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CONTINUITY OF THE QUENCHING TIME IN A SEMILINEAR HEAT EQUATION WITH NEUMANN BOUNDARY CONDITION

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Abstract. This paper concerns the study of a semilinear parabolic equation subject to Neumann boundary conditions and positive initial datum. Under some assumptions, we show that the solution of the above problem quenches in a finite time and estimate its quenching time. We also prove the continuity of the quenching time as a function of the initial datum. Finally, we give some numerical results to illustrate our analysis.

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1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider the following initial-boundary value problem

$$u_t = \Delta u - f(u) \quad \text{in} \quad \Omega \times (0,T), \tag{1}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{2}$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \overline{\Omega}, \tag{3}$$

where Δ is the Laplacian, ν is the exterior normal unit vector on $\partial\Omega$. For the nonlinear term $f(u)$, our standing assumptions are the following
(A1) $f : (0,\infty) \to (0,\infty)$ is a C^1 convex, nonincreasing function satisfying $\lim_{s\to 0^+} f(s) = \infty$, $\int_0^\alpha \frac{\mathrm{d}\sigma}{f(\sigma)} < \infty$ for any positive real α, and $\int_0^\infty \frac{\mathrm{d}\sigma}{f(\sigma)} = \infty$.
(A2) There exists a positive constant C_0 such that

$$-sf'(H(s)) \leq C_0 \quad \text{for} \quad s \geq 0, \tag{4}$$

where $H(s)$ is the inverse of the function $F(s)$ defined as follows

$$F(s) = \int_0^s \frac{\mathrm{d}\sigma}{f(\sigma)}.$$

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For the initial datum, we make the following hypotheses
(A3): $u_0 \in C^2(\overline{\Omega})$, $u_0(x) > 0$ in $\overline{\Omega}$, and there exists a positive constant B such that

$$\Delta u_0(x) - f(u_0(x)) \leq -Bf(u_0(x)) \quad \text{in} \quad \Omega. \tag{5}$$

It is worth noting that, if we choose $f(s) = s^{-p}$ with p a positive real, then it is not hard to see that f satisfies the different assumptions listed in the introduction of the paper. Also, in this case, we note that $F(s) = \frac{s^{p+1}}{p+1}$, $H(s) = ((p+1)s)^{1/(p+1)}$, and $-sf'(H(s)) = \frac{p}{p+1}$.
Here $(0,T)$ is the maximal time interval of existence of the solution u, and by a solution we mean the following.

Definition 1. *A solution of* (1)–(3) *is a function* $u(x,t)$ *continuous verifying* (1)–(3), $u(x,t) > 0$ *in* $\overline{\Omega} \times [0,T)$, *and twice continuously differentiable in* x *and once in* t *in* $\Omega \times (0,T)$.

The time T may be finite or infinite. When T is infinite, then we say that the solution u exists globally. When T is finite, then the solution u develops a quenching in a finite time, namely,

$$\lim_{t \to T} u_{\min}(t) = 0,$$

where $u_{\min}(t) = \min_{x \in \overline{\Omega}} u(x,t)$. In this last case, we say that the solution u quenches in a finite time, and the time T is called the quenching time of the solution u. Since the pioneering work of Kawarada in [25] regarding the phenomenon of quenching, solutions of semilinear parabolic equations which quench in a finite time have been the subject of investigation of many authors (see, [2]–[4], [7], [8], [12], [14], [15], [22], [26], [28]–[30], [33], [37], and the references cited therein). In particular, in [7], the problem (1)–(3) has been studied. By standard methods, it is well known that, making use of the assumptions made on the paper, one easily proves the existence and uniqueness of solutions (see, [7, Theorems 2.1 and 2.2], [16, Chapter 7, Theorem 13 (and its corollary)], [27, Theorem 7.4]). It is also shown that the solution of (1)–(3) quenches in a finite time and its quenching time has been estimated (see, [7]). In this paper, we are interested in the continuity of the quenching time as a function of the initial datum. More precisely, we consider the following initial-boundary value problem

$$v_t = \Delta v - f(v) \quad \text{in} \quad \Omega \times (0, T_h), \tag{6}$$

$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0, T_h), \tag{7}$$

$$v(x,0) = u_0^h(x) \quad \text{in} \quad \overline{\Omega}, \tag{8}$$

where $u_0^h \in C^2(\overline{\Omega})$, $u_0^h(x) \geq u_0(x)$ in Ω, and $\lim_{h\to 0} \|u_0^h - u_0\|_\infty = 0$. Here $(0,T_h)$ is the maximal time interval on which the solution v of (6)–(8) exists. Definition 1 is valid for the solution v of (6)–(8) and we assume that this solution is sufficiently regular. It is worth noting that the regularity of solutions increases with respect to the regularity of initial data, and one may apply without difficulties the maximum principle (see, [16, Chapter 7, Theorem 13 (and its corollary)], [27, Chapter 5, Theorem 53], [35, Chapter 4]). In the present paper, we prove that if h is small enough, then the solution v of (6)–(8) quenches in a finite time and its quenching time T_h goes to T as h goes to zero, where T is the quenching time of the solution u of (1)–(3). In addition we provide an upper bound of $|T_h - T|$ in terms of $\|u_0^h - u_0\|_\infty$. This work is in parts the continuity of our earlier study in [8] where we considered a particular case of the problem (1)–(3) choosing $f(s) = s^{-p}$ with p a positive constant. Under some hypotheses, we showed that the time T_h goes to T as h tends to zero. Let us point out that in [8], we have not found an upper bound of $|T_h - T|$. Similar results have been obtained in [5], [10], [18]–[20], [23], [24], [32], [36], where the authors considered analogous problems within the framework of the phenomenon of blow-up (we say that a solution blows up in a finite time if it reaches the value infinity in a finite time). The remainder of the paper is written in the following manner. In the next section, we present some results concerning quenching solutions for our subsequent use. In the third section, we analyze the continuity of the quenching time as a function of the initial datum and finally, in the last section, we show some computational results to illustrate the theory given in the paper.

2. QUENCHING TIME

In this section, under some assumptions, we show that the solution v of (6)–(8) quenches in a finite time and estimate its quenching time.

We begin by proving the following result which is inspired of an idea of Friedman and McLeod in [17].

THEOREM 2. *Suppose that there exists a constant $A \in (0,1]$ which is independent of h such that the initial datum at* (8) *satisfies*

$$\Delta u_0^h(x) - f(u_0^h(x)) \leq -Af(u_0^h(x)) \quad \text{in} \quad \Omega. \tag{9}$$

Then, the solution v of (6)–(8) *quenches in a finite time, and an upper bound of its quenching time is $\frac{F(u_{0min}^h)}{A}$.*

Proof. Let T_h be a time up to which v remains strictly positive everywhere. Our aim is to show that T_h is finite and $\frac{F(u_{0min}^h)}{A}$ is one of its upper bound. Introduce the function $J(x,t)$ defined as follows

$$J(x,t) = v_t(x,t) + Af(v(x,t)) \quad \text{in} \quad \overline{\Omega} \times [0, T_h).$$

A straightforward computation yields

$$J_t - \Delta J = (v_t - \Delta v)_t + Af'(v)v_t - A\Delta f(v) \quad \text{in} \quad \Omega \times (0, T_h). \tag{10}$$

Again by a direct calculation, we observe that

$$\Delta f(v) = f''(v)|\nabla v|^2 + f'(v)\Delta v \quad \text{in} \quad \Omega \times (0, T_h),$$

which implies that $\Delta f(v) \geq f'(v)\Delta v$ in $\Omega \times (0, T_h)$. Using this estimate and (10), we arrive at

$$J_t - \Delta J \leq (v_t - \Delta v)_t + Af'(v)(v_t - \Delta v) \quad \text{in} \quad \Omega \times (0, T_h). \tag{11}$$

It follows from (6) and (11) that

$$J_t - \Delta J \leq -f'(v)v_t - Af'(v)f(v) \quad \text{in} \quad \Omega \times (0, T_h).$$

Taking into account the expression of J, we find that

$$J_t - \Delta J \leq -f'(v)J \quad \text{in} \quad \Omega \times (0, T_h).$$

Employing the condition (7), it is not hard to see that

$$\frac{\partial J}{\partial \nu} = \left(\frac{\partial v}{\partial \nu}\right)_t + Af'(v)\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0, T_h),$$

and due to (9), we discover that

$$J(x, 0) = \Delta u_0^h(x) - f(u_0^h(x)) + Af(u_0^h(x)) \leq 0 \quad \text{in} \quad \Omega.$$

We infer from the maximum principle that

$$J(x, t) \leq 0 \quad \text{in} \quad \Omega \times (0, T_h),$$

or equivalently

$$v_t(x, t) + Af(v(x, t)) \leq 0 \quad \text{in} \quad \Omega \times (0, T_h). \tag{12}$$

This estimate may be rewritten in the following manner

$$\frac{\mathrm{d}v}{f(v)} \leq -A\mathrm{d}t \quad \text{in} \quad \Omega \times (0, T_h). \tag{13}$$

Integrate the above inequality over $(0, T_h)$ to obtain

$$T_h \leq \frac{1}{A} \int_{v(x,T_h)}^{v(x,0)} \frac{\mathrm{d}\sigma}{f(\sigma)} \quad \text{for} \quad x \in \Omega. \tag{14}$$

From (12), we observe that v is nonincreasing with respect to the second variable, which implies $0 \leq v(x, T_h) \leq v(x, 0)$ in Ω. It is easy to see that

$$\int_{v(x,T_h)}^{v(x,0)} \frac{\mathrm{d}\sigma}{f(\sigma)} \leq \int_0^{v(x,0)} \frac{\mathrm{d}\sigma}{f(\sigma)} \quad \text{for} \quad x \in \Omega.$$

We deduce from (14) that $T_h \leq \frac{1}{A} \int_0^{u_{0min}^h} \frac{\mathrm{d}\sigma}{f(\sigma)}$ or equivalently

$$T_h \leq \frac{F(u_{0min}^h)}{A}.$$

Consequently, v quenches in a finite time because the quantity on the right hand side of the above inequality is finite. This finishes the proof. □

REMARK 3. Let $t \in (0, T_h)$. Integrating the inequality in (12) from t to T_h, we get

$$T_h - t \leq \frac{1}{A} \int_0^{v(x,t)} \frac{\mathrm{d}\sigma}{f(\sigma)} \quad \text{for} \quad x \in \Omega.$$

We deduce that

$$T_h - t \leq \frac{F(v_{\min}(t))}{A} \quad \text{for} \quad t \in (0, T_h). \quad \square$$

REMARK 4. In view of the condition (5) and reasoning as in the proof of Theorem 2, it is not hard to see that there exists a positive constant C such that $u_{\min}(t) \geq H(C(T-t))$ for $t \in (0,T)$. $\square$

We also need the following result which shows an upper bound of $u_{\min}(t)$ for $t \in (0,T)$.

THEOREM 5. *Let u be solution of* (1)–(3). *Then, the following estimate holds*

$$u_{\min}(t) \leq H(T-t) \quad \textit{for} \quad t \in (0,T).$$

Proof. To prove the above estimate, we proceed as follows. Introduce the function $w(t)$ defined as follows $w(t) = u_{\min}(t)$ for $t \in [0,T)$ and let t_1, $t_2 \in [0,T)$. Then, there exist x_1, $x_2 \in \Omega$ such that $w(t_1) = u(x_1,t_1)$ and $w(t_2) = u(x_2,t_2)$. Applying Taylor's expansion, we observe that

$$w(t_2) - w(t_1) \geq u(x_2,t_2) - u(x_2,t_1) = (t_2-t_1)u_t(x_2,t_2) + o(t_2-t_1),$$

$$w(t_2) - w(t_1) \leq u(x_1,t_2) - u(x_1,t_1) = (t_2-t_1)u_t(x_1,t_1) + o(t_2-t_1),$$

which implies that $w(t)$ is Lipschitz continuous. Further, if $t_2 > t_1$, then

$$\frac{w(t_2)-w(t_1)}{t_2-t_1} \geq u_t(x_2,t_2) + o(1) = \Delta u(x_2,t_2) - f(u(x_2,t_2)) + o(1).$$

Obviously, it is not hard to see that $\Delta u(x_2,t_2) \geq 0$. Letting $t_1 \to t_2$, we obtain $w'(t) \geq -f(w(t))$ for a.e. $t \in (0,T)$ or equivalently $\frac{\mathrm{d}w}{f(w)} \geq -\mathrm{d}t$ for a.e $t \in (0,T)$. Integrate the above inequality over (t,T) to obtain $T-t \geq \int_0^{w(t)} \frac{\mathrm{d}\sigma}{f(\sigma)}$ for $t \in (0,T)$. Since $w(t) = u_{\min}(t)$, we arrive at $u_{\min}(t) \leq H(T-t)$ for $t \in (0,T)$ and the proof is complete. $\square$

REMARK 6. Regarding the last part of the proof of Theorem 5, one sees that $T \geq \int_0^{u_{0min}} \frac{\mathrm{d}\sigma}{f(\sigma)}$. Thus, we have a lower bound of the quenching time of the solution u of (1)–(3). In the same way, it is not hard to see that $\int_0^{u^h_{0min}} \frac{\mathrm{d}\sigma}{f(\sigma)}$ is a lower bound of the quenching time of the solution v of (4)–(6). $\square$

3. CONTINUITY OF THE QUENCHING TIME

This section is dedicated to our main result. Our aim consists in proving that, if h is small enough, then the solution v of (6)–(8) quenches in a finite time and its quenching time T_h goes to T as h tends to zero. We also provide an upper bound of $|T_h - T|$ in terms of $\|u_0^h - u_0\|_\infty$. Our result regarding the continuity of the quenching time is stated in the following theorem.

Theorem 7. *Suppose that the problem* (1)–(3) *has a solution* u *which quenches at the time* T. *Then, under the assumption of Theorem* 2, *the solution* v *of* (6)–(8) *quenches in a finite time* T_h, *and there exist positive constants* α *and* γ *such that for* h *small enough, the following estimate holds*

$$|T_h - T| \leq \alpha \|u_0^h - u_0\|_\infty^\gamma.$$

Proof. We know from Theorem 2 that the solution v quenches in a finite time T_h. Now, to achieve our objective, it remains to demonstrate the above estimate. We begin by proving that $T_h \geq T$. In order to obtain this result, we proceed as follows. Since $u_0^h(x) \geq u_0(x)$ in Ω, we know from the maximum principle that $v \geq u$ as long as all of them are defined. This implies that $T_h \geq T$, and consequently, we have $T_h - T = |T_h - T|$. In order to show the remaining part of the proof, we proceed by introducing the error function $e(x,t)$ defined as follows

$$e(x,t) = v(x,t) - u(x,t) \quad \text{in} \quad \overline{\Omega} \times [0,T).$$

Let t_0 be any positive quantity satisfying $t_0 < T$. A routine computation reveals that

$$e_t - \Delta e = -f'(\theta)e \quad \text{in} \quad \Omega \times (0,t_0),$$

$$\tfrac{\partial e}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0,t_0),$$

$$e(x,0) = u_0^h(x) - u_0(x) \quad \text{in} \quad \overline{\Omega},$$

where θ is an intermediate value between u and v. Due to the fact that $v(x,t) \geq u(x,t)$ in $\Omega \times (0,t_0)$, then making use of Remark 6, it is easy to check that

$$\theta(x,t) \geq u_{\min}(t) \geq H(C(T-t)) \quad \text{in} \quad \Omega \times (0,t_0), \tag{15}$$

which implies that

$$e_t \leq \Delta e - f'(H(C(T-t)))e \quad \text{in} \quad \Omega \times (0,t_0).$$

In view of the condition (4), we observe that there exists a positive constant C_1 such that

$$e_t \leq \Delta e + \tfrac{C_1}{T-t}e \quad \text{in} \quad \Omega \times (0,t_0).$$

Let $Z(t)$ be the solution of the following ODE

$$Z'(t) = \tfrac{C_1 Z(t)}{T-t} \quad \text{for} \quad t \in (0,t_0), \quad Z(0) = \|u_0^h - u_0\|_\infty.$$

When we solve the above ODE, we observe that its solution $Z(t)$ is given explicitly by

$$Z(t) = T^{C_1}\|u_0^h - u_0\|_\infty (T-t)^{-C_1} \quad \text{for} \quad t \in [0,t_0).$$

On the other hand, an application of the maximum principle renders

$$e(x,t) \leq Z(t) = C_2\|u_0^h - u_0\|_\infty (T-t)^{-C_1} \quad \text{in} \quad \Omega \times [0,t_0),$$

where $C_2 = T^{C_1}$. Fix a a positive constant and let $t_1 \in (0, T)$ be a time such that $\|e(\cdot, t_1)\|_\infty \leq C_2 \|u_0^h - u_0\|_\infty (T - t_1)^{-C_1} = a$ for h small enough. This implies that

$$T - t_1 = \left(\frac{C_2 \|u_0^h - u_0\|_\infty}{a}\right)^{\frac{1}{C_1}}. \tag{16}$$

Making use of Remark 3 and the triangle inequality, it is easy to see that

$$|T_h - t_1| \leq \frac{F(v_{\min}(t_1))}{A} \leq \frac{F(u_{\min}(t_1) + \|e(\cdot, t_1)\|_\infty)}{A}.$$

Since $\|e(\cdot, t_1)\|_\infty \leq a$ and due to the fact that the function $F : [0, \infty) \to [0, \infty)$ is increasing, we infer from Theorem 5 that

$$|T_h - t_1| \leq \frac{1}{A} F(H(T - t_1) + a). \tag{17}$$

Having in mind that H is the inverse of F, we deduce that $H : [0, \infty) \to [0, \infty)$ is also increasing. We recall that $\lim_{s\to\infty} F(s) = \infty$, which implies that $\lim_{s\to\infty} H(s) = \infty$. Introduce the function φ defined as follows

$$\varphi(x) = H(x(T - t_1)), \quad x \in [0, \infty).$$

It is clear that $\varphi(x)$ is increasing for $x \in [0, \infty)$. In addition $\lim_{x\to\infty} \varphi(x) = \infty$. According to the fact that $\varphi(1) + a$ belongs to $(0, \infty)$, we conclude that there exists a positive constant C_3 such that $\varphi(1) + a \leq \varphi(C_3)$, which implies that $H(T - t_1) + a \leq H(C_3(T - t_1))$. Recalling that $F(x)$ is increasing for $x \in [0, \infty)$, we deduce that $\frac{1}{A} F(H(T - t_1) + a) \leq \frac{1}{A} F(H(C_3(T - t_1)))$. Exploiting the above inequality and (17), we find that

$$|T_h - t_1| \leq \frac{1}{A} F(H(C_3(T - t_1))) = \frac{C_3}{A} |T - t_1|. \tag{18}$$

We deduce from (18) and the triangle inequality that

$$|T - T_h| \leq |T - t_1| + |T_h - t_1| \leq |T - t_1| + \frac{C_3}{A} |T - t_1|,$$

which leads us to

$$|T - T_h| \leq (1 + \frac{C_3}{A}) |T - t_1|.$$

Use the equality (16) to complete the rest of the proof. □

4. NUMERICAL RESULTS

In this section, we give some computational experiments to confirm the theory given in the previous section. We consider the radial symmetric solution of the following initial-boundary value problem

$$u_t = \Delta u - u^{-p} \quad \text{in} \quad B \times (0, T),$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad S \times (0, T),$$

$$u(x, 0) = u_0(x) \quad \text{in} \quad \overline{B},$$

where p is a positive constant, $u_0(x) = 4 + 3\cos(\pi\|x\|) + \frac{\varepsilon}{2+\cos(\pi\|x\|)}$, with ε a nonnegative parameter, $B = \{x \in \mathbb{R}^N;\ \|x\| < 1\}$, $S = \{x \in \mathbb{R}^N; \|x\| = 1\}$, $1 \le N \le 3$. The above problem may be rewritten in the following form

$$u_t = u_{rr} + \frac{N-1}{r}u_r - u^{-p}, \quad r \in (0,1), \quad t \in (0,T), \tag{19}$$

$$u_r(0,t) = 0, \quad u_r(1,t) = 0, \quad t \in (0,T), \tag{20}$$

$$u(r,0) = \varphi(r), \quad r \in [0,1], \tag{21}$$

where we take $\varphi(r) = 4 + 3\cos(\pi r) + \frac{\varepsilon}{2+cos(\pi r)}$. We start by the construction of an adaptive scheme as follows. Let I be a positive integer and let $h = 1/I$. Define the grid $x_i = ih$, $0 \le i \le I$, and approximate the solution u of (19)–(21) by the solution $U_h^{(n)} = (U_0^{(n)}, ..., U_I^{(n)})^T$ of the following explicit scheme

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = N\frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} - (U_0^{(n)})^{-p},$$

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \frac{(N-1)}{ih}\frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h} - (U_i^{(n)})^{-p},\ 1 \le i \le I-1,$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2} + (N-1)\frac{U_I^{(n)} - U_{I-1}^{(n)}}{h} - (U_I^{(n)})^{-p},$$

$$U_i^{(0)} = \varphi_i,\ 0 \le i \le I.$$

After a little transformation, the above equations become

$$U_0^{(n+1)} \ge (1 - 2\frac{N\Delta t_n}{h^2} - \Delta t_n (U_{hmin}^{(n)})^{-p-1})U_0^{(n)} + 2\frac{N\Delta t_n}{h^2}U_1^{(n)},$$

$$U_i^{(n+1)} \ge (\frac{\Delta t_n}{h^2} - \frac{(N-1)}{ih}\frac{\Delta t_n}{2h})U_{i-1}^{(n)} + (1 - 2\frac{\Delta t_n}{h^2} - \Delta t_n(U_{hmin}^{(n)})^{-p-1})U_i^{(n)}$$
$$+ (\frac{\Delta t_n}{h^2} + \frac{(N-1)}{ih}\frac{\Delta t_n}{2h})U_{i+1}^{(n)}, \quad 1 \le i \le I-1,$$

$$U_I^{(n+1)} \ge (2\frac{\Delta t_n}{h^2} - (N-1)\frac{\Delta t_n}{h})U_{I-1}^{(n)}$$
$$+ (1 - 2\frac{\Delta t_n}{h^2} + (N-1)\frac{\Delta t_n}{h} - \Delta t_n(U_{hmin}^{(n)})^{-p-1})U_I^{(n)},$$

with $U_{hmin}^{(n)} = \min_{0\le i\le I} U_i^{(n)}$. Let us notice that, if $\Delta t_n \le \frac{h^2(U_{hmin}^{(n)})^{p+1}}{h^2+2N(U_{hmin}^{(n)})^{p+1}}$, then one easily sees by induction that $U_{hmin}^{(n)}$ is positive for $n \ge 0$. Thus, the above condition is the CFL condition that ensures the stability of our scheme. It is important to note that if one chooses $\Delta t_n = \min\{\frac{(1-h^2)h^2}{2N}, h^2(U_{hmin}^{(n)})^{p+1}\}$, then the above CFL condition is fulfilled. Consequently, in the sequel, we shall pick the above time step for our explicit scheme. An important fact concerning the phenomenon of quenching is that, if the solution u quenches at the time T, then when the time t approaches the quenching time T, the solution u decreases to zero rapidly. Thus, in order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches the quenching time T, we need to adapt the size of the time step. This is the reason why we have chosen the above time step. For this time step, our explicit scheme becomes an adaptive scheme which is one of suitable schemes for problems whose solutions quench in a finite time. We also approximate the solution u of (19)-(21) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\frac{U_0^{(n+1)}-U_0^{(n)}}{\Delta t_n} = N\frac{2U_1^{(n+1)}-2U_0^{(n+1)}}{h^2} - (U_0^{(n)})^{-p-1}U_0^{(n+1)}$$
$$\frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n+1)}-2U_i^{(n+1)}+U_{i-1}^{(n+1)}}{h^2} + \frac{(N-1)}{ih}\frac{U_{i+1}^{(n+1)}-U_{i-1}^{(n+1)}}{2h}$$
$$- (U_i^{(n)})^{-p-1}U_i^{(n+1)},\ 1 \le i \le I-1,$$
$$\frac{U_I^{(n+1)}-U_I^{(n)}}{\Delta t_n} = \frac{2U_{I-1}^{(n+1)}-2U_I^{(n+1)}}{h^2} + (N-1)\frac{U_I^{(n+1)}-U_{I-1}^{(n+1)}}{h} - (U_I^{(n)})^{-p-1}U_I^{(n+1)},$$
$$U_i^{(0)} = \varphi_i,\ 0 \le i \le I.$$

As in the case of the explicit scheme, here again, we transform our scheme to an adaptive scheme by choosing $\Delta t_n = h^2(U_{hmin}^{(n)})^{p+1}$. The implicit scheme gives the following equations

$$(1 + 2\tfrac{N\Delta t_n}{h^2} + \Delta t_n(U_0^{(n)})^{-p-1})U_0^{(n+1)} - 2\tfrac{N\Delta t_n}{h^2}U_1^{(n+1)} = U_0^{(n)}, \tag{22}$$

$$\begin{aligned}&(\tfrac{(N-1)}{ih}\tfrac{\Delta t_n}{2h} - \tfrac{\Delta t_n}{h^2})U_{i-1}^{(n+1)} + (1 + 2\tfrac{\Delta t_n}{h^2} + \Delta t_n(U_i^{(n)})^{-p-1})U_i^{(n+1)} - \\ &- (\tfrac{\Delta t_n}{h^2} + \tfrac{(N-1)}{ih}\tfrac{\Delta t_n}{2h})U_{i+1}^{(n+1)} = U_i^{(n)},\ 1 \le i \le I-1,\end{aligned} \tag{23}$$

$$\begin{aligned}&((N-1)\tfrac{\Delta t_n}{h} - 2\tfrac{\Delta t_n}{h^2})U_{I-1}^{(n+1)} + \\ &+ (1 + 2\tfrac{\Delta t_n}{h^2} - (N-1)\tfrac{\Delta t_n}{h} + \Delta t_n(U_I^{(n)})^{-p-1})U_I^{(n+1)} = U_I^{(n)}.\end{aligned} \tag{24}$$

The above scheme leads us to the following tridiagonal linear system

$$A_h^{(n)}U_h^{(n+1)} = U_h^{(n)},$$

where $A_h^{(n)}$ is a $(I+1)\times(I+1)$ tridiagonal matrix defined as follows

$$A_h^{(n)} = \begin{pmatrix} a_0 & b_0 & 0 & \cdots & 0 \\ c_1 & a_1 & b_1 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & b_{I-1} \\ 0 & \cdots & 0 & c_I & a_I \end{pmatrix}$$

with

$$
\begin{aligned}
a_0 &= 1 + 2\tfrac{N\Delta t_n}{h^2} + \Delta t_n (U_0^{(n)})^{-p-1},\\
a_I &= 1 + 2\tfrac{\Delta t_n}{h^2} - (N-1)\tfrac{\Delta t_n}{h} + \Delta t_n (U_I^{(n)})^{-p-1},\\
a_i &= 1 + 2\tfrac{\Delta t_n}{h^2} + \Delta t_n (U_i^{(n)})^{-p-1},\ i = 1, ..., I-1,\\
b_0 &= -2\tfrac{N\Delta t_n}{h^2}, \quad b_i = -(\tfrac{\Delta t_n}{h^2} + \tfrac{(N-1)}{ih}\tfrac{\Delta t_n}{2h}),\ i = 1, ..., I-1,\\
c_i &= (\tfrac{(N-1)}{ih}\tfrac{\Delta t_n}{2h} - \tfrac{\Delta t_n}{h^2}),\ i = 1, ..., I-1,\\
c_I &= -2\tfrac{\Delta t_n}{h^2} + (N-1)\tfrac{\Delta t_n}{h}.
\end{aligned}
$$

It is not hard to see that

$$(A_h^n)_{ii} > 0, \quad (A_h^n)_{ij} \le 0, \quad i \neq j, \quad (A_h^n)_{ii} > \sum_{i\neq j} |(A_h^n)_{ij}|.$$

These inequalities imply that the matrix $A_h^{(n)}$ is invertible and its inverse $(A_h^{(n)})^{-1}$ is a positive matrix. Consequently, it is easy to see that, if $U_h^{(n)}$ is positive, then $U_h^{(n+1)}$ exists and is also positive. Thus, since $U_h^{(0)} = \varphi_h$ is positive, we show by induction that $U_h^{(n)}$ exists and is positive. It is not hard to see that $u_{rr}(0,t) = \lim_{r\to 0} \frac{u_r(r,t)}{r}$. Hence, if $r = 0$, then we see that

$$u_t(0,t) = N u_{rr}(0,t) - (u(0,t))^{-p}, \quad t \in (0,T),$$

These observations have been taken into account in the construction of our schemes at the first node. We need the following definition.

Definition 8. *We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme quenches in a finite time if* $\lim_{n\to\infty} U_{hmin}^{(n)} = 0$, *and the series $\sum_{n=0}^{\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{\infty} \Delta t_n$ is called the numerical quenching time of the discrete solution $U_h^{(n)}$.*

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical quenching time $t_n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when $U_{hmin}^{(n)} \le 10^{-10}$. The order (s) of the method is computed from

$$s = \frac{\log((t_{2h} - t_h)/(t_{4h} - t_{2h}))}{\log(2)}.$$

Numerical experiments for $p = 1, N = 2$

First case: $\varepsilon = 0$

I	t_n	n	CPU time	s
16	3.604286	5415	12	-
32	3.731558	21476	71	-
64	3.796654	84141	523	0.97
128	3.828011	335561	3782	1.04

Table 1. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

I	t_n	n	CPU time	s
16	3.604107	5325	13	-
32	3.731511	21121	87	-
64	3.796641	84721	1106	0.97
128	3.830302	331834	7718	0.95

Table 2. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method.

Second case: $\varepsilon = 1/50$

I	t_n	n	CPU time	s
16	3.642767	5453	12	-
32	3.770560	21626	55	-
64	3.835916	84784	557	0.96
128	3.870205	335873	3684	0.93

Table 3. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

I	t_n	n	CPU time	s
16	3.642591	5453	15	-
32	3.770514	21636	89	-
64	3.835904	84784	972	0.97
128	3.870181	335867	7821	0.93

Table 4. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method.

Third case: $\varepsilon = 1/100$

Remark 9. If we consider the problem (19)-(21) in the case where $p = 1$ and the initial datum $\varphi(r) = 4 + 3\cos(\pi r) + \frac{\varepsilon}{2+\cos(\pi r)}$ with $\varepsilon = 0$, then we see that the numerical quenching time of the discrete solution for the explicit scheme or the implicit scheme is approximately equal to that in which the initial datum increases slightly, that is when ε is a small positive real (see, Tables 1-6 for an illustration). This result confirms the theory regarding the continuity of the quenching time as a function of the initial datum. □

In what follows, we give some plots to illustrate our analysis. In Figures 1, 2 and 3 we can appreciate that the discrete solution quenches in a finite time.

I	t_n	n	CPU time	s
16	3.623502	5433	12	-
32	3.751034	21556	60	-
64	3.816361	84462	492	0.97
128	3.850163	333798	3554	0.95

Table 5. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

I	t_n	n	CPU time	s
16	3.623324	5433	12	-
32	3.750988	21556	69	-
64	3.816249	84462	1034	0.97
128	3.849163	332769	7763	0.959

Table 6. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method.

We also remark that the representation of the discrete solution when $\varepsilon = 0$ is practically the same that the one when $\varepsilon = 1/50$ or $\varepsilon = 1/100$.

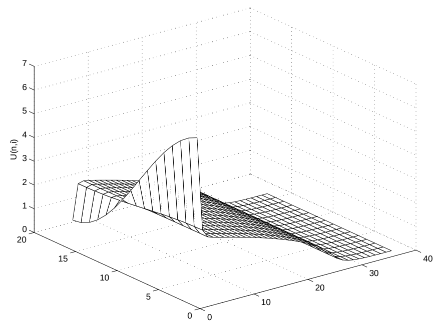

Fig. 1. Evolution of the discrete solution, $\varepsilon = 0$

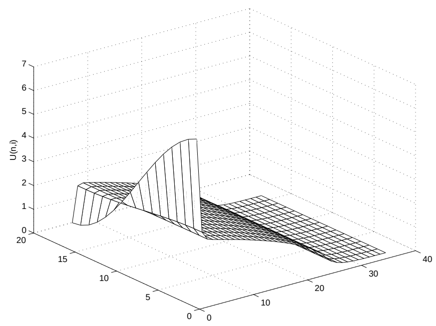

Fig. 2. Evolution of the discrete solution, $\varepsilon = 1/50$

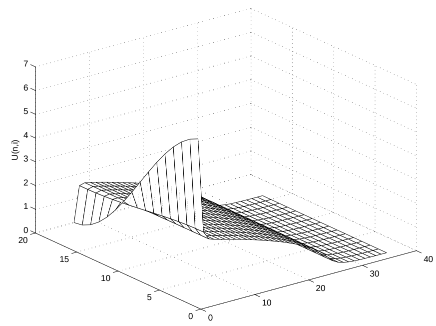

Fig. 3. Evolution of the discrete solution, $\varepsilon = 1/100$

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