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LOCAL CONVERGENCE OF NEWTON'S METHOD USING KANTOROVICH CONVEX MAJORANTS

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Abstract. We are concerned with the problem of approximating a solution of an operator equation using Newton's method. Recently in the elegant work by Ferreira and Svaiter [6] a semilocal convergence analysis was provided which makes clear the relationship of the majorant function with the operator involved. However these results cannot provide information about the local convergence of Newton's method in their present form. Here we have rectified this problem by using two flexible majorant functions. The radius of convergence is also found. Finally, under the same computational cost, we show that our radius of convergence is larger, and the error estimates on the distances involved is finer than the corresponding ones [1], [11]–[13].

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1. INTRODUCTION

In this study we are concerned with the problem of approximating a solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where F is a continuously Fréchet–differentiable operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = Q(x)$, for some suitable operator Q, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers

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(single algebraic equations with single unknowns). Excpet in special cases, the most commonly used solution methods are iterative–when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

The most popular method for generating a sequence $\{x_n\}$ approximating $x^\star$ is undoubtedly Newton's method:

$$x_0 \in \mathcal{D}, \qquad x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \quad (n \geq 0). \tag{1.2}$$

Here $F'(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$, denotes the Fréchet–derivative of operator F [4], [8].

There is an extensive literature on local as well as semilocal convergence theorems for Newton's method, see, e.g. [1]–[4], and the references there.

In particuar, we are motivated by the elegant work of Ferreira and Svaiter [6] where a semilocal convergence was provided for Kantorovich's theorem [8] which makes clear the relationship of the majorant function and operator F. However the main result (see Theorem 2 in [6]) cannot provide in its present form information about the local convergence of Newton's method. Here, we rectify this problem. We introduce two flexible majorant functions to provide a local convergence for Newton's method (1.2). The radius of convergence is also given.

Finally, under the same computational cost we show that for special choices of the majorants functions involved, our radius of convergence is larger, and the error estimates on the distances involved is finer than the corresponding ones [1], [11]–[13].

2. LOCAL CONVERGENCE ANALYSIS OF NEWTON'S METHOD (1.2)

We need a result from convex analysis [9], [10]:

Proposition 2.1. *Let $\mathcal{I} \subset (-\infty, +\infty)$ be an interval, and $g : \mathcal{I} \longrightarrow (-\infty, +\infty)$ be convex.*

(1) *For any $s_0 \in \operatorname{int}(\mathcal{I})$, the correspondence $s \longrightarrow \frac{g(s_0)-g(s)}{s_0-s}$, $s \in I$, $s \neq s_0$, is increasing, and there exist in $(-\infty, +\infty)$*

$$D^- g(s_0) = \lim_{s \to s_0^-} \frac{g(s_0)-g(s)}{s_0-s} = \sup_{s<s_0} \frac{g(s_0)-g(s)}{s_0-s}.$$

(2) *If $s, v, t \in \mathcal{I}$, $s < t$, and $s \leq t \leq v$, then*

$$g(t) - g(s) \leq (g(v) - g(s)) \, \frac{t-s}{v-s}.$$

We now state a portion of a theorem (see Theorem 2 in [6]) due to Ferreira and Svaiter, needed for what follows:

THEOREM 2.2. *Let* $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ *be a continuous operator, continuously Fréchet–differentiable on* int $(\mathcal{D})$. *Take* $x_0 \in \text{int}\,(\mathcal{D})$ *with* $F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$. *Suppose there exist* $R > 0$, *and a continuously differentiable function* $f : [0, R) \longrightarrow (-\infty, +\infty)$, *such that* $U(x_0, R) = \{x \in \mathcal{X}, : \parallel x - x_0 \parallel < R\} \subset \mathcal{D}$, *and*

$\mathcal{H}1$ $\parallel F'(x_0)^{-1}\ (F'(y) - F'(x)) \parallel \leq f'(\parallel y - x \parallel + \parallel x - x_0 \parallel) - f'(\parallel x - x_0 \parallel)$,
for $x, y \in U(x_0, R)$, $\quad \parallel x - x_0 \parallel + \parallel y - x \parallel < R$,

$\mathcal{H}2$ $\parallel F'(x_0)^{-1}\ F(x_0) \parallel \leq f(0)$,

$\mathcal{H}3$ $f(0) > 0$,

$\mathcal{H}4$ $f'(0) = -1$,

$\mathcal{H}5$ f' is convex and strictly increasing and $f(t) = 0$ for some $t \in (0, R)$.

Then f *has a smallest zero* $t_\star$ *in* $(0, R)$, *the sequences generated by Newton's method* (1.2) *for solving* $f(t) = 0$, *and* $F(x) = 0$ *with starting point* $t_0 = 0$ *and* x_0, *respectively,*

$$t_{n+1} = t_n - f'(t_n)^{-1}\ f(t_n), \qquad x_{n+1} = x_n - F'(x_n)^{-1}\ F(x_n), \quad n \geq 0$$

are well defined, $\{t_n\}$ *is strictly increasing, is contained in* $[0, t_\star)$, *and converges to* $t_\star$, $\{x_n\}$ *is contained in* $U(x_0, t_\star)$, *and converges to a point* $x^\star$ *in* $\overline{U}(x_0, t_\star)$, *which is the unique zero of* F *in* $\overline{U}(x_0, t_\star)$.

Theorem 2.2 provides a semilocal convergence result for Newton's method, and cannot give us information about the local convergence of Newton's method in this form. Indeed for e.g. when $x_0 = x^\star$, hypotheses ($\mathcal{H}2$) and ($\mathcal{H}3$) are contradicting each other. In what follows we rectify this problem.

We state the main local convergence result for Newton's method (1.2):

THEOREM 2.3. *Let* $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ *be a continuous operator, continuously Fréchet–differentiable on* int $(\mathcal{D})$. *Suppose that there exist:*
$x^\star \in \text{int}\,(\mathcal{D})$ *with* $F'(x^\star)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, *and* $F(x^\star) = 0$;
$R > 0$, *and continuously differentiable function* f_0, *and* $f : [0, R) \longrightarrow (-\infty, +\infty)$, *such that* $U(x^\star, R) \subset \mathcal{D}$,
and

$$\parallel F'(x^\star)^{-1}\ (F'(x) - F'(x^\star)) \parallel \leq f_0'(\parallel x - x^\star \parallel) - f_0'(0), \tag{2.1}$$

$$\parallel F'(x^\star)^{-1}\ (F'(y) - F'(x)) \parallel \leq f'(\parallel y - x \parallel + \parallel x - x^\star \parallel) - f'(\parallel x - x^\star \parallel), \tag{2.2}$$

for $x, y \in U(x^\star, R)$, *and* $\parallel x - x^\star \parallel + \parallel y - x \parallel < R$;
functions f_0' *and* f' *are convex and strictly increasing with*

$$f_0'(0) = f'(0) = -1, \tag{2.3}$$

$$f_0'(t) \leq f'(t) \leq 0, \quad t \in [0, R]; \tag{2.4}$$

let $x, y \in U(x^\star, R)$, *and* $0 \leq t < v < R$, *then for all* $x \in \overline{U}(x^\star, R)$, $\parallel y - x \parallel \leq v - t$, *define function* $r_{f_0, f} : [0, R)^4 \longrightarrow [0, +\infty)$ *by*

$$r_{f_0, f} = r_{f_0, f}(t, v, \parallel y - x \parallel, \parallel x - x^\star \parallel) = -\frac{e(t,v)\ \|y-x\|}{(v-t)^2\ f_0'(\|x-x^\star\|)}, \tag{2.5}$$

and set

$$t^\star = \sup\{t \in [0,R] \,:\, r_{f_0,\,f} \leq 1\}, \tag{2.6}$$

where

$$e(t,v) = f(v) - f(t) - f'(t)\,(v-t). \tag{2.7}$$

Then, sequence $\{x_n\}$ *generated by Newton's method* (1.2)*, is well defined, remains in* $U(x^\star, t^\star)$ *for all* $n \geq 0$*, and converges to* $x^\star$ *Q–linearly, so that*

$$\| x_{n+1} - x^\star \| \leq \tfrac{1}{2} \, \| x_n - x^\star \|, \tag{2.8}$$

provided that $x_0 \in U(x^\star, t^\star)$*.*

Moreover, if

$$f_0'(t^\star) < 0, \tag{2.9}$$

then the following estimate holds for all $n \geq 0$

$$\| x_{n+1} - x^\star \| \leq \tfrac{D^- f'(t^\star)}{-2\, f_0'(t^\star)} \, \| x_n - x^\star \|^2 \,. \tag{2.10}$$

Furthermore, $x^\star$ *is the unique zero of* F *in* $U(x^\star, t^\star)$*.*

From now one we assume hypotheses of Theorem 2.3 hold, with the exception of (2.9), which will be considered to hold only when explicitly stated.

We shall show Theorem 2.3 through a series of lemmas:

Lemma 2.4. *If* $x \in \overline{U}(x^\star, t)$*,* $t \in [0, t^\star)$*, then*

$$F'(x)^{-1} \in L(\mathcal{Y}, \mathcal{X}),$$

and

$$\| F'(x)^{-1} \, F'(x^\star) \| \leq -\tfrac{1}{f_0'(\|x-x^\star\|)} \leq -\tfrac{1}{f'(t)}. \tag{2.11}$$

Proof. Let $x \in \overline{U}(x^\star, t)$, $t \in [0, t^\star)$.

Using hypotheses (2.1), (2.3), and (2.4), we obtain in turn

$$\begin{aligned} \| F'(x^\star)^{-1} \, (F'(x) - F'(x^\star)) \| &\leq f_0'(\| x - x^\star \|) - f_0'(0) \\ &\leq f_0'(\| x - x^\star \|) + 1 \\ &\leq f_0'(t) + 1 < 1. \end{aligned} \tag{2.12}$$

It follows from (2.12), and the Banach Lemma on invertible operators [4], [8] that $F'(x)^{-1} \in L(\mathcal{Y}, \mathcal{X})$, so that (2.11) holds true.

That completes the proof of Lemma 2.4. □

Lemma 2.5. *Let* $x, y \in U(x^\star, R)$*, and* $0 \leq t < v < R$*.*

If $x \in \overline{U}(x^\star, t)$*, and* $\| y - x \| \leq v - t$*, then for*

$$E(z,w) = F(w) - F(z) - F'(z)\,(w-z), \; z \in U(x^\star, t), w \in \mathcal{D},$$

we have the following estimates

$$\| F'(x^\star)^{-1} \, E(x,y) \| \leq e(t,v) \, \tfrac{\|y-x\|^2}{(v-t)^2}, \tag{2.13}$$

and

$$r_{f_0,\, f} \leq 1, \tag{2.14}$$

where funtion $r_{f_0,\, f}$ *is given by* (2.5).

Proof. Using the convexity of f', hypothesis (2.2), and the definition of operator E we obtain in turn:

$$\begin{aligned}
&\| F'(x^\star)^{-1}\, E(x,y) \| \leq \\
&\leq \int_0^1 \| F'(x^\star)^{-1} \Big(F'(x+\theta\,(y-x)) - F'(x) \Big) \| \, \| y-x \| \ \mathrm{d}\theta \\
&\leq \int_0^1 \Big(f'(\| x-x^\star \| + \theta \ \| y-x \|) - f'(\| x-x^\star \|) \Big) \ \| y-x \| \ \mathrm{d}\theta \\
&\leq \int_0^1 \Big(f'(t+\theta\,(v-t)) - f'(t) \Big) \ \tfrac{\| y-x \|}{v-t} \ \mathrm{d}\theta,
\end{aligned} \tag{2.15}$$

which implies estimates (2.13).

In view of hypothesis (2.4), we get

$$\frac{\displaystyle\int_0^1 \Big(f'(t+\theta\,(v-t)) - f'(t) \Big)\, \mathrm{d}\theta}{-f_0'(t)} \leq 1 \tag{2.16}$$

which together with (2.5) imply (2.14). That completes the proof of Lemma 2.5. □

As in [6] let us denote by $\eta_{f_0,\, f}$ and N_F the maps:

$$\begin{aligned}
\eta_{f_0,\, f} \ : \ [0, t^\star) \times (t, R) \ &\longrightarrow \ (-\infty, +\infty) \\
(t, v) \ &\longrightarrow \ t - \tfrac{e(t,v)}{f_0'(t)},
\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}
N_F \ : \ U(x^\star, t^\star) \ &\longrightarrow \ \mathcal{Y} \\
x \ &\longrightarrow \ x - F'(x)^{-1}\, F(x).
\end{aligned} \tag{2.18}$$

According to (2.4) and Lemma 2.4, we have: $f_0'(t) \neq 0$, $F'(x)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ respectively.

Let $x \in U(x^\star, t^\star)$, then $N_F(x)$ may not belong to $U(x^\star, t^\star)$ or even not belong in the domain of F. That is, we can only guarantee, on $U(x^\star, t^\star)$, well definedness of only the first iteration. Therefore, we need additional results to guarantee that Newton iterations can be repeated indefinitely.

Let us define subsets of $U(x^\star, t^\star)$ on which Newton's method (2.18) is "well behaved":

$$\begin{aligned}
&K(t) = K(t,v) = \{ x \in \overline{U}(x^\star, t), \ t \in [0, t^\star], \ v \in (t, R), \\
&y \in \overline{U}(x, v-t) \ : \| F'(y)^{-1}\, E(x,y) \| \leq r_{f_0,f} \ \| x-y \| \},
\end{aligned} \tag{2.19}$$

and

$$K = \cup_{t\in[0,t^\star]} K(t). \tag{2.20}$$

LEMMA 2.6. *If $t \in [0, t^\star)$, $v \in (t, R)$, then the following hold true:*

$$K(t) \subset U(x^\star, t^\star),$$

and

$$N_F(K(t)) \subset K(\eta_{f_0,f}(t)).$$

Proof. Simply replace x_0 by $x^\star$ in the proof of Lemma 8 in [6] (see also Proposition 4 in [6]). That completes the proof of Lemma 2.6. □

In view of (1.2) and (2.18) we have:

$$x_{n+1} = N_F(x_n) \qquad (n \geq 0). \tag{2.21}$$

Proof of Theorem 2.3. According to Lemmas 2.4–2.6, it is only left to show $x_n \in U(x^\star, t^\star)$ $(n \geq 1)$, $\lim\limits_{n\to\infty} x_n = x^\star$, so that estimates (2.8) and (2.10) hold true for all $n \geq 0$.

By hypothesis $x_0 \in U(x^\star, t^\star)$. Let us assume $x_k \in U(x^\star, t^\star)$ for all $k \leq n$. We shall show $x_{k+1} \in U(x^\star, t^\star)$. Using (2.21), and Lemma 2.5 for $y = x^\star$, $x = x_n$, we get

$$\| x_{k+1} - x^\star \| \leq \| x_k - x^\star \| < t^\star, \tag{2.22}$$

which show $x_{k+1} \in U(x^\star, t^\star)$, and $\lim\limits_{k\to\infty} x_k = x^\star$.

The proof of estimates (2.8) and (2.10) is given as in [6] with function f_0' replacing f' in the denominator of the estimates involved.

Finally, to show uniqueness in $U(x^\star, t^\star)$, let $y^\star$ be a zero of F in $U(x^\star, t^\star)$. Define linear operator $\mathcal{M}$ by

$$\mathcal{M} = \int_0^1 F'(x^\star + \theta\,(y^\star - x^\star))\,\mathrm{d}\theta, \tag{2.23}$$

Using (2.1), and the estimate (2.12) for $x^\star + \theta\,(y^\star - x^\star) \in U(x^\star, t^\star)$, replacing x, we conclude $\mathcal{M}^{-1}$ exists. It then follows from the identity

$$F(y^\star) - F(x^\star) = \mathcal{M}\,(y^\star - x^\star), \tag{2.24}$$

that $x^\star = y^\star$. That completes the proof of Theorem 2.3.

3. APPLICATIONS

EXAMPLE 3.1. Let us assume there exist $L > 0$ such that the Lipschitz condition

(3.1)

$$\| F'(x^\star)^{-1}\,(F'(y) - F'(x)) \| \leq L\ \| x - y \| \ \text{ holds for all } x, y \in \overline{U}(x_0, R) \subseteq \mathcal{D}.$$

Define scalar majorant function $f\ :\ [0, R] \longrightarrow (-\infty, +\infty)$ by

$$f(t) = \tfrac{L}{2}\,t^2 - t + \beta \qquad \text{for some } \beta \geq 0, \tag{3.2}$$

and set

$$f_0(t) = f(t) \qquad t \in [0, R]. \tag{3.3}$$

It then follows from (2.16) that we can set:

$$t^\star_R = R = \frac{2}{3\,L}, \tag{3.4}$$

which is the radius of convergence obtained by Rheinboldt [11], [4].

It follows from (3.1) that there exists $L_0 > 0$ such that:

$$\| F'(x^\star)^{-1}\,(F'(x) - F'(x^\star)) \| \leq L_0 \ \| x - x^\star \|, \quad \text{for all} \quad x \in \overline{U}(x_0, R). \tag{3.5}$$

Clearly

$$L_0 \leq L \tag{3.6}$$

holds and $\frac{L}{L_0}$ can be arbitrarily large [2]–[4].

Let us define function f_0 by

$$f_0(t) = \frac{L_0}{2}\, t^2 - t + \beta. \tag{3.7}$$

It then follows from (2.16) that we can set:

$$t^\star_A = R = \frac{2}{2\,L_0 + L}. \tag{3.8}$$

By comparing (3.4) with (3.8) we conclude:

$$t^\star_R \leq t^\star_A. \tag{3.9}$$

Note that if strict inequality holds in (3.6), then so does in (3.9).

Example 3.2. Let $f : [0, R) \longrightarrow (-\infty, +\infty)$ be a twice continuously differentiable function with f' convex. Then F satisfies (2.2) if and only if:

$$\| F'(x^\star)^{-1}\ F''(x) \| \leq f''(\| x - x^\star \|) \quad \text{for all } x \in \mathcal{D}, \quad \text{such that } x \in U(x^\star, R) \tag{3.10}$$

(see Lemma 14 in [6] or [13]).

Let us define function f on $[0, R)$ by

$$f(t) = \frac{\gamma\, t^2}{1 - \gamma\, t} \ - t + \beta. \tag{3.11}$$

where $R < \frac{1}{\gamma}$, for some $\gamma > 0$.

If for example F is an analytic operator, then (3.10) is satisfied for

$$\gamma^\star = \sup_{k \geq 2} \| \ \frac{F'(x^\star)^{-1} F^{(k)}(x^\star)}{k!} \ \|^{\frac{1}{k-1}} \ . \tag{3.12}$$

Smale [12], and Wang [13] have used (3.11) to provided a convergence analysis for Newton's method (1.2).

In particular Wang [13] showed convergence for F being only twice Fréchet continuously differentiable for γ satisfying

$$\gamma^\star \leq \gamma. \tag{3.13}$$

We have also used (3.11) to provide a convergence analysis for the Secant method [5] (see also [4]).

Let us also define function f_0 by

(3.14) $$f_0(t) = f(t) \qquad t \in [0, R).$$

By solving (2.16) we obtain for analytic operators F Smale's radius of convergence [12]:

(3.15) $$t_S^\star = \frac{5-\sqrt{13}}{6\,\gamma^\star},$$

and for twice Fréchet continuously differentiable operator F Wang's [13]:

(3.16) $$t_W^\star = \frac{5-\sqrt{13}}{6\,\gamma}.$$

In what follows we shall show that we can enlarge radii given by (3.15) and (3.16).

We can see that for function f given by (3.11), condition (3.10) or equivalently (2.2) imply that there exists $\gamma_0 > 0$ satisfying

(3.17) $$\gamma_0 \le \gamma,$$

so that function $f_0 : [0, \frac{1}{\gamma_0}) \longrightarrow (-\infty, +\infty)$ satisfies condition (2.1) for $R \in [0, \frac{1}{\gamma_0})$.

Note also that $\frac{\gamma}{\gamma_0}$ can be arbitrarily large [2]–[4]. It follows by (3.17) that there exists $a \in [0, 1]$ such that

(3.18) $$\gamma_0 = a\,\gamma.$$

Set

(3.19) $$b = 1 - a,$$

and define scalar polynomial P_a by

(3.20) $$P_a(t) = 3\,a^2\,t^3 + a\,(6\,b - a)\,t^2 + (3\,b^2 - 2\,a\,b - 1)\,t - b^2.$$

By the definition of polynomial P_a and for fixed a, we get

(3.21) $$P_a(0) = -b^2 \le 0, \qquad \text{and} \quad P_a(1) = 1.$$

Using (3.21) and the intermediate value theorem we conclude that there exists $t_a \in [0, 1)$ such that $P_a(t_a) = 0$. Let us denote by t_a the minimal number in $[0, 1)$ satisfying $P_a(t_a) = 0$.

Define

(3.22) $$t_a^\star = \frac{1-t_a}{\gamma}.$$

In particular for $a = 1$, $t_1 = \frac{1+\sqrt{13}}{6}$, and consequently

(3.23) $$t_a^\star = \frac{5-\sqrt{13}}{6\,\gamma} = t_W^\star.$$

It is simple algebra to show that for all $a \in [0, 1]$, $P_a(t_1) \ge 0$, which implies

(3.24) $$t_a \le t_1.$$

and

(3.25) $$t_1^\star \le t_a^\star.$$

We also note that strict inequality holds in (3.24), and (3.25) for $a \neq 1$.

As an example, let $a = \frac{1}{2}$. Then we obtain

$$t_{1/2} = .65185 < t_1 = \tfrac{1+\sqrt{13}}{6} = .76759,$$

and

$$t_1^\star = \tfrac{.23241}{\gamma} < \tfrac{.34815}{\gamma} = t_{1/2}^\star. \tag{3.26}$$

Finally note that clearly if strict inequality holds in (2.4), i.e., in (3.6) or (3.17), then our estimates on $\| x_{n+1} - x^\star \|$ $(n \geq 0)$ are finer (more precise) than the corresponding ones in [1], [11], [13] (see e.g. (2.10)).

These results are also obtained under the same computational cost since in practice the evaluation of L (or γ) requires that L_0 (or γ_0).

Remark 3.3. As noted in [1], [5], [6], [7], [10], [12] the local results obtained here can be used for projection method such us Arnold's, the generalized minimum residual method (GMRES), the generalized conjugate residual method (GCR), for combined Newton/finite projection methods, and in connection with the mesh independence principle to develop the cheapest and most efficient mesh refinement strategies.

Remark 3.4. The local results obtained can also be used to solve equation of the form $F(x) = 0$, where F' satisfies the autonomous differential equation [4], [8]:

$$F'(x) = T(F(x)), \tag{3.27}$$

where $T : \mathcal{Y} \longrightarrow \mathcal{X}$ is a known continuous operator.

Since $F'(x^\star) = T(F(x^\star)) = T(0)$, we can apply our results without actually knowing the solution of $x^\star$ of equation $F(x) = 0$.

As an example, let $\mathcal{X} = \mathcal{Y} = (-\infty, +\infty)$, $\mathcal{D} = \overline{U}(0,1)$, and define function F on $\mathcal{D}$ by

$$F(x) = \mathrm{e}^x - 1. \tag{3.28}$$

Then, for $x^\star = 0$, we can set $T(x) = x + 1$ in (3.27).

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