n+1} = x_n - A_n \left[\sum\limits_{k=0}^{r} \left(\mathbf{I}_Y - f'(x_n) A_n\right)^k \right] f(x_n), \\ A_{n+1} = A_n \sum\limits_{k=0}^{q} \left(\mathbf{I}_Y - f'(x_{n+1}) A_n\right)^k, \end{cases} \tag{13}$$

with $x_0 \in D$ and $A_0 \in (Y,X)^*$, which are the initial elements of the proceeding and these elements are arbitrarily chosen. Also, in the relations (11)-(13), we use the numbers $r \in \mathbb{N}\cup\{0\}$ and $q \in \mathbb{N}$, in order to really generate a certain unconstant sequence $(A_n)_{n \in \mathbb{N}\cup\{0\}}$, the use a certain number $q \geq 1$ is necessary.

It has been shown [1], [2] that the order of the convergence speed of the approximant sequence $(x_n)_{n\in\mathbb{N}\cup\{0\}} \subseteq D$ to a solution of the equation $f(x) = \theta_Y$ is 2, that is there exists $K, L > 0$ so that for any $n \in \mathbb{N}\cup\{0\}$ we have the inequalities:

$$\begin{cases} \|f(x_{n+1})\|_Y \leq K \|f(x_n)\|_Y^2, \\ \|x_{n+1} - x_n\|_X \leq L \|f(x_n)\|_Y. \end{cases} \tag{14}$$

This order is not dependent of the natural numbers $r \in \mathbb{N}\cup\{0\}$ and $q \in \mathbb{N}$.

For this reason special interest is given to the case of $r = 0,\ q = 1$, for the simplicity of the calculations needed. In this case the relations (13) will be written under the form:

$$\left\{\begin{array}{l} x_{n+1}=x_n-A_n f\left(x_n\right), \\ A_{n+1}=A_n\left(2\mathbf{I}_Y-f'\left(x_{n+1}\right)A_n\right), \end{array}\right. \tag{15}$$

a method proposed by S. Ul'm [14], and also to the case of $r=0$, $q=2$ in which the choice of the initial elements $x_0\in D$ and $A_0\in(Y,X)^*$ is most favorable in order to ensure the convergence [1].

For the detailed study of these methods one can refer to the papers [3], [5], [6], [7] and [8].

The results established through the method (13) can be extended. In order to this it is necessary to use an operator $Q:D\to X$ with $Q(D)\subseteq D$. With the help of this operator we build a sequence $(x_n)_{n\in\mathbb{N}}\subseteq D$ starting from an arbitrary element $x_0\in D$ and for a certain $n\in\mathbb{N}\cup\{0\}$, $x_n\in D$ being known, we will determine the element $x_{n+1}\in D$ that verifies the equality:

$$f'\left(x_n\right)\left(x_{n+1}-x_n\right)+f\left(Q\left(x_n\right)\right)=0. \tag{16}$$

If for any $n\in\mathbb{N}\cup\{0\}$ there exists the mapping $\left[f'\left(x_n\right)\right]^{-1}\in(Y,X)^*$, we can say that the sequence $(x_n)_{n\in\mathbb{N}\cup\{0\}}$ is generated starting from $x_0\in D$ through the recurrence relation:

$$x_{n+1}=Q\left(x_n\right)-\left[f'\left(x_n\right)\right]^{-1}f\left(Q\left(x_n\right)\right). \tag{17}$$

Such a method is known under the name of the iterative method of the Traub type.

Evidently, if $Q=\mathbf{I}_X$ the iterative method generated from (17) goes back to the method of Newton-Kantorovich (2)-(3).

We apply to the methods of the Traub type same modification proceeding that was used in the cases of the methods of the Newton-Kantorovich type. Thus we will obtain the pair of sequences $(x_n)_{n\in\mathbb{N}\cup\{0\}}\subseteq D$ and $(A_n)_{n\in\mathbb{N}\cup\{0\}}\subseteq$ $\subseteq(Y,X)^*$ generated from the recurrence relations:

$$\left\{\begin{array}{l} x_{n+1}=Q\left(x_n\right)-A_n\left[\sum\limits_{k=0}^{r}\left(\mathbf{I}_Y-f'\left(x_n\right)A_n\right)^k\right]f\left(Q\left(x_n\right)\right), \\ A_{n+1}=A_n\left[\sum\limits_{k=0}^{q}\left(\mathbf{I}_Y-f'\left(x_{n+1}\right)A_n\right)^k\right]. \end{array}\right. \tag{18}$$

It is known that if the operator $Q:D\to X$ has the order p in connection with the function $f:D\to Y$, namely there exist the numbers $K,L>0$ so that for any $x\in D$ the following relations are true:

$$\left\{\begin{array}{l} \left\|f\left(Q\left(x_n\right)\right)\right\|_Y\le K\left\|f\left(x_n\right)\right\|_Y^p, \\ \left\|Q\left(x\right)-x\right\|_X\le L\left\|f\left(x_n\right)\right\|_Y, \end{array}\right. \tag{19}$$

then the order of the convergence speed of the sequence $(x_n)_{n\in\mathbb{N}\cup\{0\}}\subseteq D$ generated through the relations (16) and (18) is $p+1$ (see [4], [6], [8], [12] and [13]).

Furthermore, as we have shown in [7], given a sequence $(y_n)_{n\in\mathbb{N}\cup\{0\}} \subseteq D$ with a **convergence speed of the order** $p \geq 1$ **in connection with the function** $f : D \to Y$ **and the sequence** $(x_n)_{n\in\mathbb{N}\cup\{0\}}$ meaning the numbers $K_0, L_0 > 0$ exist so that, for any $n \in \mathbb{N}\cup\{0\}$ the following inequalities are true:

$$\begin{cases} \|f(y_n)\|_Y \leq K_0 \|f(x_n)\|_Y^p, \\ \|y_n - x_n\|_X \leq L_0 \|f(x_n)\|_Y, \end{cases} \tag{20}$$

then the sequence $(x_n)_{n\in\mathbb{N}\cup\{0\}} \subseteq D$ generated by the relation:

$$x_{n+1} = y_n - \left[f'(x_n)\right]^{-1} f(y_n), \tag{21}$$

or through the relations:

$$\begin{cases} x_{n+1} = y_n - A_n \left[\sum\limits_{k=0}^{r} \left(\mathbf{I}_Y - f'(x_n) A_n\right)^k\right] f(y_n), \\ A_{n+1} = A_n \sum\limits_{k=0}^{q} \left(\mathbf{I}_Y - f'(x_{n+1}) A_n\right)^k \end{cases} \tag{22}$$

has the convergence speed order $p+1$ in connection with the same function $f : D \to Y$.

In the paper [3] we have considered a mapping $Q : D \times (Y,X)^* \to X$ with the property that for any $A \in (Y,X)^*$ the inclusion $Q(D,A) \subseteq D$ takes place. Using this mapping we will build an iterative method for the approximation of the solution of the equation (1), with the sequences $(x_n)_{n\in\mathbb{N}\cup\{0\}}$ and $(A_n)_{n\in\mathbb{N}\cup\{0\}}$ that are obtained form the recurrence relations:

$$\begin{cases} x_{n+1} = Q(x_n, A_n) - A_n \left[\sum\limits_{k=0}^{r} \left(\mathbf{I}_Y - f'(x_n) A_n\right)^k\right] f(Q(x_n, A_n)), \\ A_{n+1} = A_n \sum\limits_{k=0}^{q} \left(\mathbf{I}_Y - f'(x_{n+1}) A_n\right)^k, \end{cases} \tag{23}$$

starting from the elements $x_0 \in D$ and $A_0 \in (Y,X)^*$ arbitrarily chosen.

In order to ensure the convergence of the approximation sequence $(x_n)_{n\in\mathbb{N}\cup\{0\}} \subseteq D$ generated through the relations (23), we will replace the relations (19) with others more adequate to the mapping $Q : D \times (Y,X)^* \to X$. Thus we will suppose that the numbers $m \in \mathbb{N}$ and $M > 0$, and, for any $i = \overline{1,m}$, the numbers $p_i, q_i, K_i > 0$ with $p_i + q_i > 1$ exist so that for any $x \in D$ and any $A \in (Y,X)^*$ the following inequalities are true:

$$\begin{cases} \|f(Q(x,A))\|_Y \leq \sum\limits_{i=1}^{m} K_i \|f(x)\|_Y^{p_i} \cdot \|\mathbf{I}_Y - f'(x) A\|^{q_i}, \\ \|Q(x,A) - x\|_X \leq M \|f(x)\|_Y. \end{cases} \tag{24}$$

In this paper we will study the more general case of an iterative proceeding in which a sequence $(y_n)_{n\in\mathbb{N}\cup\{0\}} \subseteq D$ is given and a sequence $(x_n)_{n\in\mathbb{N}\cup\{0\}} \subseteq D$ is built using the recurrence relation (22).

However, we will replace the inequalities (20). Supposing that the numbers $m \in \mathbb{N}$ and $L > 0$ exist, and for any $i = \overline{1,m}$ the numbers $p_i, q_i, K_i > 0$ exist with $p_i + q_i \geq 1$ so that for any $n \in \mathbb{N}\cup\{0\}$ the following inequalities are true:

$$\left\{ \begin{array}{l} \left\| f(y_n) \right\|_Y \leq \sum\limits_{i=1}^{m} K_i \left\| f(x_n) \right\|_Y^{p_i} \cdot \left\| \mathbf{I}_Y - f'(x_n) A_n \right\|^{q_i}, \\ \left\| y_n - x_n \right\|_X \leq M \left\| f(x_n) \right\|_Y. \end{array} \right. \tag{25}$$

It is clear that if there exists the mapping $Q : D \times (Y,X)^* \to X$ so that for any $n \in \mathbb{N}\cup\{0\}$ we have $y_n = Q(x_n, A_n)$, the proposed iterative proceeding goes back to the iterative proceeding generated through the relations (23), while the inequalities (24) are verified.

In order to simplify the writing we will introduce the function:

$$g : [0,+\infty[\times [0,+\infty[\to \mathbb{R},\ g(u,v) = \sum_{i=1}^{m} K_i u^{p_i} v^{q_i}$$

where the number $m \in \mathbb{N}$ and for any $i = \overline{1,m}$ the elements $K_i, p_i, q_i > 0$ with $p_i + q_i > 1$ are the same for which the inequalities (25) are verified. It is evident that for any $u, v \geq 0$ it results $g(u,v) \geq 0$.

We will consider as well the function:

$$h : [0,+\infty[\to \mathbb{R},\ h(t) = 1 + t + t^2 + ... + t^r = \tfrac{1-t^{r+1}}{1-t}.$$

Finally, for $x_0 \in D$ and $R > 0$, we will note by $\mathcal{S}(x_0, R)$ the closed ball with the center in the point x_0 and the radius R, so we have:

$$\mathcal{S}(x_0, R) = \left\{ x \in \mathbb{R} / \left\| x - x_0 \right\|_X \leq R \right\}.$$

We suppose that the following hypothesis is fulfilled:

I. *The function $f : D \to Y$ admits a Fréchet differential on every point $x \in D$, this differential being $f'(x) \in (X,Y)^*$. The function $f' : D \to (X,Y)^*$ verifies Lipschitz's condition, namely there exists a constant $L > 0$ so that for any $x \in D$ the following inequality is true:*

$$\left\| f'(x) - f'(y) \right\| \leq L \left\| x - y \right\|_X. \tag{26}$$

In the same context we suppose that for any $x \in D$ the mapping $f'(x) \in$ $\in (X,Y)^*$ is invertible, therefore the inverse mapping $[f'(x)]^{-1} \in (Y,X)^*$ exists. We also suppose the existence of a number $B > 0$ so that for any $x \in D$ we have the inequality:

$$\left\| \left[f'(x) \right]^{-1} \right\| \leq B. \tag{27}$$

The main result of this paper is the following:

Theorem 4. *If the following hypotheses are true:*

i) *the linear normed space* $(X, \|\cdot\|_X)$ *is a Banach space and the function* $f : D \to Y$ *verifies the hypothesis* I)*.*

ii) *the numbers* $C_1, C_2 > 0,\ r \in \mathbb{N} \cup \{0\}$ *and* $q \in \mathbb{N}$ *exist and the following conditions are verified:*

$$\left\{ \begin{array}{l} \frac{L}{2} B^2 (1 + C_2)^2 g^2 (C_1, C_2) h (C_2) + \\ \quad + g (C_1, C_2) \left[C_2^{r+1} + LMBC_1 (1 + C_2) h (C_2) \right] \leq C_1 \\ C_2 + LMBC_1 (1 + C_2) + LB^2 (1 + C_2)^2 g (C_1, C_2) h (C_2) \leq C_2^{\frac{1}{q+1}}, \end{array} \right. \tag{28}$$

together with the inequality:

$$d = \max \left\{ \tfrac{1}{C_1} \|f (x_0)\|_Y , \tfrac{1}{C_2} \left\| \mathbf{I}_Y - f' (x_0) A_0 \right\| \right\} < 1 \tag{29}$$

and the inclusion relation $\mathcal{S} (x_0, R) \subseteq D$*, where:*

$$R = [2MC_1 + B (1 + C_2) g (C_1, C_2) h (C_2)] \tfrac{d}{1 - d^{\alpha - 1}}, \tag{30}$$

with the number α *has the value:*

$$\min \{2 (p_1 + q_1), ..., 2 (p_m + q_m), p_1 + q_1 + 1, ..., p_m + q_m + 1, q + 1\},$$

then:

j) *the sequences* $(x_n)_{n \in \mathbb{N} \cup \{0\}}$, $(y_n)_{n \in \mathbb{N} \cup \{0\}} \subseteq D$ *and* $(A_n)_{n \in \mathbb{N} \cup \{0\}} \subseteq (Y, X)^*$ *that for any* $n \in \mathbb{N} \cup \{0\}$ *verify the relations* (22) *and the inequalities* (25) *and they are convergent;*

jj) *the equation* (1) *has a solution* $\overline{x} \in \mathcal{S} (x_0, R)$ *and* $\overline{x} = \lim\limits_{n \to \infty} x_n = \lim\limits_{n \to \infty} y_n$*;*

jjj) *the mapping* $\overline{A} = [f' (\overline{x})]^{-1} \in (Y, X)^*$ *exists and* $\overline{A} = \lim\limits_{n \to \infty} A_n$*;*

jv) *for any* $n \in \mathbb{N} \cup \{0\}$ *the following inequalities are true:*

$$\|x_{n+1} - x_n\|_X \leq [MC_1 + B (1 + C_2) g (C_1, C_2) h (C_2)] d^{\alpha^n}, \tag{31}$$

$$\|A_{n+1} - A_n\| \leq B (1 + C_2) \frac{\beta d^{\alpha^n} - \left(\beta d^{\alpha^n}\right)^{q+1}}{1 - \beta d^{\alpha^n}}, \tag{32}$$

where:

$$\beta = C_1 + BLMC_1 (1 + C_2) + B^2 L (1 + C_2)^2 h (C_2) g (C_1, C_2),$$

$$\|\overline{x} - x_n\|_X \leq [MC_1 + B (1 + C_2) g (C_1, C_2) h (C_2)] \frac{d^{\alpha^n}}{1 - d^{\alpha^n (\alpha - 1)}}, \tag{33}$$

$$\left\|\overline{A} - A_n\right\| \leq B (1 + C_2) \sum_{k=1}^{q} \frac{\left(\beta d^{\alpha^n}\right)^k}{1 - d^{k \alpha^n (\alpha - 1)}}, \tag{34}$$

$$\begin{array}{c} \|\overline{x} - y_n\|_X \leq \\ \leq [2MC_1 + B (1 + C_2) g (C_1, C_2) h (C_2)] \frac{d^{\alpha^n}}{1 - d^{\alpha^n (\alpha - 1)}}. \end{array} \tag{35}$$

The proof of this theorem can be found in [9].

2. THE ORDER OF CONVERGENCE SPEED. ITERATIVE METHODS OF THE (??) TYPE THAT ARE OPTIMAL WITH REGARD TO THE ORDER OF CONVERGENCE SPEED

For accuracy we have the following:

Definition 5. *Let us consider the linear normed spaces* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$, *the set* $D \subseteq X$, *the function* $f : D \to Y$, *the sequence* $(x_n)_{n\in\mathbb{N}} \subseteq D$ *and the real number* $\alpha \geq 1$.

a) *We say that the sequence* $(x_n)_{n\in\mathbb{N}} \subseteq D$ *has* ***the order of convergence speed equal with*** α ***with regard to the function*** $f : D \to Y$, *if there exist the constants* $C, M > 0$ *and* $d \in]o, 1[$ *so that for any* $n \in \mathbb{N}$ *we have the inequalities:*

$$\begin{cases} \|f(x_n)\|_Y \leq Cd^{\alpha^n}, \\ \|x_{n+1} - x_n\|_X \leq M \|f(x_n)\|_Y . \end{cases} \tag{36}$$

b) *In addition to the initial data, we further consider a sequence* $(y_n)_{n\in\mathbb{N}}$ $\subseteq D$ *together with a sequence* $(A_n)_{n\in\mathbb{N}} \subseteq (Y, X)^*$ *and we suppose that the function* $f : D \to Y$ *is Fréchet differentiable. We say that the sequence* $(y_n)_{n\in\mathbb{N}}$ *has* ***the order*** $\alpha > 1$ ***with regard to the pair of sequences*** $(x_n)_{n\in\mathbb{N}} \subseteq D, (A_n)_{n\in\mathbb{N}} \subseteq (Y, X)^*$ ***and the function*** $f : D \to Y$, *if there exists a number* $m \in \mathbb{N}$ *and for any* $i = \overline{1, m}$ *the numbers* $K_i, p_i, q_i > 0$ *so that we have* $\alpha = \min\{p_1 + q_1, ..., p_m + q_m\}$, *and the inequalities* (25) *are verified.*

In connection with this definition we have the following:

Remark 6. If the sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ has the order $\alpha > 1$ of the convergence speed with regard to the continuous function $f : D \to Y$, the linear normed space $(X, \|\cdot\|_X)$ is a Banach space and we have the inclusion $\mathcal{S}(x_0, R) \subseteq D$ where $R = \frac{MCd}{1-d^{\alpha-1}}$, then the equation $f(x) = \theta_Y$ has a solution $\overline{x} \in D$ and for any $n \in \mathbb{N}$ the following inequalities are true:

$$\begin{cases} \|x_{n+1} - x_n\|_X \leq MCd^{\alpha^n}, \\ \|\overline{x} - x_n\|_X \leq \frac{MCd^{\alpha^n}}{1-d^{\alpha^n(\alpha-1)}}, \end{cases} \tag{37}$$

where $M, C > 0$ and $d \in]0, 1[$ are the same numbers for which the inequalities (36) are true. $\square$

Indeed, the first inequality from (37) is a direct consequence of the inequalities (36).

Then, for any $n, m \in \mathbb{N}$ we have:

$$\|x_{n+m} - x_n\|_X \leq MC \sum_{i=n}^{n+m-1} d^{\alpha^i} = MCd^{\alpha^n} \sum_{j=0}^{m-1} d^{\alpha^{n-j}-\alpha^n} <$$

$$< MCd^{\alpha^n} \sum_{j=0}^{\infty} \left[d^{\alpha^n(\alpha-1)}\right]^j = \tfrac{MCd^{\alpha^n}}{1-d^{\alpha^n(\alpha-1)}}.$$

As $d < 1$, it is clear that $\lim\limits_{n\to\infty} \frac{MCd^{\alpha^n}}{1-d^{\alpha^n(\alpha-1)}} = 0$, therefore the sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ is a Cauchy sequence in the Banach space $(X, \|\cdot\|_X)$, so there exists $\overline{x} = \lim\limits_{n\to\infty} x_n \in X$ and therefore the second inequality from (37) is verified.

This inequality for $n = 0$ will be written as:

$$\|\overline{x} - x_0\|_X \leq \tfrac{MCd}{1-d^{\alpha-1}} = R.$$

From this it result that $\overline{x} \in \mathcal{S}(x_0, R)$ or $\overline{x} \in D$.

Remark 7. Let us consider the sequences $(x_n)_{n\in\mathbb{N}\cup\{0\}} \subseteq D$ and $(A_n)_{n\in\mathbb{N}\cup\{0\}} \subseteq (Y, X)^*$ together with a differentiable function $f : D \to Y$. We suppose that there exists a number $m \in \mathbb{N}$ and, for any $j = \overline{1, m}$, the numbers $K_j, p_j, q_j > 0$, so that for any $n \in \mathbb{N}$ the following relations are true:

$$\text{(38)} \qquad \begin{cases} \|f(x_{n+1})\|_Y \leq \sum\limits_{j=1}^{m} K_j \|f(x_n)\|_Y^{p_j} \cdot \|\mathbf{I}_Y - f'(x_n) A_n\|^{q_j}, \\ \|x_{n+1} - x_n\|_X \leq L \|f(x_n)\|_Y. \end{cases}$$

If:

i) $(\|\mathbf{I}_Y - f'(x_n) A_n\|)_{n\in\mathbb{N}\cup\{0\}}$ is a decreasing sequence;
ii) $p = \min\{p_1, ..., p_m\} - 1$;
iii) the number $C > 0$ exists that:

$$\begin{cases} \|f(x_0)\|_Y \leq C, \\ \sum\limits_{j=1}^{m} K_j c^{p_j-1} \|\mathbf{I}_Y - f'(x_0) A_0\|^{q_j} < C, \end{cases}$$

then the sequence $(x_n)_{n\in\mathbb{N}\cup\{0\}}$ has the convergence speed order equal to p with regard to the function $f : D \to Y$. □

Indeed, we will show that the first inequality from (36) is verified with $d = \frac{1}{C} \|f(x_0)\|_Y < 1$ and $\alpha = p$.

For $n = 0$ from the definition of d, it results that the respective relation is true.

Supposing this relation true for a certain $n \in \mathbb{N}$, we will deduce that:

$$\|f(x_{n+1})\|_Y \le \sum_{j=1}^{m} K_j C^{p_j} d^{p_j \cdot p^n} \|\mathbf{I}_Y - f'(x_0) A_0\|^{q_j} \le$$
$$\le \left(\sum_{j=1}^{m} K_j C^{p_j} d^{p_j} \|\mathbf{I}_Y - f'(x_0) A_0\|^{q_j} \right) d^{p^{n+1}} \le C d^{p^{n+1}},$$

which justifies the statement from the present remark.

Remark 8. We consider the same data as for the remark 7, the inequalities (38) being verified.
If:

i) there exists a number $r \in \mathbb{N} \cup \{0\}$ so that for any $n \in \mathbb{N} \cup \{0\}$ we have:

$$A_{n+1} = S^{(r+1)}_{f'(x_{n+1})}(A_n),$$

that is:

$$A_{n+1} = A_n \sum_{k=0}^{r} \left(\mathbf{I}_Y - f'(x_{n+1}) A_n\right)^k;$$

ii) $\beta = \min\{p_1 + q_1, ..., p_m + q_m\} > 1$;

iii) the mapping $f' : D \to (X, Y)^*$ verifies Lipschitz's condition namely that there exists the number $L > 0$ so that for any $x, y \in D$ the following inequality is verified:

$$\left\|f'(x) - f'(y)\right\| \le L \|x - y\|_X;$$

iv) $B > 0$ exists so that for any $x \in D$ the mapping $[f'(x)]^{-1} \in (Y, X)^*$ exists and the following inequality is true:

$$\left\| [f'(x)]^{-1} \right\| \le B;$$

v) the numbers $C_1, C_2 > 0$ exist so that:

$$\begin{cases} \sum_{j=1}^{m} K_j C_1^{p_j - 1} C_2^{q_j} \le 1, \\ \left[C_2 + LMBC_1 (1 + C_2) \frac{1 - C_2^{r+1}}{1 - C_2} \right]^{r+1} \le C_2, \\ d = \max\left\{ \frac{1}{C_1} \|f(x_0)\|_Y, \frac{1}{C_2} \|I_Y - f'(x_0) A_0\| \right\} < 1; \end{cases}$$

vi) the inclusion $\mathcal{S}(x_0, R) \subseteq D$ takes place with $R = \frac{MCd}{1 - d^{\alpha - 1}}$, where $\alpha = \min\{p_1 + q_1, ..., p_m + q_m, r + 1\}$,

then the sequence $(x_n)_{n \in \mathbb{N} \cup \{0\}}$ has the convergence speed order equal to α, the optimum choice of the number r being:

$$r = [\min\{p_1 + q_1, ..., p_m + q_m\}] - 1$$

(here $[a]$ represents the integer part of the real number a), a case in which:

$$\alpha = \min\{p_1 + q_1, ..., p_m + q_m\}.$$

□

Indeed, we show that for any $n \in \mathbb{N} \cup \{0\}$ the following relations are true:

$$\left\{\begin{array}{l} x_n \in \mathcal{S}\left(x_0, R\right), \\ \left\|f\left(x_n\right)\right\|_Y \leq C_1 d^{\alpha^n}, \\ \left\|I_Y - f'\left(x_n\right) A_n\right\| \leq C_2 d^{\alpha^n}, \\ \left\|A_n\right\| \leq B\left(1 + C_2\right). \end{array}\right. \tag{39}$$

For $n = 0$ the first relation is evidently true, the first two inequalities result from the definition of d, and the inequality for $\|A_0\|$ result in the same manner as in the proof of the theorem 4.

We suppose that the relations (39) are true for an arbitrary number $n \in \mathbb{N}$. We will prove these relations for $n+1$ replacing n.

The relation $x_{n+1} \in \mathcal{S}\left(x_0, R\right)$ can be shown in the same manner as in the proof of the theorem 4. Afterwards:

$$\left\|f\left(x_{n+1}\right)\right\|_Y \leq \sum_{j=1}^{m} K_j C_1^{p_j} C_2^{q_j} d^{(p_j+q_j)\alpha^n}.$$

For any $j = \overline{1, m}$ it is clear that $p_j + q_j \geq \alpha$ and as $d < 1$ it is clear that $d^{(p_j+q_j)\alpha^n} \leq d^{\alpha^{n+1}}$ and in this way:

$$\left\|f\left(x_{n+1}\right)\right\|_Y \leq \left(\sum_{j=1}^{m} K_j C_1^{p_j-1} C_2^{q_j}\right) C_1 d^{\alpha^{n+1}} \leq C_1 d^{\alpha^{n+1}}.$$

It is clear as well that

$$\begin{aligned} &\left\|\mathbf{I}_Y - f'\left(x_{n+1}\right) \mathbf{S}^{(r+1)}_{f'(x_{n+1})}\left(A_n\right)\right\| \leq \\ &\leq \left(\left\|\mathbf{I}_Y - f'\left(x_n\right) A_n\right\| + LM\left\|f\left(x_n\right)\right\|_Y \left\|A_n\right\|\right)^{r+1} \leq \\ &\leq \left[C_2 + LMBC_1\left(1 + C_2\right)\right]^{r+1} d^{(r+1)\alpha^n} \leq C_2 d^{(r+1)\alpha^n} \leq C_2 d^{\alpha^{n+1}}, \end{aligned}$$

the last inequality being a consequence of the fact that $r + 1 \geq \alpha$.

Finally

$$\begin{aligned} \left\|A_{n+1}\right\| &\leq \left\|\left[f'\left(x_{n+1}\right)\right]^{-1}\right\| \left(1 + \left\|\mathbf{I}_Y - f'\left(x_{n+1}\right) A_{n+1}\right\|\right) \leq \\ &\leq B\left(1 + C_2 d^{\alpha^{n+1}}\right) \leq B\left(1 + C_2\right), \end{aligned}$$

therefore the relations (39) are true with n replaced by $n+1$.

On the basis of the principle of mathematical induction, the relations (39) are true for any $n \in \mathbb{N} \cup \{0\}$.

The inequality $\left\|f\left(x_n\right)\right\|_Y \leq C_1 d^{\alpha^n}$ together with the second inequality from (38) conduct to the conclusion of this remark.

REMARK 9. Let us consider now two sequences $(x_n)_{n\in\mathbb{N}\cup\{0\}}, (y_n)_{n\in\mathbb{N}\cup\{0\}} \subseteq$ $\subseteq D$ together with the sequence $(A_n)_{n\in\mathbb{N}} \subseteq (Y,X)^*$, so that, for any $n \in$ $\in \mathbb{N}\cup\{0\}$, the inequalities (25) are true.
If

i) the sequence $(x_n)_{n\in\mathbb{N}\cup\{0\}}$ has the convergence speed order α with regard to a function $f : D \to Y$;
ii) the sequences of real numbers $(\|f(x_n)\|)_{n\in\mathbb{N}\cup\{0\}}$ and $(\|\mathbf{I}_Y - f'(x_n)A_n\|)_{n\in\mathbb{N}\cup\{0\}}$ are decreasing;
iii) $p = \min\{p_1, ..., p_m\}$ where $p_1, ..., p_m > 0$ are the same values that verify the relations (25),

then the sequence $(y_n)_{n\in\mathbb{N}\cup\{0\}}$ also has the convergence speed order α with regard to the function $f : D \to Y$. $\square$

Indeed, we deduce from the hypotheses the elements $C, M > 0$ and $d \in]0,1[$ exist so that for any $n \in \mathbb{N}\cup\{0\}$ the inequalities (36) are true.
On account of the inequalities (25) we will deduce that:

$$\|f(y_n)\|_Y \le \sum_{j=1}^{m} K_j d^{p_j\alpha^n} \left\|\mathbf{I}_Y - f'(x_0)A_0\right\|^{q_j}.$$

From the definition of the number $p > 1$ it is clear that for any $j = \overline{1,m}$ one has $p_j \ge p$ and because $d < 1$ it results that $d^{p_j} \le d^p = \overline{d} < 1$, therefore:

$$\|f(y_n)\|_Y \le C\overline{d}^{\alpha^n},$$

where

$$C = \sum_{j=1}^{m} K_j \left\|\mathbf{I}_Y - f'(x_0)A_0\right\|^{q_j}.$$

Also

$$\begin{aligned}&\|y_{n+1} - y_n\|_X \le \|y_{n+1} - x_{n+1}\|_X + \|x_{n+1} - x_n\|_X + \|x_n - y_n\|_X \le \\ &\le M\left(\|f(x_{n+1})\|_Y + \|f(x_n)\|_Y\right) + L\|f(x_n)\|_Y \le (2M+L)\|f(x_n)\|_Y = \\ &= M_1\|f(x_n)\|_Y,\end{aligned}$$

(with $M_1 = 2M + L$) from where the statements from the present remark can be concluded.

REMARK 10. Let us consider the same data as in the case of the remark 9, the inequalities (25) being verified.
If

i) there exists a number $r \in \mathbb{N}\cup\{0\}$ so that for any $n \in \mathbb{N}\cup\{0\}$ we have the relation:
$$A_{n+1} = \mathbf{S}^{(r+1)}_{f'(x_{n+1})}(A_n);$$
ii) the number $\gamma = \min\limits_{j=\overline{1,m}}(p_j + q_j)$, where $p_1, ...p_m, q_1, ..., q_m$ are the same from (25), verifies the inequality $\gamma > 1$;

iii) the mapping $f' : D \to (X,Y)^*$ verifies Lipschitz's condition, namely there exists a number $L > 0$ so that for any $x, y \in D$ the following inequality is true:

$$\left\| f'(x) - f'(y) \right\| \le L \left\| x - y \right\|_X ;$$

iv) there exists a number $B > 0$ so that for any $x \in D$ the mapping $[f'(x)]^{-1} \in (Y,X)^*$ exists and the inequality:

$$\left\| \left[f'(x) \right]^{-1} \right\| \le B$$

is true;

v) the sequence $(x_n)_{n \in \mathbb{N} \cup \{0\}} \subseteq D$ has the convergence speed order equal to $\alpha \in \left]1, r+1\right[$ with regard to the function $f : D \to Y$, the inequalities (36) being true with $C, M > 0$ and $d \in \left]0,1\right[$;

vi) a real number $\overline{C}$ exists so that:

$$\left\| \mathbf{I}_Y - f'(x_0) A_0 \right\| < \overline{C}, \;\; \left[\overline{C} + LBMC \left(1 + \overline{C} \right) \right]^{r+1} \le \overline{C},$$

then the sequence $(y_n)_{n \in \mathbb{N} \cup \{0\}}$ also has the convergence speed order α with regard to the function $f : D \to Y$. $\square$

Indeed, from the hypotheses on the sequence $(x_n)_{n \in \mathbb{N} \cup \{0\}}$ we deduce that the numbers $C, M > 0$ and $d \in \left]0,1\right[$ exist so that for any $n \in \mathbb{N} \cup \{0\}$ the following inequalities are true

$$\begin{cases} \left\| f(x_n) \right\|_Y \le C d^{\alpha^n}, \\ \left\| x_{n+1} - x_n \right\|_X \le M \left\| f(x_n) \right\|_Y . \end{cases}$$

Let us consider the number

$$\delta = \max \left\{ d, \tfrac{1}{\overline{C}} \left\| I_Y - f'(x_0) A_0 \right\| \right\}$$

and evidently $\delta < 1$.

We will show that, for any $n \in \mathbb{N}$, we have the inequalities:

$$\left\| \mathbf{I}_Y - f'(x_n) A_n \right\| \le \overline{C} \delta^{\alpha^n}; \;\; \left\| A_n \right\| \le B \left(1 + \overline{C} \right) \tag{40}$$

For $n = 0$ the first inequality is a consequence of the definition of δ, while the second will be shown in the same manner as in the case of the remark 8.

We suppose that the the inequalities (40) are true for a certain number $n \in \mathbb{N}$ and then we prove them for $n+1$ replacing n.

It is clear that

$$\begin{aligned} &\left\| \mathbf{I}_Y - f'(x_{n+1}) A_{n+1} \right\| = \\ &= \left\| \mathbf{I}_Y - f'(x_{n+1}) \mathbf{S}^{(r+1)}_{f'(x_{n+1})} (A_n) \right\| \le \left\| \mathbf{I}_Y - f'(x_{n+1}) A_n \right\|^{r+1} \le \\ &\le \left[\left\| \mathbf{I}_Y - f'(x_n) A_n \right\| + \left\| f'(x_n) - f'(x_{n+1}) \right\| \cdot \left\| A_n \right\| \right] \le \\ &\le \left[\overline{C} \delta^{\alpha^n} + L \left\| x_{n+1} - x_n \right\|_X \cdot \left\| A_n \right\| \right]^{r+1} . \end{aligned}$$

Also

$$\|x_{n+1}-x_n\|_X \leq M\,\|f(x_n)\|_Y \leq MCd^{\alpha^n} \leq MC\delta^{\alpha^n},$$

therefore:

$$\left\|\mathbf{I}_Y - f'(x_{n+1})\,A_{n+1}\right\| \leq \left[\overline{C} + LMBC\left(1+\overline{C}\right)\right]\delta^{(r+1)\alpha^n} \leq \overline{C}\delta^{(r+1)\alpha^n} \leq$$
$$\leq \overline{C}\delta^{\alpha^{n+1}}.$$

Therefore

$$\|f(y_n)\|_Y \leq \sum_{j=1}^{m} K_j\,\|f(x_n)\|_Y^{p_j}\cdot\left\|\mathbf{I}_Y - f'(x_n)\,A_n\right\|^{q_j} \leq$$
$$\leq \sum_{j=1}^{m} K_j C^{p_j}\overline{C}^{q_j}\delta^{(p_j+q_j)\alpha^n} \leq \left(\sum_{j=1}^{m} K_j C^{p_j}\overline{C}^{q_j}\right)\delta^{\gamma\alpha^n}.$$

We will note by $Q = \sum\limits_{j=1}^{m} K_j C^{p_j}\overline{C}^{q_j}$ and $\Delta = \delta^{\gamma}$; it is clear that we have the numbers $Q > 0$ and $\Delta \in \,]0,1[$ and:

$$\|f(y_n)\|_Y \leq Q\Delta^{\alpha^n}.$$

We also have the inequality:

$$\|y_{n+1}-y_n\|_X \leq M_1\,\|f(x_n)\|_Y$$

similar to the case of the remark 9.

Therefore the sequence $(y_n)_{n\in\mathbb{N}\cup\{0\}}$ has the convergence speed order α with regard to the function $f : D \to Y$.

3. SPECIAL CASES

Let us consider the case in which we choose the sequence $(y_n)_{n\in\mathbb{N}\cup\{0\}} \subseteq X$ defined through:

$$y_n = x_n - \mathbf{S}^{(p+1)}_{f'(x_n)}(A_n)\,f(x_n) \tag{41}$$

with a certain $p \in \mathbb{N}$. In this case the relations of recurrence (22) become:

$$\begin{cases} x_{n+1} = & x_n - \mathbf{S}^{(p+1)}_{f'(x_n)}(A_n)\,f(x_n) - \\ & -\mathbf{S}^{(r+1)}_{f'(x_n)}(A_n)\,f\left(x_n - \mathbf{S}^{(p+1)}_{f'(x_n)}(A_n)\,f(x_n)\right), \\ A_{n+1} = & \mathbf{S}^{(q+1)}_{f'(x_{n+1})}(A_n)\,. \end{cases} \tag{42}$$

In this case the function $g : [0,+\infty[\, \times [0,+\infty[\, \to \mathbb{R}$ is defined through:

$$g(u,v) = \tfrac{LB^2}{2}u^2(1+v)^2\sum_{k=0}^{p} v^k + uv^{p+1} \tag{43}$$

where $L, B > 0$ are the constants that result from the verification of the hypothesis I) also needed in this case.

The convergence of the pair of sequences $(x_n)_{n\in\mathbb{N}\cup\{0\}} \subseteq D$ and $(A_n)_{n\in\mathbb{N}\cup\{0\}} \subseteq (Y,X)^*$ is expressed through the following:

Theorem 11. *If the following statements are true:*

i) *the linear normed space* $(X, \|\cdot\|_X)$ *is a Banach space and with regard to the function* $f : D \to Y$ *the hypothesis* I) *is true;*

ii) *the numbers* $C_1, C_2 > 0$, $r \in \mathbb{N}\cup\{0\}$ *and* $q \in \mathbb{N}$ *exist and the system* (28), *of the inequality* (29) *together with the relation of inclusion* $\mathcal{S}(x_0, R) \subseteq D$, *where* R *is expressed through the equality* (30) *and* $\alpha = \min\{3, q+1\}$ *are verified;*

then:

j) *if the sequence* $(y_n)_{n\in\mathbb{N}\cup\{0\}}$ *is defined through the relation* (41), *then for any* $n \in \mathbb{N}\cup\{0\}$ *the following relation, of the same type as the relation* (25), *are true:*

$$\left\{\begin{array}{ll}\|f(y_n)\|_Y \leq & \frac{LB^2}{2}(1+\delta_n)^2\|f(x_n)\|_Y^2 \sum\limits_{k=0}^{p}\delta_n^k + \\ & +\delta_n^{p+1}\cdot\|f(x_n)\|_Y\,, \\ \|y_n - x_n\|_X \leq & B(1+C_2)\,h(C_2)\,\|f(x_n)\|_Y\,,\end{array}\right. \tag{44}$$

where $\delta_n = \|\mathbf{I}_Y - f'(x_n)A_n\|$;

jj) *the conclusions* j)-jv) *of the theorem* 4 *with* $g(u,v)$ *defined through* (42) *and* $M = B(1+C_2)\,h(C_2)$ *are true.*

Proof. We apply the theorem 4 in the case in which the sequence $(y_n)_{n\in\mathbb{N}\cup\{0\}}$ is generated by the relation (41). For this it is enough sufficient to show that for any $n \in \mathbb{N}\cup\{0\}$ the relations (44) are true.

Through mathematical induction we will show that for any $n \in \mathbb{N}\cup\{0\}$ the propositions a)-c) from the proof of the theorem 4 followed by the inequalities (44) are true.

The relations a)-c) for a certain number $n \in \mathbb{N}\cup\{0\}$ are proved in the same manner as in the proof of the theorem 4.

Afterwards for the same number $n \in \mathbb{N}\cup\{0\}$ we have:

$$\|y_n - x_n\|_X \leq \left\|\mathbf{S}^{(p+1)}_{f'(x_n)}(A_n)\right\| \cdot \|f(x_n)\|_Y\,.$$

But:

$$\left\|\mathbf{S}^{(p+1)}_{f'(x_n)}(A_n)\right\| \leq \|A_n\| \sum_{k=0}^{p} \|I_Y - f'(x_n)A_n\|^k$$

and:

$$\begin{aligned}\|A_n\| &= \left\|[f'(x_n)]^{-1} + A_n - [f'(x_n)]^{-1}\right\| \leq \\ &\leq \left\|[f'(x_n)]^{-1}\right\|(1 + \|I_Y - f'(x_n)A_n\|) \leq B(1 + \|I_Y - f'(x_n)A_n\|)\,.\end{aligned}$$

From here:

$$\begin{aligned}&\left\|f\left(y_n\right)\right\|_Y \leq \\ &\leq \left\|f\left(y_n\right)-f\left(x_n\right)-f'\left(x_n\right)\left(y_n-x_n\right)\right\|_Y+\left\|f\left(x_n\right)+f'\left(x_n\right)\left(y_n-x_n\right)\right\|_Y,\end{aligned}$$

for which:

$$\begin{aligned}&\left\|f\left(y_n\right)-f\left(x_n\right)-f'\left(x_n\right)\left(y_n-x_n\right)\right\|_Y \leq \tfrac{L}{2}\left\|y_n-x_n\right\|_X^2 \leq \\ &\leq \tfrac{LB^2}{2}\left(1+\left\|I_Y-f'\left(x_n\right)A_n\right\|\right)^2\left\|f\left(x_n\right)\right\|_Y^2 \cdot \sum_{k=1}^{p}\left\|I_Y-f'\left(x_n\right)A_n\right\|^k,\end{aligned}$$

and:

$$\begin{aligned}&\left\|f\left(x_n\right)+f'\left(x_n\right)\left(y_n-x_n\right)\right\|_Y \leq \left\|I_Y-S_{f'(x_n)}^{(p+1)}\left(A_n\right)\right\| \cdot \left\|f\left(x_n\right)\right\|_Y+ \\ &\quad+\left\|I_Y-f'\left(x_n\right)A_n\right\|^{p+1} \cdot \left\|f\left(x_n\right)\right\|_Y.\end{aligned}$$

In this way:

$$\begin{aligned}&\left\|f\left(y_n\right)\right\|_Y \leq \\ &\leq \tfrac{LB^2}{2}\left(1+\left\|I_Y-f'\left(x_n\right)A_n\right\|\right)^2\left\|f\left(x_n\right)\right\|_Y^2 \cdot \sum_{k=1}^{p}\left\|I_Y-f'\left(x_n\right)A_n\right\|^k+ \\ &\quad+\left\|I_Y-f'\left(x_n\right)A_n\right\|^{p+1} \cdot \left\|f\left(x_n\right)\right\|_Y.\end{aligned}$$

For the finishing of the second inequality from (44) we have:

$$\left\|y_n-x_n\right\|_X \leq \left[B\left(1+C_2\right)\sum_{k=0}^{p}C_2^k\right]\left\|f\left(x_n\right)\right\|_Y = B\left(1+C_2\right)h\left(C_2\right)\left\|f\left(x_n\right)\right\|_Y.$$

The theorem is thus proven. □

Remark 12. From the conclusions of the theorem 4 it results that the convergence speed of the sequence $(x_n)_{n\in\mathbb{N}\cup\{0\}}$ does not depend on r and p. So the order of this speed is $\alpha=3$ for any $q\geq 2$. In this way, for the simplicity of the calculation, the most efficient methods are the ones for which $r=p=0$ and $q=2$. The relations (42) will become:

$$\left\{\begin{array}{l} x_{n+1}=x_n-A_n f\left(x_n\right)-A_n f\left(x_n-A_n f\left(x_n\right)\right), \\ A_{n+1}=A_n\left[3I_Y-3f'\left(x_{n+1}\right)A_n+\left(f'\left(x_{n+1}\right)A_n\right)^2\right]. \end{array}\right. \tag{45}$$

□

If we introduce this values in the expression of $g(u,v)$ and $h(v)$ we will obtain:

$$g(u,v)=\tfrac{LB^2}{2}u^2(1+v)^2+uv, \quad h(v)=1.$$

The pair $(C_1, C_2) \in [0, +\infty[\times [0, +\infty[$ must be a solution of the system in u and v :

$$\begin{cases} z\left[z\left(z-v\right)+w\right] \leq 1 \\ \left[w+z\left(2z-v\right)\right]^3 \leq v \end{cases}$$

where $z = \frac{LB^2}{2}u\left(1+v\right)^2 + v$ and $w = v + LMBu\left(1+v\right)$.

REFERENCES

[1] Diaconu, A., *Sur quelques méthodes itératives combinées (I)*, Mathematica, **22 (45)**, 2, pp. 247–261, 1980.

[2] Diaconu, A., *Sur la manière d'obtenir et sur la convergence de certaines méthdes itératives*, "Babeş-Bolyai" University, Faculty of Mathematics, Research Seminaries, Seminar of Functional Analysis and Numerical Methods, Preprint No. 1, pp. 25–74, 1987.

[3] Diaconu, A., *Sur quelques procédés itératifs de type Aitken-Steffensen*, "Babeş-Bolyai" University, Faculty of Mathematics, Research Seminars, Seminar on Mathematical Analysis Preprint No. 7, pp. 131–150, 1988.

[4] Diaconu, A., *Remarks on the convergence of some iterative methods of the Traub type*, Studia Universitatis "Babeş-Bolyai", Mathematica, **52**, 2, pp. 47–60, 1997.

[5] Diaconu, A., *The approximation of the equation's solution in linear normed spaces using approximant sequences (I)*, Rev. d'Anal. Numérique et de Théorie d'Approx., **32(1)**, pp. 21–30, 2003.

[6] Diaconu, A., *The approximation of the equation's solution in linear normed spaces using approximant sequences (II),* Studia Univ. "Babes-Bolyai", Mathematica, Volume **XLVIII**, March, pp. 55–74, 2003.

[7] Diaconu, A., *The acceleration of the convergence of certain approximant sequences*, Pure Mathematics and Applications, **11**, 2, pp. 187–207, 2000.

[8] Diaconu, A., *The acceleration of the convergence of certain approximant sequences of the solutions of certain equations in normed linear spaces*, Revue d'Analyse Numérique et de Théorie d'Approximation, **29** (1), pp. 23–41, 2000.

[9] Diaconu, A., *On the acceleration of the convergence of certaian iterative proceedings*, in press.

[10] Kantorovich, L. V., *Functional Analysis and Applied mathematics*, U.M.N. III, **5**, (28), pp. 29–185, 1948 (in Russian).

[11] Kantorovich, L. V., *Functional analysis*, Editura Ştiinţifică şi Enciclopedică, Bucureşti, 1986 (in Romanian).

[12] Păvăloiu, I., *Sur les procédés iteratifs á un ordre élevé de convergence*, Mathematica, Cluj, **12** (35), 2, pp. 309–324, 1970.

[13] Păvăloiu, I., *Introducere în teoria aproximării soluţiilor ecuaţiilor*, Editura Dacia, Cluj-Napoca, 1976 (in Romanian).

[14] Ul'm, S., *Ob oteraţionîh mettodah s posledovatel'noi approksimacii obratnovo operatora*, Izv. Acad. Nauk Estonskoi S.S.R. **16**, 4, pp. 403–411, 1967.

Received by the editors: January 26, 2006.