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POSITIVE SEMI-DEFINITE MATRICES, EXPONENTIAL CONVEXITY FOR MULTIPLICATIVE MAJORIZATION AND RELATED MEANS OF CAUCHY'S TYPE

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Abstract. In this paper, we obtain new results concerning the generalizations of additive and multiplicative majorizations by means of exponential convexity. We prove positive semi-definiteness of matrices generated by differences deduced from majorization type results which implies exponential convexity and log-convexity of these differences and also obtain Lyapunov's and Dresher's inequalities for these differences. We give some applications of additive and multiplicative majorizations. In addition, we introduce new means of Cauchy's type and establish their monotonicity.

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1. INTRODUCTION AND PRELIMINARIES

For fixed $n \geq 2$ let

$$\mathbf{x} = (x_1, ..., x_n), \quad \mathbf{y} = (y_1, ..., y_n)$$

denote two n-tuples. Let

$$x_{[1]} \geq x_{[2]} \geq ... \geq x_{[n]}, \quad y_{[1]} \geq y_{[2]} \geq ... \geq y_{[n]},$$
$$x_{(1)} \leq x_{(2)} \leq ... \leq x_{(n)}, \quad y_{(1)} \leq y_{(2)} \leq ... \leq y_{(n)}$$

be their ordered components.

Definition 1.1. (cf. [10], p. 319) *$\boldsymbol{y}$ is said to majorize $\boldsymbol{x}$ (or $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$), in symbol, $\boldsymbol{y} \succ \boldsymbol{x}$, if*

$$\sum_{i=1}^{m} x_{[i]} \leq \sum_{i=1}^{m} y_{[i]} \tag{1.1}$$

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holds for $m = 1, 2, ..., n-1$ *and*

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i. \tag{1.2}$$

Note that (1.1) *is equivalent to*

$$\sum_{i=n-m+1}^{n} x_{(i)} \leq \sum_{i=n-m+1}^{n} y_{(i)}$$

for $m = 1, 2, ..., n-1$.

Parallel to the concept of additive majorization is the notion of multiplicative majorization (also termed log-majorization).

Definition 1.2. *Let* $\boldsymbol{x}$, $\boldsymbol{y}$ *be two positive* n*-tuples,* $\boldsymbol{y}$ *is said to be multiplicatively majorized by* $\boldsymbol{x}$*, denoted by* $\boldsymbol{x} \prec_{\times} \boldsymbol{y}$ *if*

$$\prod_{i=1}^{m} x_{[i]} \leq \prod_{i=1}^{m} y_{[i]} \tag{1.3}$$

holds for $m = 1, 2, ..., n-1$ *and*

$$\prod_{i=1}^{n} x_i = \prod_{i=1}^{n} y_i. \tag{1.4}$$

Note that (1.3) *is equivalent to*

$$\prod_{i=n-m+1}^{n} x_{(i)} \leq \prod_{i=n-m+1}^{n} y_{(i)}$$

holds for $m = 1, 2, ..., n-1$.

To differentiate the two types of majorization, we sometimes use the symbol $\prec_{+}$ rather than $\prec$ to denote (additive) majorization.

The following theorem is well-known as the majorization theorem and a convenient reference for its proof is in the book of Marshall and Olkin (1979) ([6], p.11) (see [10], p.320):

Theorem 1.3. *Let* I *be an interval in* $\mathbb{R}$ *and* $\boldsymbol{x}$, $\boldsymbol{y}$ *be two* n*-tuples such that* $x_i, y_i \in I$ $(i = 1, ..., n)$*. Then*

$$\sum_{i=1}^{n} \phi(x_i) \leq \sum_{i=1}^{n} \phi(y_i) \tag{1.5}$$

holds for every continuous convex function $\phi : I \to \mathbb{R}$ *iff* $\boldsymbol{y} \succ \boldsymbol{x}$ *holds.*

Remark 1.4. [5] If $\phi(x)$ is a strictly convex function then equality in (1.5) is valid iff $x_{[i]} = y_{[i]}$, $i = 1, ..., n$. □

The following theorem can be regarded as a generalization of the majorization theorem and is proved by Fuchs (1947) in [4] (see [10], p. 323):

THEOREM 1.5. *Let $\boldsymbol{x}$, $\boldsymbol{y}$ be two decreasing n-tuples and $\boldsymbol{p} = (p_1, ..., p_n)$ be a real n-tuple such that*

$$\sum_{i=1}^{k} p_i\, x_i \;\le\; \sum_{i=1}^{k} p_i\, y_i \;\; for \;\; k = 1, ..., n-1; \tag{1.6}$$

and

$$\sum_{i=1}^{n} p_i\, x_i \;=\; \sum_{i=1}^{n} p_i\, y_i. \tag{1.7}$$

Then for every continuous convex function $\phi : I \to \mathbb{R}$, we have

$$\sum_{i=1}^{n} p_i\, \phi(x_i) \;\le\; \sum_{i=1}^{n} p_i\, \phi(y_i). \tag{1.8}$$

Let $x(\tau)$, $y(\tau)$ be two real-valued functions defined on an interval $[a,b]$ such that $\int_a^s x(\tau)\mathrm{d}\tau$, $\int_a^s y(\tau)\mathrm{d}\tau$ both exist for all s $\in [a,b]$.

DEFINITION 1.6. (cf. [10], p. 324) *$y(\tau)$ is said to majorize $x(\tau)$, in symbol, $y(\tau) \succ x(\tau)$, for $\tau \in [a,b]$ if they are decreasing in $\tau \in [a,b]$ and*

$$\int_a^s x(\tau)\,\mathrm{d}\tau \le \int_a^s y(\tau)\,\mathrm{d}\tau \quad for \;\; s \in [a,b], \tag{1.9}$$

and equality in (1.9) *holds for $s = b$.*

The following theorem can be regarded as majorization theorem in integral case (see [10], p. 325):

THEOREM 1.7. *$y(\tau) \succ x(\tau)$ for $\tau \in [a,b]$ iff they are decreasing in $[a,b]$ and*

$$\int_a^b \phi(x(\tau))\,\mathrm{d}\tau \;\le\; \int_a^b \phi(y(\tau))\,\mathrm{d}\tau \tag{1.10}$$

holds for every ϕ that is continuous and convex in $[a,b]$ such that the integrals exist.

The following theorem is a simple consequence of Theorem 12.14 in [11] (see [10], p. 328):

THEOREM 1.8. *Let $x(\tau)$, $y(\tau) : [a,b] \to \mathbb{R}$, $x(\tau)$ and $y(\tau)$ are continuous and increasing and let $G : [a,b] \to \mathbb{R}$ be a function of bounded variation.*

(a) *If*

$$\int_\nu^b x(\tau)\,\mathrm{d}G(\tau) \;\le\; \int_\nu^b y(\tau)\,\mathrm{d}G(\tau) \;\; for \; all \;\; \nu \in [a,b], \tag{1.11}$$

and

$$\int_a^b x(\tau)\,\mathrm{d}G(\tau) \;=\; \int_a^b y(\tau)\,\mathrm{d}G(\tau) \tag{1.12}$$

hold then for every continuous convex function f, we have

$$\int_a^b f(x(\tau))\,\mathrm{d}G(\tau) \le \int_a^b f(y(\tau))\,\mathrm{d}G(\tau). \tag{1.13}$$

(b) *If* (1.11) *holds then* (1.13) *holds for every continuous increasing convex function f.*

Let $F(\tau)$, $G(\tau)$ be two continuous and increasing functions for $\tau \ge 0$ such that $F(0) = G(0) = 0$ and define

$$\overline{F}(\tau) = 1 - F(\tau), \qquad \overline{G}(\tau) = 1 - G(\tau) \quad \textit{for} \quad \tau \ge 0. \tag{1.14}$$

Definition 1.9. (cf. [10], p. 330) *$\overline{F}(\tau)$ is said to majorize $\overline{G}(\tau)$, in symbol, $\overline{F}(\tau) \succ \overline{G}(\tau)$, for $\tau \in [0, +\infty)$ if*

$$\int_0^s \overline{G}(\tau)\,\mathrm{d}\tau \le \int_0^s \overline{F}(\tau)\,\mathrm{d}\tau \qquad \textit{for all} \ \ s > 0, \tag{1.15}$$

and

$$\int_0^\infty \overline{G}(\tau)\,\mathrm{d}\tau = \int_0^\infty \overline{F}(\tau)\,\mathrm{d}\tau < \infty. \tag{1.16}$$

The following result was obtained by Boland and Proschan (1986) [3] (see [10], p. 331):

Theorem 1.10. *$\overline{F}(\tau) \succ \overline{G}(\tau)$ for $\tau \in [0, +\infty)$ holds iff*

$$\int_0^\infty \phi(\tau)\,\mathrm{d}F(\tau) \le \int_0^\infty \phi(\tau)\,\mathrm{d}G(\tau) \tag{1.17}$$

holds for all convex functions ϕ, provided the integrals are finite.

Definition 1.11. *A function $h : (a, b) \to \mathbb{R}$ is exponentially convex function if it is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \, h\,(x_i + x_j) \ge 0$$

for all $n \in \mathbb{N}$ and all choices $\xi_i \in \mathbb{R}$ and $x_i \in (a, b)$, $i = 1, ..., n$ such that $x_i + x_j \in (a, b)$, $1 \le i, j \le n$.

The following proposition is given in [2]:

Proposition 1.12. *Let $h : (a, b) \to \mathbb{R}$. The following propositions are equivalent.*

(i) *h is exponentially convex.*

(ii) *h is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \, h\left(\tfrac{x_i + x_j}{2}\right) \ge 0,$$

for every $n \in \mathbb{N}$, every $\xi_i \in \mathbb{R}$ and every $x_i, x_j \in (a, b)$, $1 \le i, j \le n$.

COROLLARY 1.13. *If h is exponentially convex then*

$$det\left[h\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n} \geq 0,$$

for every $n \in \mathbb{N}$ and every $x_i \in (a,b)$, $i = 1,...,n$.

COROLLARY 1.14. *If $h : (a,b) \to \mathbb{R}^+$ is exponentially convex function then h is a* log*-convex function.*

The following lemma is equivalent to definition of convex function (see [10], p. 2):

LEMMA 1.15. *If f is convex on an interval $I \subseteq \mathbb{R}$, then*

$$f(s_1)(s_3 - s_2) + f(s_2)(s_1 - s_3) + f(s_3)(s_2 - s_1) \geq 0,$$

holds for every $s_1 < s_2 < s_3$, $s_1, s_2, s_3 \in I$.

In [1], the following result is proved:

THEOREM 1.16. *Let $\mathbf{x}$ and $\mathbf{y}$ be two positive n-tuples, $\mathbf{y} \succ \mathbf{x}$,*

$$\Lambda_t = \Lambda_t(\mathbf{x};\mathbf{y}) := \sum_{i=1}^{n} \varphi_t(y_i) - \sum_{i=1}^{n} \varphi_t(x_i),$$

and all $x_{[i]}$'s and $y_{[i]}$'s are not equal.

Then the following statements are valid:

(a) *For every $n \in \mathbb{N}$ and $s_1,...,s_n \in \mathbb{R}$, the matrix $\left[\Lambda_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{n}$ is a positive semi-definite matrix. Particularly*

$$\det\left[\Lambda_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{k} \geq 0 \tag{1.18}$$

for $k = 1,...,n$.

(b) *The function $s \to \Lambda_s$ is exponentially convex.*

(c) *The function $s \to \Lambda_s$ is a* log*-convex on $\mathbb{R}$ and the following inequality holds for $-\infty < r < s < t < \infty$:*

$$\Lambda_s^{t-r} \leq \Lambda_r^{t-s}\,\Lambda_t^{s-r}. \tag{1.19}$$

Similar results and corresponding Cauchy means are proved in [1] with a stronger condition that $\mathbf{x}$ and $\mathbf{y}$ are positive n-tuples.

In this paper we give results for generalizations of additive majorization as in [1] and multiplicative majorization. Moreover, several applications of majorization are obtained by using following important example

$$\Big(\sum_{i=1}^{n} x_i, 0, ..., 0\Big) \succ (x_1, ..., x_n).$$

We also give some applications of additive and multiplicative majorizations. In this connection, the following remark in [7] is important:

"majorization theory is the underlying mathematical theory on which the framework hings. It allows the transformation of the originally complicated matrix-valued non-convex problem into a simple scalar problem."

It was shown in [7] that additive majorization relation plays a key role in the design of linear MIMO transceivers, whereas the multiplicative majorization relation is the basis for nonlinear decision-feedback MIMO transceivers.

2. THE CASE OF NON-NEGATIVE SEQUENCES AND FUNCTIONS

Lemma 2.1. *Define the function*

$$\overline{\varphi}_s(x) := \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 1; \\ x \log x, & s = 1, \end{cases} \tag{2.1}$$

where $s \in \mathbb{R}^+$.
Then $\overline{\varphi}''_s(x) = x^{s-2}$, *that is,* $\overline{\varphi}_s(x)$ *is convex for* $x > 0$.

In our results we use the notation $0 \log 0 = 0$.

Theorem 2.2. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two non-negative n-tuples,* $\boldsymbol{y} \succ \boldsymbol{x}$,

$$\overline{\Lambda}_t = \overline{\Lambda}_t(\mathbf{x};\, \mathbf{y}) := \sum_{i=1}^{n} \overline{\varphi}_t(y_i) - \sum_{i=1}^{n} \overline{\varphi}_t(x_i),$$

and all $x_{[i]}$*'s and* $y_{[i]}$*'s are not equal*
Then the following statements are valid:

(a) *For every* $n \in \mathbb{N}$ *and* $s_1, ..., s_n \in \mathbb{R}^+$, *the matrix* $\left[\overline{\Lambda}_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{n}$ *is a positive semi-definite matrix. Particularly*

$$\det \left[\overline{\Lambda}_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{k} \geq 0 \tag{2.2}$$

for $k = 1, ..., n$.

(b) *The function* $s \to \overline{\Lambda}_s$ *is exponentially convex.*

(c) *The function* $s \to \overline{\Lambda}_s$ *is a* log-*convex on* $\mathbb{R}^+$ *and the following inequality holds for* $0 < r < s < t < \infty$:

$$\left(\overline{\Lambda}_s\right)^{t-r} \leq \left(\overline{\Lambda}_r\right)^{t-s} \left(\overline{\Lambda}_t\right)^{s-r}. \tag{2.3}$$

Proof. (a) Consider the function

$$\mu(x) = \sum_{i,j}^{k} u_i u_j \, \varphi_{s_{ij}}(x)$$

for $k = 1, ..., n$, $x > 0$, $u_i \in \mathbb{R}$, $s_{ij} \in \mathbb{R}^+$, where $s_{ij} = \frac{s_i + s_j}{2}$ and $\varphi_{s_{ij}}$ is defined in (2.1).

We have

$$\mu''(x) = \sum_{i,j}^{k} u_i u_j \, x^{s_{ij}-2} = $$
$$= \left(\sum_{i}^{k} u_i \, x^{\frac{s_i}{2}-1} \right)^2 \geq 0, \quad x \geq 0.$$

This shows that μ is a convex function for $x \geq 0$.

Using Theorem 1.3,

$$\sum_{m=1}^{n} \mu(y_m) - \sum_{m=1}^{n} \mu(x_m) \geq 0.$$

This implies

$$\sum_{m=1}^{n} \left(\sum_{i,j}^{k} u_i u_j \, \overline{\varphi}_{s_{ij}}(y_m) \right) - \sum_{m=1}^{n} \left(\sum_{i,j}^{k} u_i u_j \, \overline{\varphi}_{s_{ij}}(x_m) \right) \geq 0,$$

or equivalently

$$\sum_{i,j}^{k} u_i u_j \, \overline{\Lambda}_{s_{ij}} \geq 0.$$

From last inequality, it follows that the matrix $\left[\overline{\Lambda}_{\frac{s_i+s_j}{2}} \right]_{i,j=1}^{n}$ is a positive semi-definite matrix, that is, (2.2) is valid.

(b) Note that $\overline{\Lambda}_s$ is continuous for $s \in \mathbb{R}^+$. Then by using Proposition 1.12, we get exponentially convexity of the function $s \to \overline{\Lambda}_s$.

(c) Since $\overline{\varphi}_t(x)$ is continuous and strictly convex function for $x > 0$ and all $x_{[i]}$'s and $y_{[i]}$'s are not equal, therefore by Theorem 1.3 with $\phi = \overline{\varphi}_t$, we have

$$\sum_{i=1}^{n} \overline{\varphi}_t(y_i) > \sum_{i=1}^{n} \overline{\varphi}_t(x_i).$$

This implies

$$\overline{\Lambda}_t = \overline{\Lambda}_t(\mathbf{x};\mathbf{y}) = \sum_{i=1}^{n} \overline{\varphi}_t(y_i) - \sum_{i=1}^{n} \overline{\varphi}_t(x_i) > 0,$$

that is, $\overline{\Lambda}_t$ is positive-valued function.

A simple consequence of Corollary 1.14 is that $\overline{\Lambda}_s$ is log-convex, then by definition

$$\log \left(\overline{\Lambda}_s \right)^{t-r} \leq \log \left(\overline{\Lambda}_r \right)^{t-s} + \log \left(\overline{\Lambda}_t \right)^{s-r},$$

which is equivalent to (2.3). □

As in [1], we define the following means of Cauchy type.

$$M_{t,s} = \left(\frac{\overline{\Lambda}_t}{\overline{\Lambda}_s}\right)^{\frac{1}{t-s}}, \qquad t,\, s \in \mathbb{R}^+, \;\; s \neq t. \tag{2.4}$$

$$M_{s,s} = \exp\left(\frac{\sum_{i=1}^n {y_i}^s \,\log y_i - \sum_{i=1}^n {x_i}^s \,\log x_i}{\sum_{i=1}^n {y_i}^s - \sum_{i=1}^n {x_i}^s} - \frac{2s-1}{s(s-1)}\right), \;\; s \neq 1.$$

$$M_{1,1} = \exp\left(\frac{\sum_{i=1}^n y_i\,(\log y_i)^2 - \sum_{i=1}^n x_i\,(\log x_i)^2}{2\left(\sum_{i=1}^n y_i\,\log y_i - \sum_{i=1}^n x_i\,\log x_i\right)} - 1\right).$$

THEOREM 2.3. *Let $t, s, u, v \in \mathbb{R}^+$ such that $t \le u,\ s \le v$, then the following inequality is valid*

$$M_{t,s} \le M_{u,v}. \tag{2.5}$$

Proof. Since $\overline{\Lambda}_t$ is log-convex, therefore by (2.4) we get (2.5). □

THEOREM 2.4. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two non-negative decreasing n-tuples, $\boldsymbol{p} = (p_1, ..., p_n)$ be a real n-tuple and let*

$$\overline{\lambda}_t = \overline{\lambda}_t(\mathbf{x}, \mathbf{y}; \mathbf{p}) := \sum_{i=1}^n p_i\,\overline{\varphi}_t(y_i) - \sum_{i=1}^n p_i\,\overline{\varphi}_t(x_i),$$

such that conditions (1.6) *and* (1.7) *are satisfied and $\overline{\lambda}_t$ is positive. Then the following statements are valid:*

(a) *For every $n \in \mathbb{N}$ and $s_1, ..., s_n \in \mathbb{R}^+$, the matrix $\left[\overline{\lambda}_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^n$ is a positive semi-definite matrix. Particularly*

$$\det\left[\overline{\lambda}_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^k \ge 0 \tag{2.6}$$

for $k = 1, ..., n$.

(b) *The function $s \to \overline{\lambda}_s$ is exponentially convex.*

(c) *The function $s \to \overline{\lambda}_s$ is a* log*-convex on $\mathbb{R}^+$ and the following inequality holds for $0 < r < s < t < \infty$:*

$$\left(\overline{\lambda}_s\right)^{t-r} \le \left(\overline{\lambda}_r\right)^{t-s} \left(\overline{\lambda}_t\right)^{s-r}. \tag{2.7}$$

Proof. As in the proof of Theorem 2.2, we use Theorem 1.5 instead of Theorem 1.3. □

As in [1], we define the following means of Cauchy type.

$$\widetilde{M}_{t,s} = \left(\frac{\overline{\lambda}_t}{\overline{\lambda}_s}\right)^{\frac{1}{t-s}}, \qquad t,\, s \in \mathbb{R}^+, \;\; s \neq t. \tag{2.8}$$

$$\widetilde{M}_{s,s} = \exp\left(\frac{\sum_{i=1}^n p_i\,{y_i}^s \,\log y_i - \sum_{i=1}^n p_i\,{x_i}^s \,\log x_i}{\sum_{i=1}^n p_i\,{y_i}^s - \sum_{i=1}^n p_i\,{x_i}^s} - \frac{2s-1}{s(s-1)}\right), \;\; s \neq 1.$$

$$\widetilde{M}_{1,1} = \exp\left(\frac{\sum_{i=1}^n p_i\,y_i\,(\log y_i)^2 - \sum_{i=1}^n p_i\,x_i\,(\log x_i)^2}{2\left(\sum_{i=1}^n p_i\,y_i\,\log y_i - \sum_{i=1}^n p_i\,x_i\,\log x_i\right)} - 1\right).$$

THEOREM 2.5. *Let $t, s, u, v \in \mathbb{R}^+$ such that $t \le u$, $s \le v$, then the following inequality is valid*

$$\widetilde{M}_{t,s} \le \widetilde{M}_{u,v}. \tag{2.9}$$

Proof. Since $\overline{\lambda}_t$ is log-convex, therefore by (2.8) we get (2.9). □

COROLLARY 2.6. *Let $\mathbf{x}$ be non-negative n-tuple and*

$$F_t = F_t(\mathbf{x}) := \overline{\varphi}_t\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} \overline{\varphi}_t(x_i),$$

Then the following statements are valid:

(a) *For every $n \in \mathbb{N}$ and $s_1, ..., s_n \in \mathbb{R}^+$, the matrix $\left[F_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{n}$ is a positive semi-definite matrix. Particularly*

$$\det\left[F_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{k} \ge 0 \tag{2.10}$$

for $k = 1, ..., n$.

(b) *The function $s \to F_s$ is exponentially convex.*

(c) *The function $s \to F_s$ is a* log*-convex on $\mathbb{R}^+$ and the following inequality holds for $0 < r < s < t < \infty$:*

$$F_s^{t-r} \le F_r^{t-s} F_t^{s-r}. \tag{2.11}$$

Proof. Set $\mathbf{y} = \big(\sum_{i=1}^{n} x_i, 0, ..., 0\big)$ and $\mathbf{x} = (x_1, ..., x_n)$ in Theorem 2.2, we get our required results. □

We define the following means of Cauchy type.

(2.12)

$$\Delta_{t,r}(x) = \left(\frac{F_t(x)}{F_r(x)}\right)^{\frac{1}{t-r}}, \quad t, r \in \mathbb{R}^+, \ r \ne t.$$

$$\Delta_{r,r}(x) = \exp\left(\frac{\big(\sum_{i=1}^{n} x_i\big)^r \log\big(\sum_{i=1}^{n} x_i\big) - \sum_{i=1}^{n} x_i{}^r \log x_i}{\big(\sum_{i=1}^{n} x_i\big)^r - \sum_{i=1}^{n} x_i{}^r} - \frac{2r-1}{r(r-1)}\right), \ r \ne 1.$$

$$\Delta_{1,1}(x) = \exp\left(\frac{(\sum_{i=1}^{n} x_i)(\log(\sum_{i=1}^{n} x_i))^2 - \sum_{i=1}^{n} x_i (\log x_i)^2}{2\Big((\sum_{i=1}^{n} x_i)(\log(\sum_{i=1}^{n} x_i)) - \sum_{i=1}^{n} x_i \log x_i\Big)} - 1\right).$$

COROLLARY 2.7. *Let $t, r, u, v \in \mathbb{R}^+$ such that $t \le u$, $r \le v$, then the following inequality is valid*

$$\Delta_{t,r}(x) \le \Delta_{u,v}(x). \tag{2.13}$$

Proof. Since $F_t(x)$ is log-convex, therefore by (2.12) we get (2.13). □

We define the following Cauchy means, which are similar to [9] for $p_i = 1$, $i = 1, ..., n$.

$$\Upsilon^s_{t,r}(x) \;=\; \left(\frac{r(r-s)}{t(t-s)}.\frac{\left(\sum_{i=1}^n x_i^s\right)^{\frac{t}{s}} - \sum_{i=1}^n x_i^t}{\left(\sum_{i=1}^n x_i^s\right)^{\frac{r}{s}} - \sum_{i=1}^n x_i^r}\right)^{\frac{1}{t-r}}, \tag{2.14}$$

$t,\ r,\ s \in \mathbb{R}^+,\ t \neq r,\ t \neq s,\ r \neq s.$

$$\Upsilon^s_{s,r}(x) \;=\; \left(\frac{r(r-s)}{s^2}.\frac{\sum_{i=1}^n x_i^s \log\left(\sum_{i=1}^n x_i^s\right) - s\sum_{i=1}^n x_i^s \log x_i}{\left(\sum_{i=1}^n x_i^s\right)^{\frac{r}{s}} - \sum_{i=1}^n x_i^r}\right)^{\frac{1}{s-r}},\ \ r \neq s.$$

$$\Upsilon^s_{r,r}(x) = \exp\left(\frac{\left(\sum_{i=1}^n x_i^s\right)^{\frac{r}{s}} \log\left(\sum_{i=1}^n x_i^s\right) - s\sum_{i=1}^n {x_i}^r \log x_i}{s\left(\left(\sum_{i=1}^n x_i^s\right)^{\frac{r}{s}} - \sum_{i=1}^n {x_i}^r\right)} - \frac{2r-s}{r(r-s)}\right),$$

$$r \neq s.$$

$$\Upsilon^s_{s,s}(x) \;=\; \exp\left(\frac{\left(\sum_{i=1}^n x_i^s\right)\left(\log\left(\sum_{i=1}^n x_i^s\right)\right)^2 - s^2\sum_{i=1}^n {x_i}^s (\log x_i)^2}{2s\left(\left(\sum_{i=1}^n x_i^s\right)\log\left(\sum_{i=1}^n x_i^s\right) - s\sum_{i=1}^n {x_i}^s \log x_i\right)} - \frac{1}{s}\right).$$

Corollary 2.8. *Let $t, r, u, v \in \mathbb{R}^+$ such that $t \leq u$, $r \leq v$, then the following inequality is valid*

$$\Upsilon^s_{t,r}(x) \;\leq\; \Upsilon^s_{u,v}(x) \tag{2.15}$$

Proof. Let

$$F_t(x) := \begin{cases} \frac{1}{t(t-1))}\left(\left(\sum_{i=1}^n x_i\right)^t - \sum_{i=1}^n x_i^t\right), & \text{t} \neq 1; \\ \sum_{i=1}^n x_i \log\left(\sum_{i=1}^n x_i\right) - \sum_{i=1}^n x_i \log x_i, & \text{t} = 1. \end{cases} \tag{2.16}$$

Using corollary 2.7, we have

$$\left(\frac{r(r-1)}{t(t-1)}.\frac{\left(\sum_{i=1}^n x_i\right)^t - \sum_{i=1}^n x_i^t}{\left(\sum_{i=1}^n x_i\right)^r - \sum_{i=1}^n x_i^r}\right)^{\frac{1}{t-r}} \leq \left(\frac{u(u-1)}{v(v-1)}.\frac{\left(\sum_{i=1}^n x_i\right)^v - \sum_{i=1}^n x_i^v}{\left(\sum_{i=1}^n x_i\right)^u - \sum_{i=1}^n x_i^u}\right)^{\frac{1}{v-u}}.$$

Since $s > 0$ by substituting $x_i = x_i^s$, $t = \frac{t}{s}$, $r = \frac{r}{s}$, $u = \frac{u}{s}$ and $v = \frac{v}{s}$ in above inequality, we get

$$\left(\frac{r(r-s)}{t(t-s)}.\frac{\left(\sum_{i=1}^n x_i^s\right)^{\frac{t}{s}} - \sum_{i=1}^n x_i^t}{\left(\sum_{i=1}^n x_i^s\right)^{\frac{r}{s}} - \sum_{i=1}^n x_i^r}\right)^{\frac{s}{t-r}} \leq \left(\frac{u(u-s)}{v(v-s)}.\frac{\left(\sum_{i=1}^n x_i^s\right)^{\frac{v}{s}} - \sum_{i=1}^n x_i^v}{\left(\sum_{i=1}^n x_i^s\right)^{\frac{u}{s}} - \sum_{i=1}^n x_i^u}\right)^{\frac{s}{v-u}}.$$

By raising power $\frac{1}{s}$, we get (2.15). □

Remark 2.9. Let us note that in [8], the following function $\phi_t = t F_t$ was considered. It was proved that

$$\phi_s^{t-r} \;\leq\; \phi_r^{t-s}\ \phi_t^{s-r}. \tag{2.17}$$

In [9], it was proved that this implies

$$\digamma_s^{t-r} \leq \frac{s^{t-r}}{r^{t-s}t^{s-r}}\, \digamma_r^{t-s}\, \digamma_t^{s-r}.$$

Since $\frac{s^{t-r}}{r^{t-s}t^{s-r}} < 1$, we have that (2.17) is better than (2.11). □

Theorem 2.10. *Let $x(\tau)$ and $y(\tau)$ be two non-negative real-valued functions defined on an interval $[a,b]$, decreasing in $[a,b]$, $y(\tau) \succ x(\tau)$ and*

$$\overline{\beta}_t(x(\tau);\, y(\tau)) := \int_a^b \overline{\varphi}_t(y(\tau))\,\mathrm{d}\tau - \int_a^b \overline{\varphi}_t(x(\tau))\,\mathrm{d}\tau,$$

and $\overline{\beta}_t$ is positive.

Then the following statements are valid:

(a) *For every $n \in \mathbb{N}$ and $s_1, ..., s_n \in \mathbb{R}^+$, the matrix $\left[\overline{\beta}_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^n$ is a positive semi-definite matrix. Particularly*

$$\det\left[\overline{\beta}_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^k \geq 0 \tag{2.18}$$

for $k = 1, ..., n$.

(b) *The function $s \to \overline{\beta}_s$ is exponentially convex.*

(c) *The function $s \to \overline{\beta}_s$ is a* log-*convex on $\mathbb{R}^+$ and the following inequality holds for $0 < r < s < t < \infty$:*

$$\left(\overline{\beta}_s\right)^{t-r} \leq \left(\overline{\beta}_r\right)^{t-s} \left(\overline{\beta}_t\right)^{s-r}. \tag{2.19}$$

Proof. As in the proof of Theorem 2.2, we use Theorem 1.7 instead of Theorem 1.3. □

Theorem 2.11. *Let $x(\tau), y(\tau) : [a,b] \to \mathbb{R}$, $x(\tau)$ and $y(\tau)$ are non-negative continuous and increasing, $G : [a,b] \to \mathbb{R}$ be a function of bounded variation and*

$$\overline{\Gamma}_t(x(\tau),\, y(\tau);\, G(\tau)) := \int_a^b \overline{\varphi}_t(y(\tau))\,\mathrm{d}G(\tau) - \int_a^b \overline{\varphi}_t(x(\tau))\,\mathrm{d}G(\tau)$$

such that conditions (1.11) and (1.12) are satisfied and $\overline{\Gamma}_t$ is positive.

Then the following statements are valid:

(a) *For every $n \in \mathbb{N}$ and $s_1, ..., s_n \in \mathbb{R}^+$, the matrix $\left[\overline{\Gamma}_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^n$ is a positive semi-definite matrix. Particularly*

$$\det\left[\overline{\Gamma}_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^k \geq 0 \tag{2.20}$$

for $k = 1, ..., n$.

(b) *The function $s \to \overline{\Gamma}_s$ is exponentially convex.*

(c) *The function* $s \to \overline{\Gamma}_s$ *is a* log-*convex on* $\mathbb{R}^+$ *and the following inequality holds for* $0 < r < s < t < \infty$:

$$\left(\overline{\Gamma}_s\right)^{t-r} \le \left(\overline{\Gamma}_r\right)^{t-s} \left(\overline{\Gamma}_t\right)^{s-r}. \tag{2.21}$$

Proof. As in the proof of Theorem 2.2, we use Theorem 1.8 instead of Theorem 1.3. □

THEOREM 2.12. *Let* $F(\tau)$ *and* $G(\tau)$ *are non-negative continuous and increasing functions defined on an interval* $[0, +\infty)$ *such that* $F(0) = G(0) = 0$, $\overline{F}(\tau) \succ \overline{G}(\tau)$, $\overline{F}(\tau)$ *and* $\overline{G}(\tau)$ *are defined in* (1.14),

$$\theta_t(\tau,\, G(\tau);\, F(\tau)) := \int_0^\infty \overline{\varphi}_t(\tau)\, \mathrm{d}G(\tau) - \int_0^\infty \overline{\varphi}_t(\tau)\, \mathrm{d}F(\tau),$$

and θ_t *is positive.*

Then the following statements are valid:

(a) *For every* $n \in \mathbb{N}$ *and* $s_1, ..., s_n \in \mathbb{R}^+$, *the matrix* $\left[\theta_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^n$ *is a positive semi-definite matrix. Particularly*

$$\det\left[\theta_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^k \ge 0 \tag{2.22}$$

for $k = 1, ..., n$.

(b) *The function* $s \to \theta_s$ *is exponentially convex.*

(c) *The function* $s \to \theta_s$ *is a* log-*convex on* $\mathbb{R}^+$ *and the following inequality holds for* $0 < r < s < t < \infty$:

$$\theta_s^{t-r} \le \theta_r^{t-s}\, \theta_t^{s-r}. \tag{2.23}$$

Proof. As in the proof of Theorem 2.2, we use Theorem 1.10 instead of Theorem 1.3. □

As in [1], we define the following means of Cauchy type.

$$\begin{aligned}
\overline{\theta}_{t,s} &= \left(\tfrac{\theta_t}{\theta_s}\right)^{\frac{1}{t-s}}, \qquad t,\, s \in \mathbb{R}^+, \quad s \ne t. \\
\overline{\theta}_{s,s} &= \exp\left(\tfrac{\int_0^\infty \tau^s\, \log\tau\, \mathrm{d}G(\tau) - \int_0^\infty \tau^s\, \log\tau\, \mathrm{d}F(\tau)}{\int_0^\infty \tau^s\, \mathrm{d}G(\tau) - \int_0^\infty \tau^s\, \mathrm{d}F(\tau)} - \tfrac{2s-1}{s(s-1)}\right), \quad s \ne 1. \\
\overline{\theta}_{1,1} &= \exp\left(\tfrac{\int_0^\infty \tau\, \log^2\tau\, \mathrm{d}G(\tau) - \int_0^\infty \tau\, \log^2\tau\, \mathrm{d}F(\tau)}{2\left(\int_0^\infty \tau\, \log\tau\, \mathrm{d}G(\tau) - \int_0^\infty \tau\, \log\tau\, \mathrm{d}F(\tau)\right)} - 1\right).
\end{aligned} \tag{2.24}$$

THEOREM 2.13. *Let* $t, s, u, v \in \mathbb{R}^+$ *such that* $t \le u$, $s \le v$, *then the following inequality is valid*

$$\overline{\theta}_{t,s} \le \overline{\theta}_{u,v}. \tag{2.25}$$

Proof. Since θ_t is log-convex, therefore by (2.24) we get (2.25). □

Remark 2.14. As in [1], we can use Theorem 2.2, Theorem 2.4, Corollary 2.6, Theorem 2.10, Theorem 2.11 and Theorem 2.12 to obtain corresponding Cauchy means. □

3. MULTIPLICATIVE MAJORIZATION

Lemma 3.1. *Given $t \in \mathbb{R}$, define the function*

$$\psi_t(x) := \begin{cases} \frac{1}{t^2}\, e^{tx}, & t \neq 0; \\ \frac{1}{2}\, x^2, & t = 0, \end{cases} \tag{3.1}$$

Then $\psi_t''(x) = e^{tx}$, that is, $\psi_t(x)$ is convex for $x \in \mathbb{R}$.

Theorem 3.2. *Let $\mathbf{x}$ and $\mathbf{y}$ be two real n-tuples, $\boldsymbol{y} \succ \boldsymbol{x}$,*

$$\xi_t = \xi_t(\mathbf{x};\ \mathbf{y}) := \sum_{i=1}^{n} \psi_t(y_i) - \sum_{i=1}^{n} \psi_t(x_i),$$

and all $x_{[i]}$'s and $y_{[i]}$'s are not equal.
Then the following statements are valid:

(a) *For every $n \in \mathbb{N}$ and $s_1, ..., s_n \in \mathbb{R}$, the matrix $\left[\xi_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{n}$ is a positive semi-definite matrix. Particularly*

$$\det\left[\xi_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{k} \geq 0 \tag{3.2}$$

for $k = 1, ..., n$.

(b) *The function $s \to \xi_s$ is exponentially convex.*

(c) *The function $s \to \xi_s$ is a* log-*convex on $\mathbb{R}$ and the following inequality holds for $-\infty < r < s < t < \infty$:*

$$\xi_s^{t-r} \leq \xi_r^{t-s}\, \xi_t^{s-r}. \tag{3.3}$$

Proof. As in the proof of Theorem 2.2, we use ψ_t instead of $\overline{\varphi}_t$. □

As in [1], we define the following means of Cauchy type.

$$\Theta_{t,s} = \left(\tfrac{\xi_t}{\xi_s}\right)^{\frac{1}{t-s}}, \qquad t,\, s \in \mathbb{R}, \quad s \neq t. \tag{3.4}$$

$$\Theta_{s,s} = \exp\left(\frac{\sum_{i=1}^n y_i\, e^{sy_i} - \sum_{i=1}^n x_i\, e^{sx_i}}{\sum_{i=1}^n e^{sy_i} - \sum_{i=1}^n e^{sy_i}} - \frac{2}{s}\right), \quad s \neq 0.$$

$$\Theta_{0,0} = \exp\left(\frac{\sum_{i=1}^n y_i^3 - \sum_{i=1}^n x_i^3}{3\left(\sum_{i=1}^n y_i^2 - \sum_{i=1}^n x_i^2\right)}\right).$$

Theorem 3.3. *Let $t, s, u, v \in \mathbb{R}$ such that $t \leq u$, $s \leq v$, then the following inequality is valid*

$$\Theta_{t,s} \leq \Theta_{u,v}. \tag{3.5}$$

Proof. Since ξ_t is log-convex, therefore by (3.4) we get (3.5). □

Theorem 3.4. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two decreasing real n-tuples,* $\boldsymbol{p} = (p_1, ..., p_n)$ *be a real n-tuple and let*

$$\overline{\xi}_t = \overline{\xi}_t(\mathbf{x}, \mathbf{y}; \mathbf{p}) := \sum_{i=1}^{n} p_i\, \psi_t(y_i) - \sum_{i=1}^{n} p_i\, \psi_t(x_i),$$

such that conditions (1.6) *and* (1.7) *are satisfied and* $\overline{\xi}_t$ *is positive.*

Then the following statements are valid:

(a) *For every* $n \in \mathbb{N}$ *and* $s_1, ..., s_n \in \mathbb{R}$*, the matrix* $\left[\overline{\xi}_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{n}$ *is a positive semi-definite matrix. Particularly*

$$\det \left[\overline{\xi}_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{k} \geq 0 \tag{3.6}$$

for $k = 1, ..., n$.

(b) *The function* $s \to \overline{\xi}_s$ *is exponentially convex.*

(c) *The function* $s \to \overline{\xi}_s$ *is a* log*-convex on* $\mathbb{R}$ *and the following inequality holds for* $-\infty < r < s < t < \infty$ *:*

$$\left(\overline{\xi}_s\right)^{t-r} \leq \left(\overline{\xi}_r\right)^{t-s} \left(\overline{\xi}_t\right)^{s-r}. \tag{3.7}$$

Proof. As in the proof of Theorem 2.2, we use Theorem 1.5 instead of Theorem 1.3 and ψ_t instead of $\overline{\varphi}_t$. □

As in [1], we define the following means of Cauchy type.

$$\overline{\Theta}_{t,s} = \left(\frac{\overline{\xi}_t}{\overline{\xi}_s}\right)^{\frac{1}{t-s}}, \qquad t, s \in \mathbb{R}, \quad s \neq t. \tag{3.8}$$

$$\overline{\Theta}_{s,s} = \exp\left(\frac{\sum_{i=1}^n p_i\, y_i\, \mathrm{e}^{sy_i} - \sum_{i=1}^n p_i\, x_i\, \mathrm{ee}^{sx_i}}{\sum_{i=1}^n p_i\, \mathrm{e}^{sy_i} - \sum_{i=1}^n p_i\, \mathrm{e}^{sx_i}} - \frac{2}{s}\right), \quad s \neq 0.$$

$$\overline{\Theta}_{0,0} = \exp\left(\frac{\sum_{i=1}^n p_i\, y_i^3 - \sum_{i=1}^n p_i\, x_i^3}{3\left(\sum_{i=1}^n p_i\, y_i^2 - \sum_{i=1}^n p_i\, x_i^2\right)}\right).$$

Theorem 3.5. *Let* $t, s, u, v \in \mathbb{R}$ *such that* $t \leq u$*,* $s \leq v$*, then the following inequality is valid*

$$\overline{\Theta}_{t,s} \leq \overline{\Theta}_{u,v}. \tag{3.9}$$

Proof. Since $\overline{\xi}_t$ is log-convex, therefore by (3.8) we get (3.9). □

Corollary 3.6. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two positive n-tuples,* $\boldsymbol{x} \prec_{\times} \boldsymbol{y}$ *,*

$$\Omega_t(\log \mathbf{x}; \log \mathbf{y}) = \xi_t(\mathbf{x};\ \mathbf{y}) := \begin{cases} \frac{1}{t^2}\left(\sum_{i=1}^n y_i^t - \sum_{i=1}^n x_i^t\right), & t \neq 0; \\ \frac{1}{2}\left(\sum_{i=1}^n \log^2 y_i - \sum_{i=1}^n \log^2 x_i\right), & t = 0, \end{cases}$$

and all $x_{[i]}$*'s and* $y_{[i]}$*'s are not equal.*

Then the following statements are valid:

(a) *For every* $n \in \mathbb{N}$ *and* $s_1, ..., s_n \in \mathbb{R}$, *the matrix* $\left[\Omega_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{n}$ *is a positive semi-definite matrix. Particularly*

$$\det\left[\Omega_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{k} \geq 0 \tag{3.10}$$

for $k = 1, ..., n$.

(b) *The function* $s \to \Omega_s$ *is exponentially convex.*

(c) *The function* $s \to \Omega_s$ *is a* log*-convex on* $\mathbb{R}$ *and the following inequality holds for* $-\infty < r < s < t < \infty$:

$$\Omega_s^{t-r} \leq \Omega_r^{t-s}\,\Omega_t^{s-r}. \tag{3.11}$$

Proof. As in the proof of Theorem 2.2, we use Theorem 1.3 for $\mathbf{x} = \log\mathbf{x}$ and $\mathbf{y} = \log\mathbf{y}$ and using ψ_t instead of $\overline{\varphi}_t$. □

As in [1], we define the following means of Cauchy type.

$$\Psi_{t,s} = \left(\frac{\Omega_t}{\Omega_s}\right)^{\frac{1}{t-s}}, \qquad t,\, s \in \mathbb{R}, \quad s \neq t. \tag{3.12}$$

$$\Psi_{s,s} = \exp\left(\frac{\sum_{i=1}^n y_i^s \log y_i - \sum_{i=1}^n x_i^s \log x_i}{\sum_{i=1}^n y_i^s - \sum_{i=1}^n x_i^s} - \frac{2}{s}\right), \quad s \neq 0.$$

$$\Psi_{0,0} = \exp\left(\frac{\sum_{i=1}^n \log^3 y_i - \sum_{i=1}^n \log^3 x_i}{3\left(\sum_{i=1}^n \log^2 y_i - \sum_{i=1}^n \log^2 x_i\right)}\right).$$

Corollary 3.7. *Let* $t, s, u, v \in \mathbb{R}$ *such that* $t \leq u$, $s \leq v$, *then the following inequality is valid*

$$\Psi_{t,s} \leq \Psi_{u,v}. \tag{3.13}$$

Proof. Since Ω_t is log-convex, therefore by (3.12) we get (3.13). □

Corollary 3.8. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two positive decreasing* n*-tuples,* $\boldsymbol{p} = (p_1, ..., p_n)$ *be a real* n*-tuple and let*

$$\overline{\Omega}_t(\mathbf{x}, \mathbf{y}; \mathbf{p}) =$$

$$= \overline{\xi}_t(\log \mathbf{x},\ \log \mathbf{y};\ \mathbf{p}) := \begin{cases} \frac{1}{t^2}\left(\sum_{i=1}^n p_i\, y_i^t - \sum_{i=1}^n p_i\, x_i^t\right), & t \neq 0; \\ \frac{1}{2}\left(\sum_{i=1}^n p_i \log^2 y_i - \sum_{i=1}^n p_i \log^2 x_i\right), & t = 0, \end{cases}$$

such that conditions (1.6) *and* (1.7) *are satisfied and* $\overline{\Omega}_t$ *is positive.*

Then the following statements are valid:

(a) *For every* $n \in \mathbb{N}$ *and* $s_1, ..., s_n \in \mathbb{R}$, *the matrix* $\left[\overline{\Omega}_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{n}$ *is a positive semi-definite matrix. Particularly*

$$\det\left[\overline{\Omega}_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{k} \geq 0 \tag{3.14}$$

for $k = 1, ..., n$.

(b) *The function* $s \to \overline{\Omega}_s$ *is exponentially convex.*

(c) *The function* $s \to \overline{\Omega}_s$ *is a* log*-convex on* $\mathbb{R}$ *and the following inequality holds for* $-\infty < r < s < t < \infty$:

$$(\overline{\Omega}_s)^{t-r} \leq (\overline{\Omega}_r)^{t-s} (\overline{\Omega}_t)^{s-r}. \tag{3.15}$$

Proof. As in the proof of Theorem 2.2, we use Theorem 1.5 for $\mathbf{x} = \log\mathbf{x}$ and $\mathbf{y} = \log\mathbf{y}$ and using ψ_t instead of $\overline{\varphi}_t$. □

As in [1], we define the following means of Cauchy type.

$$\overline{\Psi}_{t,s} = \left(\frac{\overline{\Omega}_t}{\overline{\Omega}_s}\right)^{\frac{1}{t-s}}, \qquad t,\, s \in \mathbb{R}, \quad s \neq t. \tag{3.16}$$

$$\overline{\Psi}_{s,s} = \exp\left(\frac{\sum_{i=1}^n p_i\, y_i^s\, \log y_i - \sum_{i=1}^n p_i\, x_i^s\, \log x_i}{\sum_{i=1}^n p_i\, y_i^s - \sum_{i=1}^n p_i\, x_i^s} - \frac{2}{s}\right), \quad s \neq 0.$$

$$\overline{\Psi}_{0,0} = \exp\left(\frac{\sum_{i=1}^n p_i\, \log^3 y_i - \sum_{i=1}^n p_i\, \log^3 x_i}{3\left(\sum_{i=1}^n p_i\, \log^2 y_i - \sum_{i=1}^n p_i\, \log^2 x_i\right)}\right).$$

Corollary 3.9. *Let* $t,\ s,\ u,\ v \in \mathbb{R}$ *such that* $t \leq u,\ s \leq v$, *then the following inequality is valid*

$$\overline{\Psi}_{t,s} \leq \overline{\Psi}_{u,v}. \tag{3.17}$$

Proof. Since $\overline{\Omega}_t$ is log-convex, therefore by (3.16) we get (3.17). □

Theorem 3.10. *Let* $x(\tau)$ *and* $y(\tau)$ *be two real-valued functions defined on an interval* $[a, b]$, *decreasing in* $[a, b]$, $y(\tau) \succ x(\tau)$ *and*

$$\Phi_t(x(\tau);\, y(\tau)) := \int_a^b \psi_t(y(\tau))\, \mathrm{d}\tau - \int_a^b \psi_t(x(\tau))\, \mathrm{d}\tau,$$

and Φ_t *is positive.*

Then the following statements are valid:

(a) *For every* $n \in \mathbb{N}$ *and* $s_1, ..., s_n \in \mathbb{R}$, *the matrix* $\left[\Phi_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^n$ *is a positive semi-definite matrix. Particularly*

$$\det\left[\Phi_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^k \geq 0 \tag{3.18}$$

for $k = 1, ..., n$.

(b) *The function* $s \to \Phi_s$ *is exponentially convex.*

(c) *The function* $s \to \Phi_s$ *is a* log*-convex on* $\mathbb{R}$ *and the following inequality holds for* $-\infty < r < s < t < \infty$:

$$\Phi_s^{t-r} \leq \Phi_r^{t-s}\, \Phi_t^{s-r}. \tag{3.19}$$

Proof. As in the proof of Theorem 2.2, we use Theorem 1.7 instead of Theorem 1.3 and ψ_t instead of $\overline{\varphi}_t$. □

THEOREM 3.11. *Let $x(\tau)$, $y(\tau) : [a,b] \to \mathbb{R}$, $x(\tau)$ and $y(\tau)$ are continuous and increasing, $G : [a,b] \to \mathbb{R}$ be a function of bounded variation and*

$$\overline{\Phi}_t(x(\tau),\, y(\tau);\, G(\tau)) \;:=\; \int_a^b \psi_t(y(\tau))\,\mathrm{d}G(\tau) - \int_a^b \psi_t(x(\tau))\,\mathrm{d}G(\tau)$$

such that conditions (1.11) *and* (1.12) *are satisfied and $\overline{\Phi}_t$ is positive.*

Then the following statements are valid:

(a) *For every $n \in \mathbb{N}$ and $s_1, ..., s_n \in \mathbb{R}$, the matrix $\left[\overline{\Phi}_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{n}$ is a positive semi-definite matrix. Particularly*

$$\det\left[\overline{\Phi}_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{k} \;\geq\; 0 \tag{3.20}$$

for $k = 1, ..., n$.

(b) *The function $s \to \overline{\Phi}_s$ is exponentially convex.*

(c) *The function $s \to \overline{\Phi}_s$ is a* log*-convex on $\mathbb{R}$ and the following inequality holds for $-\infty < r < s < t < \infty$:*

$$\left(\overline{\Phi}_s\right)^{t-r} \;\leq\; \left(\overline{\Phi}_r\right)^{t-s} \left(\overline{\Phi}_t\right)^{s-r}. \tag{3.21}$$

Proof. As in the proof of Theorem 2.2, we use Theorem 1.8 instead of Theorem 1.3 and ψ_t instead of $\overline{\varphi}_t$. □

THEOREM 3.12. *Let $F(\tau)$ and $G(\tau)$ are real continuous and increasing functions defined on an interval $[0, +\infty)$ such that $F(0) = G(0) = 0$, $\overline{F}(\tau) \succ \overline{G}(\tau)$, $\overline{F}(\tau)$ and $\overline{G}(\tau)$ are defined in (1.14),*

$$\vartheta_t(\tau,\, G(\tau);\, F(\tau)) \;:=\; \int_0^\infty \psi_t(\tau)\,\mathrm{d}G(\tau) - \int_0^\infty \psi_t(\tau)\,\mathrm{d}F(\tau),$$

and ϑ_t is positive.

Then the following statements are valid:

(a) *For every $n \in \mathbb{N}$ and $s_1, ..., s_n \in \mathbb{R}$, the matrix $\left[\vartheta_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{n}$ is a positive semi-definite matrix. Particularly*

$$\det\left[\vartheta_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{k} \;\geq\; 0 \tag{3.22}$$

for $k = 1, ..., n$.

(b) *The function $s \to \vartheta_s$ is exponentially convex.*

(c) *The function $s \to \vartheta_s$ is a* log*-convex on $\mathbb{R}$ and the following inequality holds for $-\infty < r < s < t < \infty$:*

$$\vartheta_s^{t-r} \;\leq\; \vartheta_r^{t-s}\, \vartheta_t^{s-r}. \tag{3.23}$$

Proof. As in the proof of Theorem 2.2, we use Theorem 1.10 instead of Theorem 1.3 and ψ_t instead of $\overline{\varphi}_t$. □

As in [1], we define the following means of Cauchy type.

$$\overline{\vartheta}_{t,s} = \left(\frac{\vartheta_t}{\vartheta_s}\right)^{\frac{1}{t-s}}, \qquad t,\, s \in \mathbb{R}, \quad s \neq t. \tag{3.24}$$

$$\overline{\vartheta}_{s,s} = \exp\left(\frac{\int_0^\infty \tau\, e^{s\tau}\, dG(\tau) - \int_0^\infty \tau\, e^{s\tau}\, dF(\tau)}{\int_0^\infty e^{s\tau}\, dG(\tau) - \int_0^\infty e^{s\tau}\, dF(\tau)} - \frac{2}{s}\right), \quad s \neq 0.$$

$$\overline{\vartheta}_{0,0} = \exp\left(\frac{\int_0^\infty \tau^3\, dG(\tau) - \int_0^\infty \tau^3\, dF(\tau)}{3\left(\int_0^\infty \tau^2\, dG(\tau) - \int_0^\infty \tau^2\, dF(\tau)\right)}\right).$$

THEOREM 3.13. *Let $t, s, u, v \in \mathbb{R}$ such that $t \leq u$, $s \leq v$, then the following*

$$\overline{\vartheta}_{t,s} \leq \overline{\vartheta}_{u,v}. \tag{3.25}$$

Proof. Since ϑ_t is log-convex, therefore by (3.24) we get (3.25). □

REMARK 3.14. As in [1], we can use Theorem 3.2, Theorem 3.4, Corollary 3.6, Corollary 3.8, Theorem 3.10, Theorem 3.11 and Theorem 3.12 to obtain corresponding Cauchy means. □

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