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THE CONVERGENCE OF THE EULER'S METHOD

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Abstract. In this article we study the Euler's iterative method. For this method we give a global theorem of convergence. In the last section of the paper we give a numerical example which illustrates the result exposed in this work.

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1. INTRODUCTION

We consider the problem of finding a zero of the equation

$$f(x) = 0, \tag{1}$$

where $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is an analytic function with simple roots. This zero can be determined as a fixed point of some iteration functions $g : [a, b] \to [a, b]$, by means of the one-point iteration method

$$x_{n+1} = g(x_n),\ x_0 \in [a, b],\ n = 0, 1, ...,\ n \in \mathbb{N}, \tag{2}$$

where x_0 is the starting value and g is a function of form

$$g(x) = x + \varphi(x).$$

In this article we analyze the Euler's method for approximating the solution $x^* \in [a, b]$ of the equation (1). This method is defined by the relation

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)+\sqrt{[f'(x_n)]^2-2f(x_n)f''(x_n)}},\ x_0 \in [a, b],\ n \geq 0,\ n \in \mathbb{N}. \tag{3}$$

The Euler's method has been rediscovered by several authors, see for example [1], [2], [5], [6], [7], [8], and references therein.

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2. THEOREMS OF CONVERGENCE

Next we will study sufficient conditions in order that the sequence $\{x_n\}_{n\geq 0}$ generated through (3) would be convergent, and if $x^* = \lim_{n\to\infty} x_n$, then $f(x^*) = 0$.

In order to prove the convergence of the method of form (3), we would use the next result.

THEOREM 1. ([3], [4]) *If we consider the function f, the real number $\delta > 0$ and $x_0 \in \Delta$, where $\Delta = \{x \in \mathbb{R} : |x - x_0| \leq \delta\} \subseteq [a, b]$, we could assure that the following relations hold*

a) *the function f is of class $C^s(\Delta)$, $s \geq 2$, $s \in \mathbb{N}$ and $\sup_{x\in\Delta} \left|f^{(s)}(x)\right| = M < \infty$;*

b) *we have the relation $\left|\sum_{i=0}^{s-1} \frac{1}{i!} f^{(i)}(x)\varphi^i(x)\right| \leq \gamma \left|f(x)\right|^s$ for every $x \in \Delta$, where $\gamma \in \mathbb{R}, \gamma \geq 0$;*

c) *the function φ verifies the relation $|\varphi(x)| \leq \eta\, |f(x)|$, for every $x \in \Delta$, where $\eta \in \mathbb{R}, \eta > 0$;*

d) *the numbers λ, η, M and δ verify the relations:*

$$\mu_0 = \lambda\, |f(x_0)| < 1, \text{ where } \lambda = \left(\gamma + \frac{M\eta^s}{s!}\right)^{\frac{1}{s-1}} \text{ and } \frac{\eta\mu_0}{\lambda(1-\mu_0)} \leq \delta;$$

then the sequence $\{x_n\}_{n\geq 0}$ generated by (2) has the following properties:

i) *it is convergent, and if $x^* = \lim_{n\to\infty} x_n$ then $f(x^*) = 0$ and $x^* \in \Delta$;*

ii) $|x_{n+1} - x_n| \leq \frac{\eta\mu_0^{s^n}}{\lambda}$, *for any* $n = 0, 1, ..., n \in \mathbb{N}$;

iii) $|x^* - x_n| \leq \frac{\eta\mu_0^{s^n}}{\lambda(1-\mu_0^{s^n})}, n = 0, 1, 2, ..., n \in \mathbb{N}$.

Proof. See [3], [4]. □

Based on Theorem 1, in our next result we would analyze the convergence of sequence $\{x_n\}_{n\geq 0}$ given by (3).

THEOREM 2. *If the function f, the real number $\delta > 0$ and $x_0 \in \Delta$, where $\Delta = \{x \in \mathbb{R} : |x - x_0| \leq \delta\} \subseteq [a, b]$, verify the relations*

a) *the function f is of class $C^3(\Delta)$ and $\sup_{x\in\Delta} |f'''(x)| = M < \infty$;*

b) $\left|\frac{1}{f'(x)}\right| \leq \beta$ *for every $x \in \Delta$, $\beta \in \mathbb{R}, \beta > 0$;*

c) $\frac{f(x)f''(x)}{[f'(x)]^2} \overset{not}{=} L_f(x) \leq \frac{1}{2}$ *for every $x \in \Delta$;*

d) $\lambda = \sqrt{\frac{8M}{3!}\beta^3} > 0$;

e) $\mu_0 = \lambda\, |f(x_0)| < 1$;

f) $\frac{2\beta\mu_0}{\lambda(1-\mu_0)} \leq \delta$;

then the sequence $\{x_n\}_{n\geq 0}$ generated by (3) is convergent, and if $x^* = \lim_{n\to\infty} x_n$, the next relations hold

i) $f(x^*) = 0$ *and* $x^* \in \Delta$;
ii) $x_n \in \Delta$, $n = 0, 1, 2..., n \in \mathbb{N}$;
iii) $|f(x_n)| \leq \frac{\mu_0^{3^n}}{\lambda}$, $n = 0, 1, 2, ..., n \in \mathbb{N}$;
iv) $|x^* - x_n| \leq \frac{2\beta\mu_0^{3^n}}{\lambda(1-\mu_0^{3^n})}$, $n = 0, 1, 2, ..., n \in \mathbb{N}$.

Proof. We consider the function φ of form

$$\varphi(x) = -\frac{2f(x)}{f'(x)+\sqrt{[f'(x)]^2-2f(x)f''(x)}}, \; x \in \Delta.$$

We'll show that the elements of the sequence $\{x_n\}_{n\geq 0}$ generated by (3) are in Δ.

By conditions b), c) and f) we have

$$|x_1 - x_0| = \left|\frac{f(x_0)}{f'(x_0)}\right| \left|\frac{2}{1+\sqrt{1-2L_f(x_0)}}\right| \leq 2\left|\frac{f(x_0)}{f'(x_0)}\right| \leq$$
$$\leq 2\beta\,|f(x_0)| = \frac{2\lambda\beta|f(x_0)|}{\lambda} < \frac{2\beta\mu_0}{\lambda(1-\mu_0)} \leq \delta \Rightarrow x_1 \in \Delta.$$

Applying the Taylor expansion of function f around x_0 and taking into account that

$$\varphi(x) = \frac{-f'(x)+\sqrt{[f'(x)]^2-2f(x)f''(x)}}{f''(x)} =$$
$$= -\frac{f'(x)-\sqrt{[f'(x)]^2-2f(x)f''(x)}}{f''(x)},$$

$\forall x \in \Delta$, and $\varphi(x)$ is verifying the parable

$$ax^2 + bx + c = 0, \text{ where } c = f(x), b = f'(x), a = \frac{f''(x)}{2},$$

we get

$$|f(x_1)| \leq \left|f(x_1) - \left(f(x_0) + f'(x_0)(x_1 - x_0) + \tfrac{1}{2}f''(x_0)(x_1 - x_0)^2\right)\right| +$$
$$+ \left|f(x_0) + f'(x_0)(x_1 - x_0) + \tfrac{1}{2}f''(x_0)(x_1 - x_0)^2\right| \leq$$
$$\leq \left|\frac{f'''(\xi)}{3!}(x_1 - x_0)^3\right| + \left|f(x_0) + f'(x_0)\varphi(x_0) + \tfrac{1}{2}f''(x_0)\varphi^2(x_0)\right| \leq$$
$$\leq \frac{M}{3!}\,|x_1 - x_0|^3 \leq \frac{8M\beta^3}{3!}\,|f(x_0)|^3 = \frac{\mu_0^3}{\lambda}, \; \xi \in \Delta.$$

Because $\left|\frac{1}{f'(x_1)}\right| \leq \beta$, we have that

$$|x_2 - x_1| = \left|\frac{f(x_1)}{f'(x_1)}\right| \left|\frac{2}{1+\sqrt{1-2\frac{f(x_1)f''(x_1)}{[f'(x_1)]^2}}}\right| \leq 2\left|\frac{f(x_1)}{f'(x_1)}\right| \leq 2\beta\,|f(x_1)| \leq \frac{2\beta\mu_0^3}{\lambda}.$$

From all that we have proved above, by using the induction, it results that the property iii) holds for every $n \in \mathbb{N}$,

$$|f(x_n)| \leq \frac{\mu_0^{3^n}}{\lambda}. \tag{4}$$

Analogously, from b), c) and (4) we can prove the following relation

$$(5)\qquad |x_{n+1}-x_n| = \left|\tfrac{f(x_n)}{f'(x_n)}\right| \left|\tfrac{2}{1+\sqrt{1-2\frac{f(x_n)f''(x_n)}{[f'(x_n)]^2}}}\right| \le \tfrac{2\beta\mu_0^{3^n}}{\lambda},\ n=0,1,...,n\in\mathbb{N}.$$

From (5), e) and f) we get the relation ii)

$$(6)\qquad |x_{n+1}-x_0| \le \sum_{i=0}^{n} |x_{i+1}-x_i| \le$$
$$\le \sum_{i=0}^{n} \tfrac{2\beta\mu_0^{3^i}}{\lambda} \le \tfrac{2\beta\mu_0}{\lambda}(1+\mu_0^{3-1}+\mu_0^{3^2-1}+...+\mu_0^{3^n-1})$$
$$< \tfrac{2\beta\mu_0}{\lambda(1-\mu_0)} \le \delta \Rightarrow x_{n+1}\in\Delta,\ n=0,1,2,...,n\in\mathbb{N}.$$

For the convergence of the sequence given by (3) we shall use the Cauchy's theorem. By relation (5) and e) we deduce that

$$(7)\qquad |x_{n+p}-x_n| \le \sum_{i=n}^{n+p-1} |x_{i+1}-x_i| \le \sum_{i=n}^{n+p-1} \tfrac{2\beta\mu_0^{3^i}}{\lambda} <$$
$$< \tfrac{2\beta\mu_0^{3^n}}{\lambda}(1+\mu_0^{3^{n+1}-3^n}+...+\mu_0^{3^{n+p-1}-3^n})$$
$$< \tfrac{2\beta\mu_0^{3^n}}{\lambda(1-\mu_0^{3^n})},\ p\in\mathbb{N},\ n=0,1,2,...,n\in\mathbb{N}.$$

Because $\mu_0<1$, it results that the sequence $\{x_n\}_{n\ge 0}$ is fundamental, so according to the Cauchy's theorem, it is convergent.

If $x^* = \lim\limits_{n\to\infty} x_n$, for $p\to\infty$, from the inequality (7) we obtain the relation iv)

$$(8)\qquad |x^*-x_n| \le \tfrac{2\beta\mu_0^{3^n}}{\lambda(1-\mu_0^{3^n})},\ n=0,1,2,...,n\in\mathbb{N}.$$

We show now that the relations i) hold, that is, x^* is a root of equation (1) and $x^*\in\Delta$.

From the continuity of function f and from (4) for $n\to\infty$, it results

$$0 \le |f(x^*)| \le \lim_{n\to\infty} \tfrac{\mu_0^{3^n}}{\lambda} = 0,$$

that is, $f(x^*)=0$.

From f) and the inequality (8) for $n=0$, we obtain

$$|x^*-x_0| \le \tfrac{2\beta\mu_0^{3^0}}{\lambda(1-\mu_0^{3^0})} \le \delta,$$

so, $x^*\in\Delta$. □

It is evidently that all the assumptions of Theorem 1 are verified for $s=3$, $\gamma=0$ and $\eta=2\beta$.

3. NUMERICAL EXAMPLE

We shall present a numerical example, which illustrates the result exposed in Theorem 2.

EXAMPLE 3. We used the following test functions and display the zeros x^* found.

$$f_1(x) = \ln(x^2 - 3),\ x^* = -2,\ x \in [-2.35, -1.9],$$
$$f_2(x) = x^3 - 3x^2 - 13x + 15,\ x^* = 5,\ x \in [4.5, 5.5],$$
$$f_3(x) = x^5 - 1,\ x^* = 1,\ x \in [0.88, 1.38].$$

For the derivatives of order 1, 2 and 3 of $f_i,\ i = 1, 2, 3$, we have the relations

$$f_1'(x) = \frac{2x}{-3+x^2},\ f_1''(x) = \frac{-4x^2}{(-3+x^2)^2} + \frac{2}{-3+x^2},$$
$$f_1'''(x) = \frac{16x^3}{(-3+x^2)^3} - \frac{12x}{(-3+x^2)^2},$$

from which we get $\beta = 0.536702$ and $M = 422.22$;

$$f_2'(x) = 3x^2 - 6x - 13,\ f_2''(x) = 6x - 6,\ f_2'''(x) = 6,$$

from which we get $\beta = 0.0481928$ and $M = 6$;

$$f_3'(x) = 5x^4,\ f_3''(x) = 20x^3,\ f_3'''(x) = 60x^2,$$

from which we get $\beta = 0.333503$ and $M = 114.264$.

In the Table 1 are listed the values for x_0, M, β, λ, μ_0, δ and $\frac{2\beta\mu_0}{\lambda(1-\mu_0)}$, for each test functions.

i	x_0	M	β	λ	μ_0	δ	$\frac{2\beta\mu_0}{\lambda(1-\mu_0)} < \delta$
1	-1.989	422.22	0.53670	9.32908	0.41860	0.089	0.08284
2	4.875	6	0.04819	0.02992	0.11414	0.625	0.41503
3	1.035	114.264	0.33350	2.37724	0.33873	0.353	0.14372

Table 1.

The implementations were done in Mathematica 7.0 with double precision. From the Table 1 we can conclude that all the assumptions a)–f) of Theorem 2 are verified.

In the next Table 2 we can observe that, the convergence is faster and the method (3) converges at x^*.

i	x_0	x_1	x_2	$x_3 = x^*$
1	-1.989	-2.0000063482540900	-1.9999999999999990	-2
2	4.875	5.0000611233335144	4.999999999999993	5
3	1.027	0.999958832170524	1.000000000000139	1

Table 2.

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