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SHARP BOUNDS FOR GAMMA AND DIGAMMA FUNCTION ARISING FROM BURNSIDE'S FORMULA

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Abstract. The main aim of this paper is to improve the Burnside's formula for approximating the factorial function. We prove the complete monotonicity of a function involving the gamma function to establish new lower and upper sharp bounds for the gamma and digamma function.

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1. INTRODUCTION

There are many situations of practical problems from pure mathematics, or other branches of science when we are forced to deal with large factorials. As a direct computation cannot be made even by the computer programs, approximation formulas were constructed, one of the most known and most used being the Stirling's formula:

$$n! \approx \sqrt{2\pi} \cdot n^{n+\frac{1}{2}} \mathrm{e}^{-\mathrm{n}}. \tag{1.1}$$

It was first discovered in 1733 by the French mathematician Abraham de Moivre (1667-1754) in the form

$$n! \approx \text{constant} \cdot n^{n+\frac{1}{2}} \mathrm{e}^{-\mathrm{n}}$$

with missing constant. Afterwards, the Scottish mathematician James Stirling (1692-1770) found the constant $\sqrt{2\pi}$ in formula (1.1). For proofs, interesting historical facts, and other details see [8, 10].

If in probability theory, or statistics, such approximations are satisfactory, in pure mathematics, more precise estimates are required. Although in the last decades, many authors are concerned to give new improvements of the Stirling's formula, we mention here the following approximation due to W.

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Burnside [3]:

$$n! \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{\mathrm{e}}\right)^{n+\frac{1}{2}}, \tag{1.2}$$

which gives better results than Stirling's formula (1.1). Burnside's formula, as Stirling's formula remain beautiful because of their simplicity.

In this paper, we study the complete monotonicity of the function $f : [0,\infty) \to \mathbb{R}$, given by the formula

$$f(x) = \ln\Gamma(x+1) - \ln\sqrt{2\pi} - \left(x+\tfrac{1}{2}\right)\ln\left(x+\tfrac{1}{2}\right) + x + \tfrac{1}{2}.$$

As a direct consequence, we establish new lower and upper sharp bounds for the gamma and digamma function. More precisely, we prove that for $x \geq 1$,

$$\omega\cdot\sqrt{2\pi}\left(\frac{x+\frac{1}{2}}{\mathrm{e}}\right)^{x+\frac{1}{2}} \leq \Gamma(x+1) < \sqrt{2\pi}\left(\frac{x+\frac{1}{2}}{\mathrm{e}}\right)^{x+\frac{1}{2}},$$

where the constant $\omega = \frac{2}{3\sqrt{3\pi}}\mathrm{e}^{3/2} = 0.973\,23...$ is best possible.

For $x \geq 1$, we have

$$\ln\left(x+\tfrac{1}{2}\right) - \tfrac{1}{x} < \psi(x) \leq \ln\left(x+\tfrac{1}{2}\right) - \tfrac{1}{x} + \zeta,$$

with best possible constant $\zeta = 1 - \ln\frac{3}{2} - \gamma = 0.017319...$. Here,

$$\gamma = \lim_{n\to\infty}\left(1 + \tfrac{1}{2} + ... + \tfrac{1}{n} - \ln n\right) = 0.577215...$$

is the Euler-Mascheroni constant. Our new inequality improves other known results [2, 4, 5, 6, 9] of the form

$$\ln x - \tfrac{1}{x} < \psi(x) < \ln x - \tfrac{1}{2x}\ , \quad x > 1.$$

Similar techniques were developed by the author in the very recent paper [7].

2. THE RESULTS

The gamma Γ and digamma ψ functions are defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\mathrm{d}t \quad , \quad \psi(x) = \tfrac{\mathrm{d}}{\mathrm{d}x}(\ln\Gamma(x)) = \tfrac{\Gamma'(x)}{\Gamma(x)},$$

for every positive real numbers x. The gamma function is an extension of the factorial function, since $\Gamma(n+1) = n!$, for $n = 0, 1, 2, 3\ldots$. The derivatives ψ', ψ'', ..., known as polygamma functions, have the following integral representations:

$$\psi^{(n)}(x) = (-1)^{n-1}\int_0^\infty \tfrac{t^n\mathrm{e}^{-xt}}{1-\mathrm{e}^{-t}}\mathrm{d}t \tag{2.1}$$

for $n = 1, 2, 3, \ldots$. For proofs and other details, see for example, [1, 11]. We also use the following integral representation

$$\tfrac{1}{x^n} = \tfrac{1}{(n-1)!}\int_0^\infty t^{n-1}\mathrm{e}^{-xt}\mathrm{d}t\ , \quad n \geq 1. \tag{2.2}$$

Recall that a function f is (strictly) completely monotonic in an interval I if f has derivatives of all orders in I such that $(-1)^n f^{(n)}(x) \geq 0$, (respective $(-1)^n f^{(n)}(x) > 0$) for all $x \in I$ and $n = 0, 1, 2, 3 \ldots$. It is to be noticed that every non-constant, completely monotonic function is in fact strictly completely monotonic.

Completely monotonic functions involving $\ln \Gamma(x)$ are important because they produce bounds for the polygamma functions. The famous Hausdorff-Bernstein-Widder theorem states that a function f is completely monotonic on $[0, \infty)$ if and only if it is a Laplace transform,

$$f(x) = \int_0^\infty \mathrm{e}^{-xt} \mathrm{d}\mu(t),$$

where μ is a non-negative measure on $[0, \infty)$ such that the integral converges for all $x > 0$, see [11, pp. 161]. Now we are in position to prove the following

THEOREM 1. *Let there be given* $f : [0, \infty) \to \mathbb{R}$, *by*

$$f(x) = \ln \Gamma(x+1) - \ln \sqrt{2\pi} - \left(x + \tfrac{1}{2}\right) \ln \left(x + \tfrac{1}{2}\right) + x + \tfrac{1}{2}.$$

Then $-f$ *is completely monotonic.*

Proof. We have

$$f'(x) = \psi(x+1) - \ln\left(x + \tfrac{1}{2}\right).$$

By the recurrence formula $\psi(x+1) = \psi(x) + \frac{1}{x}$, (see [1, Rel. 6.3.5, p. 258]), we obtain

$$f'(x) = \psi(x) + \tfrac{1}{x} - \ln\left(x + \tfrac{1}{2}\right),$$

then

$$f''(x) = \psi''(x) - \tfrac{1}{x^2} - \tfrac{1}{x + \frac{1}{2}}.$$

Using (2.1)–(2.2), we deduce that

$$f''(x) = \int_0^\infty \tfrac{t\mathrm{e}^{-xt}}{1-\mathrm{e}^{-t}} \mathrm{d}t - \int_0^\infty t\mathrm{e}^{-xt} \mathrm{d}t - \int_0^\infty \mathrm{e}^{-\left(x+\frac{1}{2}\right)t} \mathrm{d}t,$$

or

$$f''(x) = \int_0^\infty \left(\tfrac{t}{1-\mathrm{e}^{-t}} - t - \mathrm{e}^{-\frac{1}{2}t}\right) \mathrm{e}^{-tx} \mathrm{d}t.$$

Hence

$$f''(x) = \int_0^\infty \tfrac{\mathrm{e}^{-(x+1)t}}{1-\mathrm{e}^{-t}} \varphi(t) \, \mathrm{d}t,$$

where φ denotes the function

$$\varphi(t) = t - \mathrm{e}^{-t/2}\left(\mathrm{e}^t - 1\right).$$

As we have

$$\varphi'(t) = -\tfrac{1}{2\mathrm{e}^{t/2}}\left(\mathrm{e}^{t/2} - 1\right)^2 < 0,$$

it results that φ is strictly decreasing. For $t > 0$, we have $\varphi(t) < \varphi(0) = 0$, so $-f''$ is completely monotonic. Furthermore, f' is strictly decreasing, since $f'' < 0$.

But we have $\lim_{x\to\infty} f'(x) = 0$, so $f'(x) > 0$ and consequently, f is strictly increasing. Using the fact that $\lim_{x\to\infty} f(x) = 0$, we deduce that $f < 0$. Finally, $-f$ is completely monotonic. $\square$

As a direct consequence of the fact that f is strictly increasing, we have

$$f(1) \le f(x) < \lim_{x\to\infty} f(x) = 0,$$

for all $x \ge 1$. As $f(1) = \frac{3}{2} + \ln \frac{2}{3\sqrt{3\pi}}$, by exponentiating, we get, for $x \ge 1$:

$$\omega \cdot \sqrt{2\pi}\left(\frac{x+1/2}{\mathrm{e}}\right)^{x+1/2} \le \Gamma(x+1) < \sqrt{2\pi}\left(\frac{x+1/2}{\mathrm{e}}\right)^{x+1/2},$$

where the constant $\omega = \frac{2}{3\sqrt{3\pi}}\mathrm{e}^{3/2} = 0.973\,23...$ is best possible.

Using the fact that f' is strictly decreasing, we have

$$0 = \lim_{x\to\infty} f'(x) < f'(x) \le f'(1),$$

for all $x \ge 1$. As we have $f'(1) = 1 - \ln \frac{3}{2} - \gamma$, we obtain, for $x \ge 1$:

$$0 < \psi(x) + \frac{1}{x} - \ln\left(x + \frac{1}{2}\right) \le \zeta,$$

with best possible constant $\zeta = 1 - \ln \frac{3}{2} - \gamma = 0.017319...$.

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