Abstract. In this article we study the Euler’s iterative method. For this method we give a global theorem of convergence. In the last section of the paper we give a numerical example which illustrates the result exposed in this work.


Keywords. Euler’s method, fixed point, one-point iteration method.

1. INTRODUCTION

We consider the problem of finding a zero of the equation

\[ f(x) = 0, \]

where \( f : [a, b] \subset \mathbb{R} \to \mathbb{R} \) is an analytic function with simple roots. This zero can be determined as a fixed point of some iteration functions \( g : [a, b] \to [a, b] \), by means of the one-point iteration method

\[ x_{n+1} = g(x_n), \quad x_0 \in [a, b], \quad n = 0, 1, ..., \quad n \in \mathbb{N}, \]

where \( x_0 \) is the starting value and \( g \) is a function of form

\[ g(x) = x + \varphi(x). \]

In this article we analyze the Euler’s method for approximating the solution \( x^* \in [a, b] \) of the equation (1). This method is defined by the relation

\[ x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + \sqrt{|f'(x_n)|^2 - 2f(x_n)f''(x_n)}}, \quad x_0 \in [a, b], \quad n \geq 0, \quad n \in \mathbb{N}. \]

The Euler’s method has been rediscovered by several authors, see for example [1], [2], [5], [6], [7], [8], and references therein.

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2. THEOREMS OF CONVERGENCE

Next we will study sufficient conditions in order that the sequence \( \{x_n\}_{n \geq 0} \) generated through (3) would be convergent, and if \( x^* = \lim_{n \to \infty} x_n \), then \( f(x^*) = 0 \).

In order to prove the convergence of the method of form (3), we would use the next result.

**Theorem 1.** ([3], [4]) If we consider the function \( f \), the real number \( \delta > 0 \) and \( x_0 \in \Delta \), where \( \Delta = \{ x \in \mathbb{R} : |x - x_0| \leq \delta \} \subseteq [a, b] \), we could assure that the following relations hold

\[
\text{a) the function } f \text{ is of class } C^s(\Delta), \ s \geq 2, \ s \in \mathbb{N} \text{ and } \sup_{x \in \Delta} |f^{(s)}(x)| = M < \infty; \\
\text{b) we have the relation } \left| \sum_{i=0}^{s-1} \frac{1}{n!} f^{(i)}(x)\varphi^i(x) \right| \leq \gamma |f(x)|^s \text{ for every } x \in \Delta, \ \text{where } \gamma \in \mathbb{R}, \ \gamma \geq 0; \\
\text{c) the function } \varphi \text{ verifies the relation } |\varphi(x)| \leq \eta |f(x)|, \text{ for every } x \in \Delta, \ \text{where } \eta \in \mathbb{R}, \ \eta > 0; \\
\text{d) the numbers } \lambda, \eta, M \text{ and } \delta \text{ verify the relations:} \\
\mu_0 = \lambda |f(x_0)| < 1, \ \text{where } \lambda = \left( \gamma + \frac{M\eta^s}{s!} \right)^{\frac{1}{s}} \text{ and } \frac{\mu_0}{\lambda(1-\mu_0)} \leq \delta; \\
\text{then the sequence } \{x_n\}_{n \geq 0} \text{ generated by (2) has the following properties:} \\
i) it is convergent, and if } x^* = \lim_{n \to \infty} x_n \text{ then } f(x^*) = 0 \text{ and } x^* \in \Delta; \\
\text{ii) } |x_{n+1} - x_n| \leq \frac{\mu_0^n}{\lambda^n}, \text{ for any } n = 0, 1, ..., n \in \mathbb{N}; \\
\text{iii) } |x^* - x_n| \leq \frac{\mu_0^n}{\lambda^n}, \text{ for every } n = 0, 1, ..., n \in \mathbb{N}. \\
\text{Proof. See [3, [4].}
\]

Based on Theorem 1, in our next result we would analyze the convergence of sequence \( \{x_n\}_{n \geq 0} \) given by (3).

**Theorem 2.** If the function \( f \), the real number \( \delta > 0 \) and \( x_0 \in \Delta \), where \( \Delta = \{ x \in \mathbb{R} : |x - x_0| \leq \delta \} \subseteq [a, b] \), verify the relations

\[
\text{a) the function } f \text{ is of class } C^3(\Delta) \text{ and } \sup_{x \in \Delta} |f^{(3)}(x)| = M < \infty; \\
\text{b) } \left| \frac{1}{f(x)} \right| \leq \beta, \text{ for every } x \in \Delta, \ \beta \in \mathbb{R}, \ \beta > 0; \\
\text{c) } \frac{f(x)f''(x)}{f'(x)^2} = L_f(x) \leq \frac{1}{2}, \text{ for every } x \in \Delta; \\
\text{d) } \lambda = \sqrt{\frac{8M}{3\beta^3}} > 0; \\
\text{e) } \mu_0 = \lambda |f(x_0)| < 1; \\
\text{f) } \frac{2\beta\mu_0}{\lambda(1-\mu_0)} \leq \delta; \\
\text{then the sequence } \{x_n\}_{n \geq 0} \text{ generated by (3) is convergent, and if } x^* = \lim_{n \to \infty} x_n, \\
\text{the next relations hold}
\]
We’ll show that the elements of the sequence \( \{ x_n \} \) hold for every

\( |f(x_n)| \leq \frac{\mu_0^n}{\lambda(1-\mu_0^n)}, \quad n = 0, 1, 2, ..., n \in \mathbb{N}; \)

\( |x^* - x_n| \leq \frac{2M\beta^n}{\lambda}, \quad n = 0, 1, 2, ..., n \in \mathbb{N}. \)

**Proof.** We consider the function \( \varphi \) of form

\[
\varphi(x) = \frac{-2f(x)}{f'(x)+\sqrt{f'(x)^2-2f(x)f''(x)}}, \quad x \in \Delta.
\]

We’ll show that the elements of the sequence \( \{ x_n \} \) generated by (3) are in \( \Delta \).

By conditions b), c) and f) we have

\[
|x_1 - x_0| = \left| \frac{f(x_0)}{f'(x_0)} \right| \left| \frac{2}{1+\sqrt{1-2f'(x_0)^2}} \right| \leq 2 \left| \frac{f(x_0)}{f'(x_0)} \right| \leq 2\beta |f(x_0)| = \frac{2\lambda\beta f(x_0)}{\lambda} < \frac{2\beta\mu_0}{1-\mu_0} \leq \delta \Rightarrow x_1 \in \Delta.
\]

Applying the Taylor expansion of function \( f \) around \( x_0 \) and taking into account that

\[
\varphi(x) = \frac{-f'(x)+\sqrt{f'(x)^2-2f(x)f''(x)}}{f'(x)} = \frac{-f'(x)-\sqrt{f'(x)^2-2f(x)f''(x)}}{f''(x)},
\]

\( \forall x \in \Delta \), and \( \varphi(x) \) is verifying the parable

\[
a x^2 + bx + c = 0, \quad a = f(x), b = f'(x), a = f''(x),
\]

we get

\[
|f(x_1)| \leq |f(x_1) - (f(x_0) + f'(x_0)(x_1 - x_0) + \frac{1}{2}f''(x_0)(x_1 - x_0)^2)| + |f(x_0) + f'(x_0)(x_1 - x_0) + \frac{1}{2}f''(x_0)(x_1 - x_0)^2| \leq \frac{|f''(x)|}{3}(x_1 - x_0)^3 + |f(x_0) + f'(x_0)| \varphi(x_0) + \frac{1}{2}f''(x_0)\varphi^2(x_0) \leq \frac{M}{3} |x_1 - x_0|^3 \leq \frac{8M\beta^3}{3} |f(x_0)|^3 = \frac{\mu_0^3}{\lambda}, \quad \xi \in \Delta.
\]

Because \( \left| \frac{1}{f'(x_1)} \right| \leq \beta \), we have that

\[
|x_2 - x_1| = \left| \frac{f(x_1)}{f'(x_1)} \right| \left| \frac{2}{1+\sqrt{1-2f'(x_1)f''(x_1)}} \right| \leq 2 \left| \frac{f(x_1)}{f'(x_1)} \right| \leq 2\beta |f(x_1)| \leq \frac{2\beta\mu_0}{\lambda}.
\]

From all that we have proved above, by using the induction, it results that the property iii) holds for every \( n \in \mathbb{N} \),

(4) \[ |f(x_n)| \leq \frac{\mu_0^n}{\lambda}, \]
Analogously, from b), c) and (4) we can prove the following relation

\[ |x_{n+1} - x_n| = \left| \frac{\frac{f(x_n)}{f'(x_n)}}{1 + \sqrt{1 - 2\frac{f(x_n)f''(x_n)}{f'(x_n)^2}}} \right| \leq \frac{2\beta \mu_0^a}{\lambda}, \quad n = 0, 1, \ldots, n \in \mathbb{N}. \]

From (5), e) and f) we get the relation ii)

\[ |x_{n+1} - x_0| \leq \sum_{i=0}^{n} |x_{i+1} - x_i| \leq \sum_{i=n}^{n+p-1} \frac{2\beta \mu_0^a}{\lambda} < \frac{2\beta \mu_0^a}{\lambda}(1 + \mu_0^{3-1} + \mu_0^{2-1} + \ldots + \mu_0^{a-1}) < \frac{2\beta \mu_0^a}{\lambda(1-\mu_0)}, \quad p \in \mathbb{N}, \quad n = 0, 1, 2, \ldots, n \in \mathbb{N}. \]

For the convergence of the sequence given by (3) we shall use the Cauchy’s theorem. By relation (5) and e) we deduce that

\[ |x^* - x_n| \leq \frac{2\beta \mu_0^a}{\lambda(1-\mu_0)}, \quad n = 0, 1, 2, \ldots, n \in \mathbb{N}. \]

We show now that the relations i) hold, that is, \(x^*\) is a root of equation (1) and \(x^* \in \Delta\).

From the continuity of function \(f\) and from (4) for \(n \to \infty\), it results

\[ 0 \leq |f(x^*)| \leq \lim_{n \to \infty} \frac{\mu_0^a}{\lambda} = 0, \]

that is, \(f(x^*) = 0\).

From f) and the inequality (8) for \(n = 0\), we obtain

\[ |x^* - x_0| \leq \frac{2\beta \mu_0^{a_0}}{\lambda(1-\mu_0^{a_0})} \leq \delta, \]

so, \(x^* \in \Delta\). □

It is evidently that all the assumptions of Theorem 1 are verified for \(s = 3\), \(\gamma = 0\) and \(\eta = 2\beta\).
3. Numerical Example

We shall present a numerical example, which illustrates the result exposed in Theorem 2.

Example 3. We used the following test functions and display the zeros $x^*$ found.

$$f_1(x) = \ln(x^2 - 3), \quad x^* = -2, \quad x \in [-2.35, -1.9],$$

$$f_2(x) = x^3 - 3x^2 - 13x + 15, \quad x^* = 5, \quad x \in [4.5, 5.5],$$

$$f_3(x) = x^5 - 1, \quad x^* = 1, \quad x \in [0.88, 1.38].$$

For the derivatives of order 1, 2 and 3 of $f_i, \ i = 1, 2, 3$, we have the relations

$$f_1'(x) = \frac{2x}{3 + x^2}, \quad f_1''(x) = \frac{-4x^2}{(3 + x^2)^2} + \frac{2}{3 + x^2},$$

$$f_1'''(x) = \frac{16x^3}{(3 + x^2)^3} - \frac{12x}{(3 + x^2)^2},$$

from which we get $\beta = 0.536702$ and $M = 422.22$;

$$f_2'(x) = 3x^2 - 6x - 13, \quad f_2''(x) = 6x - 6, \quad f_2'''(x) = 6,$$

from which we get $\beta = 0.0481928$ and $M = 6$;

$$f_3'(x) = 5x^4, \quad f_3''(x) = 20x^3, \quad f_3'''(x) = 60x^2,$$

from which we get $\beta = 0.333503$ and $M = 114.264$.

In the Table 1 are listed the values for $x_0, M, \beta, \lambda, \mu_0, \delta$ and $\frac{2\beta\mu_0}{\lambda(1-\mu_0)}$, for each test functions.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$x_0$</th>
<th>$M$</th>
<th>$\beta$</th>
<th>$\lambda$</th>
<th>$\mu_0$</th>
<th>$\delta$</th>
<th>$\frac{2\beta\mu_0}{\lambda(1-\mu_0)} &lt; \delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1.989</td>
<td>422.22</td>
<td>0.53670</td>
<td>9.32908</td>
<td>0.41860</td>
<td>0.089</td>
<td>0.08284</td>
</tr>
<tr>
<td>2</td>
<td>4.875</td>
<td>6</td>
<td>0.04819</td>
<td>0.02992</td>
<td>0.11414</td>
<td>0.625</td>
<td>0.41503</td>
</tr>
<tr>
<td>3</td>
<td>1.035</td>
<td>114.264</td>
<td>0.33350</td>
<td>2.37724</td>
<td>0.33873</td>
<td>0.353</td>
<td>0.14372</td>
</tr>
</tbody>
</table>

Table 1.

The implementations were done in Mathematica 7.0 with double precision.

From the Table 1 we can conclude that all the assumptions a)–f) of Theorem 2 are verified.

In the next Table 2 we can observe that, the convergence is faster and the method (3) converges at $x^*$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$x_0$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3 = x^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1.989</td>
<td>-2.0000063482540900</td>
<td>-1.9999999999999990</td>
<td>-2</td>
</tr>
<tr>
<td>2</td>
<td>4.875</td>
<td>5.00006123335144</td>
<td>4.9999999999999993</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>1.027</td>
<td>0.9999958832170524</td>
<td>1.0000000000000139</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2.
REFERENCES


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