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OPTIMAL INEQUALITIES RELATED TO THE LOGARITHMIC, IDENTRIC, ARITHMETIC AND HARMONIC MEANS[‡]

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Abstract. The logarithmic mean $L(a,b)$, identric mean $I(a,b)$, arithmetic mean $A(a,b)$ and harmonic mean $H(a,b)$ of two positive real values a and b are defined by

$$L(a,b)=\begin{cases}\frac{b-a}{\log b-\log a}, & a\neq b,\\ a, & a=b,\end{cases}$$

$$I(a,b)=\begin{cases}\frac{1}{\mathrm{e}}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & a\neq b,\\ a, & a=b,\end{cases}$$

$A(a,b)=\frac{a+b}{2}$ and $H(a,b)=\frac{2ab}{a+b}$, respectively.

In this article, we answer the questions: What are the best possible parameters $\alpha_1,\alpha_2,\beta_1$ and β_2, such that $\alpha_1A(a,b)+(1-\alpha_1)H(a,b)\leq L(a,b)\leq\beta_1A(a,b)+(1-\beta_1)H(a,b)$ and $\alpha_2A(a,b)+(1-\alpha_2)H(a,b)\leq I(a,b)\leq\beta_2A(a,b)+(1-\beta_2)H(a,b)$ hold for all $a,b>0$?

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1. INTRODUCTION

The logarithmic mean $L(a,b)$ and identric mean $I(a,b)$ of two positive real values a and b are defined by

$$L(a,b)=\begin{cases}\frac{b-a}{\log b-\log a}, & a\neq b,\\ a, & a=b\end{cases}\tag{1.1}$$

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and

$$I(a,b)=\begin{cases}\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & a\neq b,\\ a, & a=b,\end{cases} \tag{1.2}$$

respectively. In the recent past, both mean values have been the subject of intensive research. In particular, many remarkable inequalities for logarithmic mean or identric mean can be found in the literature [1–25]. It might be surprising that the logarithmic mean has applications in physics, economics, and even in meteorology [26–28]. In [26] the authors study a variant of Jensen's functional equation involving $L(a,b)$, which appears in a heat conduction problem.

The power mean $M_p(a,b)$ of order p is defined by

$$M_p(a,b)=\begin{cases}\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}, & p\neq 0,\\ \sqrt{ab}, & p=0.\end{cases}$$

If we denote by $A(a,b)=\frac{1}{2}(a+b)$, $G(a,b)=\sqrt{ab}$ and $H(a,b)=\frac{2ab}{a+b}$ the arithmetic mean, geometric mean and harmonic mean of two positive numbers a and b, respectively, then it is well-known that

$$\begin{aligned}\min\{a,b\}&\leq H(a,b)=M_{-1}(a,b)\leq G(a,b)=M_0(a,b)\\ &\leq L(a,b)\leq I(a,b)\leq A(a,b)=M_1(a,b)\leq \max\{a,b\},\end{aligned} \tag{1.3}$$

and all inequalities are strict for $a\neq b$.

In [9, 12, 30] the authors present bounds for $L(a,b)$ in terms of $G(a,b)$ and $A(a,b)$.

$$G^{\frac{2}{3}}(a,b)A^{\frac{1}{3}}(a,b)<L(a,b)<\tfrac{2}{3}G(a,b)+\tfrac{1}{3}A(a,b)$$

for all $a,b>0$ with $a\neq b$.

The following companion of (1.3) provides inequalities for the geometric and arithmetic means of $L(a,b)$ and $I(a,b)$. A proof can be found in [7].

$$G^{\frac{1}{2}}(a,b)A^{\frac{1}{2}}(a,b)<L^{\frac{1}{2}}(a,b)I^{\frac{1}{2}}(a,b)<\tfrac{1}{2}L(a,b)+\tfrac{1}{2}I(a,b)<\tfrac{1}{2}G(a,b)+\tfrac{1}{2}A(a,b)$$

for all $a,b>0$ with $a\neq b$.

The following bounds for $L(a,b)$, $I(a,b)$, $(L(a,b)I(a,b))^{\frac{1}{2}}$, and $\frac{L(a,b)+I(a,b)}{2}$ in terms of power means are proved in [4, 6–8, 10, 24, 30].

$$\begin{aligned}&M_0(a,b)<L(a,b)<M_{\frac{1}{3}}(a,b),\\ &M_{\frac{2}{3}}(a,b)<I(a,b)<M_{\log 2}(a,b),\\ &M_0(a,b)<L^{\frac{1}{2}}(a,b)I^{\frac{1}{2}}(a,b)<M_{\frac{1}{2}}(a,b)\end{aligned}$$

and

$$M_{\frac{\log 2}{1+\log 2}}(a,b)<\tfrac{1}{2}L(a,b)+\tfrac{1}{2}I(a,b)<M_{\frac{1}{2}}(a,b)$$

for all $a, b > 0$ with $a \neq b$.

The main purpose of this paper is to answer the questions: What are the best possible parameters $\alpha_1, \alpha_2, \beta_1$ and β_2, such that $\alpha_1 A(a,b) + (1-\alpha_1)H(a,b) \leq L(a,b) \leq \beta_1 A(a,b) + (1-\beta_1)H(a,b)$ and $\alpha_2 A(a,b) + (1-\alpha_2)H(a,b) \leq I(a,b) \leq \beta_2 A(a,b) + (1-\beta_2)H(a,b)$ hold for all $a, b > 0$?

2. LEMMAS

Lemma 2.1. *The function* $g(t) = (t^2+4t+1)\log t - 3t^2 + 3 > 0$ *for* $t \in (1, +\infty)$.

Proof. By simple computation we have

$$g(1) = 0, \tag{2.1}$$

$$g'(t) = (2t+4)\log t - 5t + \tfrac{1}{t} + 4,$$

$$g'(1) = 0, \tag{2.2}$$

$$g''(t) = 2\log t + \tfrac{4}{t} - \tfrac{1}{t^2} - 3,$$

$$g''(1) = 0 \tag{2.3}$$

and

$$g'''(t) = \tfrac{2}{t} - \tfrac{4}{t^2} + \tfrac{2}{t^3} = \tfrac{2}{t^3}(t-1)^2. \tag{2.4}$$

Equation (2.4) leads to $g'''(t) > 0$ for $t \in (1, +\infty)$, then $g''(t)$ is strictly increasing in $(1, +\infty)$. Hence $g(t) > 0$ for $t \in (1, +\infty)$ follows from the monotonicity of $g''(t)$ and (2.1)–(2.3). □

Lemma 2.2. *The function* $g(t) = (5t^3 + 19t^2 + 19t + 5)\log t - 14t^3 - 6t^2 + 6t + 14 > 0$ *for* $t \in (1, +\infty)$.

Proof. By elementary computation we get

$$g(1) = 0, \tag{2.5}$$

$$g'(t) = (15t^2 + 38t + 19)\log t - 37t^2 + 7t + \tfrac{5}{t} + 25,$$

$$g'(1) = 0, \tag{2.6}$$

$$g''(t) = (30t + 38)\log t - 59t + \tfrac{19}{t} - \tfrac{5}{t^2} + 45,$$

$$g''(1) = 0, \tag{2.7}$$

$$g'''(t) = 30\log t + \tfrac{38}{t} - \tfrac{19}{t^2} + \tfrac{10}{t^3} - 29,$$

$$g'''(1) = 0 \tag{2.8}$$

and

$$g^{(4)}(t) = \tfrac{1}{t^4}(t-1)(30t^2 - 8t + 30). \tag{2.9}$$

Equation (2.9) leads to $g^{(4)}(t) > 0$ for $t \in (1, +\infty)$, then $g'''(t)$ is strictly increasing in $(1, +\infty)$. Hence $g(t) > 0$ for $t \in (1, +\infty)$ follows from the monotonicity of $g'''(t)$ and (2.5)–(2.8). □

LEMMA 2.3. *If* $g(t) = -[t^3 + (2\mathrm{e} - 1)t^2 + (2\mathrm{e} - 1)t + 1]\log t + (2\mathrm{e} - 2)t^3 + (-2\mathrm{e}+6)t^2 + (2\mathrm{e}-6)t - 2\mathrm{e} + 2$, *then there exists* $\lambda \in (1, +\infty)$ *such that* $g(t) > 0$ *for* $t \in (1, \lambda)$ *and* $g(t) < 0$ *for* $t \in (\lambda, +\infty)$.

Proof. Elementary computation yields

$$g(1) = 0, \quad \lim_{t\to+\infty} g(t) = -\infty, \tag{2.10}$$

$$g'(t) = -[3t^2 + (4\mathrm{e} - 2)t + 2\mathrm{e} - 1]\log t + (6\mathrm{e} - 7)t^2 + (-6\mathrm{e} + 13)t - \tfrac{1}{t} - 5,$$

$$g'(1) = 0, \quad \lim_{t\to+\infty} g'(t) = -\infty, \tag{2.11}$$

$$g''(t) = -(6t + 4\mathrm{e} - 2)\log t + (12\mathrm{e} - 17)t - \tfrac{2\mathrm{e}-1}{t} + \tfrac{1}{t^2} - 10\mathrm{e} + 15,$$

$$g''(1) = 0, \quad \lim_{t\to+\infty} g''(t) = -\infty, \tag{2.12}$$

$$g'''(t) = -6\log t - \tfrac{4\mathrm{e}-2}{t} + \tfrac{2\mathrm{e}-1}{t^2} - \tfrac{2}{t^3} + 12\mathrm{e} - 23,$$

$$g'''(1) = 10\mathrm{e} - 24 > 0, \quad \lim_{t\to+\infty} g'''(t) = -\infty, \tag{2.13}$$

$$g^{(4)}(t) = -\tfrac{1}{t^4}(t - 1)[6t^2 + (8 - 4\mathrm{e})t + 6]. \tag{2.14}$$

Equation (2.14) implies that $g^{(4)}(t) < 0$ for $t \in (1, +\infty)$, then $g'''(t)$ is strictly decreasing in $(1, +\infty)$.

From (2.13) and the monotonicity of $g'''(t)$ we clearly see that there exists $t_1 \in (1, +\infty)$, such that $g'''(t) > 0$ for $t \in (1, t_1)$ and $g'''(t) < 0$ for $t \in (t_1, +\infty)$. Hence we know that $g''(t)$ is strictly increasing in $[1, t_1)$ and strictly decreasing in $[t_1, +\infty)$.

The monotonicity of $g''(t)$ and (2.12) imply that there exists $t_2 \in (1, +\infty)$, such that $g''(t) > 0$ for $t \in (1, t_2)$ and $g''(t) < 0$ for $t \in (t_2, +\infty)$. Hence we know that $g'(t)$ is strictly increasing in $[1, t_2)$ and strictly decreasing in $[t_2, +\infty)$.

From (2.11) and the monotonicity of $g'(t)$ we clearly see that there exists $t_3 \in (1, +\infty)$, such that $g'(t) > 0$ for $t \in (1, t_3)$ and $g'(t) < 0$ for $t \in (t_3, +\infty)$. Hence we conclude that $g(t)$ is strictly increasing in $[1, t_3)$ and strictly decreasing in $[t_3, +\infty)$.

The monotonicity of $g(t)$ and (2.10) imply that there exists $\lambda \in (1, +\infty)$, such that $g(t) > 0$ for $t \in (1, \lambda)$ and $g(t) < 0$ for $t \in (\lambda, +\infty)$. □

3. MAIN RESULTS

THEOREM 3.1. *The double inequality*

$$\alpha_1 A(a,b) + (1-\alpha_1)H(a,b) \leq L(a,b) \leq \beta_1 A(a,b) + (1-\beta_1)H(a,b)$$

holds for all $a, b > 0$ *if and only if* $\alpha_1 \leq 0$ *and* $\beta_1 \geq \frac{2}{3}$.

Proof. If $a = b$, then $\alpha_1 A(a,b) + (1-\alpha_1)H(a,b) = L(a,b) = \beta_1 A(a,b) + (1-\beta_1)H(a,b) = a$ for all $\alpha_1, \beta_1 \in \mathbb{R}$. Next, we assume that $a \neq b$.

Firstly, we prove that $H(a,b) < L(a,b) < \frac{2}{3}A(a,b) + \frac{1}{3}H(a,b)$. From (1.3) we know that $H(a,b) < L(a,b)$ is true, so we only need to prove that $L(a,b) < \frac{2}{3}A(a,b) + \frac{1}{3}H(a,b)$.

Without loss of generality, we assume that $a > b$. Let $t = \frac{a}{b} > 1$, then simple computation leads to

$$\begin{aligned}\frac{2}{3}A(a,b) + \frac{1}{3}H(a,b) - L(a,b) &= b\left[\frac{t+1}{3} + \frac{2t}{3(t+1)} - \frac{t-1}{\log t}\right] \\ &= \frac{b[(t^2+4t+1)\log t - 3t^2+3]}{3(t+1)\log t}.\end{aligned} \tag{3.1}$$

Therefore, $L(a,b) < \frac{2}{3}A(a,b) + \frac{1}{3}H(a,b)$ follows from Lemma 2.1 and (3.1).

Secondly, we prove that the parameters $\alpha_1 \leq 0$ and $\beta_1 \geq \frac{2}{3}$ cannot be improved.

For any $0 < \varepsilon < 1$ and $0 < x < 1$, from (1.1) we have

$$\begin{aligned}&\lim_{x\to 0}[\varepsilon A(1,x) + (1-\varepsilon)H(1,x) - L(1,x)] = \\ &= \lim_{x\to 0}\left[\varepsilon \cdot \frac{1+x}{2} + (1-\varepsilon)\cdot\frac{2x}{1+x} - \frac{x-1}{\log x}\right] = \frac{\varepsilon}{2}.\end{aligned} \tag{3.2}$$

Equation (3.2) implies that for any $0 < \varepsilon < 1$, there exists $0 < \delta = \delta(\varepsilon) < 1$, such that $\varepsilon A(1,x) + (1-\varepsilon)H(1,x) > L(1,x)$ for $x \in (0,\delta)$. Hence the parameter $\alpha_1 \leq 0$ cannot be improved.

Next, for any $0 < \varepsilon < 1$ and $0 < x < 1$, from (1.1) we get

$$\begin{aligned}&L(1+x,1) - \left[(\tfrac{2}{3}-\varepsilon)A(1+x,1) + (\tfrac{1}{3}+\varepsilon)H(1+x,1)\right] = \\ &= \frac{x}{\log(1+x)} - \frac{(\frac{2}{3}-\varepsilon)x^2+4x+4}{2(x+2)} \\ &= \frac{h(x)}{2(x+2)\log(1+x)},\end{aligned} \tag{3.3}$$

where $h(x) = 2x(x+2) - [(\frac{2}{3}-\varepsilon)x^2 + 4x + 4]\log(1+x)$.

Let $x \to 0$ and using Taylor expansion we obtain

$$h(x) = \varepsilon x^3 + o(x^3). \tag{3.4}$$

Equations (3.3) and (3.4) imply that for any $0 < \varepsilon < 1$, there exists $0 < \delta = \delta(\varepsilon) < 1$, such that $L(1+x,1) > (\frac{2}{3}-\varepsilon)A(1+x,1) + (\frac{1}{3}+\varepsilon)H(1+x,1)$ for $x \in (0,\delta)$. Hence the parameter $\beta_1 \geq \frac{2}{3}$ cannot be improved. $\square$

THEOREM 3.2. *The double inequality*

$$\alpha_2 A(a,b) + (1-\alpha_2)H(a,b) \le I(a,b) \le \beta_2 A(a,b) + (1-\beta_2)H(a,b)$$

holds for all $a, b > 0$ *if and only if* $\alpha_2 \le \frac{2}{\mathrm{e}}$ *and* $\beta_2 \ge \frac{5}{6}$.

Proof. If $a = b$, then $\alpha_2 A(a,b) + (1-\alpha_2)H(a,b) = I(a,b) = \beta_2 A(a,b) + (1-\beta_2)H(a,b) = a$ for all $\alpha_2, \beta_2 \in \mathbb{R}$. Next, we assume that $a \ne b$.

Firstly, we prove that $\frac{2}{\mathrm{e}}A(a,b) + (1-\frac{2}{\mathrm{e}})H(a,b) < I(a,b)$ and the parameter $\alpha_2 \le \frac{2}{\mathrm{e}}$ cannot be improved.

Without loss of generality, we assume that $a > b$. Let $t = \frac{a}{b} > 1$, then

$$I(a,b) - \left[\tfrac{2}{\mathrm{e}}A(a,b) + (1-\tfrac{2}{\mathrm{e}})H(a,b)\right] = \tag{3.5}$$
$$= \tfrac{b}{\mathrm{e}}\left[t^{\frac{t}{t-1}} - (t+1) - (\mathrm{e}-2)\tfrac{2t}{t+1}\right].$$

Let $f(t) = \log t^{\frac{t}{t-1}} - \log\left[(t+1) + (\mathrm{e}-2)\frac{2t}{t+1}\right]$, then elementary computation yields

$$\lim_{t\to 1} f(t) = 0, \qquad \lim_{t\to+\infty} f(t) = 0 \tag{3.6}$$

and

$$f'(t) = \tfrac{g(t)}{(t+1)(t-1)^2[t^2+(2\mathrm{e}-2)t+1]}, \tag{3.7}$$

where $g(t) = -[t^3 + (2\mathrm{e}-1)t^2 + (2\mathrm{e}-1)t + 1]\log t + (2\mathrm{e}-2)t^3 + (-2\mathrm{e}+6)t^2 + (2\mathrm{e}-6)t - 2\mathrm{e} + 2$.

From (3.7) and Lemma 2.3 we know that there exists $\lambda \in (1, +\infty)$, such that $f(t)$ is strictly increasing in $(1, \lambda)$ and strictly decreasing in $(\lambda, +\infty)$. Then (3.6) and the monotonicity of $f(t)$ imply that $f(t) > 0$ for $t \in (1, +\infty)$, and from (3.5) we know that $I(a,b) > \frac{2}{\mathrm{e}}A(a,b) + (1-\frac{2}{\mathrm{e}})H(a,b)$ for $a, b > 0$ with $a \ne b$.

Next, we prove that the parameter $\alpha_2 \le \frac{2}{\mathrm{e}}$ cannot be improved.

For any $0 < \varepsilon < 1$ and $0 < x < 1$, from (1.2) we have

$$\lim_{x\to 0}\left[(\tfrac{2}{\mathrm{e}}+\varepsilon)A(1,x) + (1-\tfrac{2}{\mathrm{e}}-\varepsilon)H(1,x) - I(1,x)\right] = \tag{3.8}$$
$$= \lim_{x\to 0}\left[(\tfrac{2}{\mathrm{e}}+\varepsilon)\cdot\tfrac{1+x}{2} + (1-\tfrac{2}{\mathrm{e}}-\varepsilon)\cdot\tfrac{2x}{1+x} - \tfrac{1}{\mathrm{e}}x^{\frac{x}{x-1}}\right]$$
$$= \tfrac{\varepsilon}{2}.$$

Equation (3.8) implies that for any $0 < \varepsilon < 1$, there exists $0 < \delta = \delta(\varepsilon) < 1$, such that $(\frac{2}{\mathrm{e}}+\varepsilon)A(1,x) + (1-\frac{2}{\mathrm{e}}-\varepsilon)H(1,x) > I(1,x)$ for $x \in (0,\delta)$. Hence the parameter $\alpha_2 \le \frac{2}{\mathrm{e}}$ cannot be improved.

Secondly, we prove that $I(a,b) < \frac{5}{6}A(a,b) + \frac{1}{6}H(a,b)$ and the parameter $\beta_2 \ge \frac{5}{6}$ cannot be improved.

Let $t = \frac{a}{b} > 1$, then from (1.2) we have

$$\frac{5}{6}A(a,b) + \frac{1}{6}H(a,b) - I(a,b) = b\left[\frac{5t^2+14t+5}{12(t+1)} - \frac{1}{\mathrm{e}}t^{\frac{t}{t-1}}\right]. \tag{3.9}$$

Let $f(t) = \log\left[\frac{5t^2+14t+5}{12(t+1)}\right] - \log\left(\frac{1}{\mathrm{e}}t^{\frac{t}{t-1}}\right)$, then

$$f(1) = 0 \tag{3.10}$$

and

$$f'(t) = \frac{g(t)}{(t-1)(t^2-1)(5t^2+14t+5)}, \tag{3.11}$$

where $g(t) = (5t^3 + 19t^2 + 19t + 5)\log t - 14t^3 - 6t^2 + 6t + 14$.

From Lemma 2.2 and (3.11) together with (3.10) we clearly see that $f(t) > 0$ for $t \in (1, +\infty)$. Hence from (3.9) we know that $\frac{5}{6}A(a,b) + \frac{1}{6}H(a,b) > I(a,b)$ for $a, b > 0$ with $a \neq b$.

Next, we prove that the parameter $\beta_2 \geq \frac{5}{6}$ cannot be improved.

For any $0 < \varepsilon < 1$ and $0 < x < 1$, from (1.2) we get

$$\begin{aligned} I(1+x,1) - \left[(\tfrac{5}{6} - \varepsilon)A(1+x,1) + (\tfrac{1}{6} + \varepsilon)H(1+x,1)\right] &= \\ = \frac{1}{\mathrm{e}}(1+x)^{\frac{1+x}{x}} - \frac{(\frac{5}{6}-\varepsilon)x^2+4x+4}{2(2+x)} & \\ = \frac{h(x)}{2(2+x)}, & \end{aligned} \tag{3.12}$$

where $h(x) = \frac{2}{\mathrm{e}}(2+x)(1+x)^{\frac{1+x}{x}} - (\frac{5}{6} - \varepsilon)x^2 - 4x - 4$.

Let $x \to 0$ and using Taylor expansion we obtain

$$\begin{aligned} h(x) &= 2(2+x)\left[1 + \tfrac{1}{2}x - \tfrac{1}{24}x^2 + o(x^2)\right] - (\tfrac{5}{6} - \varepsilon)x^2 - 4x - 4 \\ &= \varepsilon x^2 + o(x^2). \end{aligned} \tag{3.13}$$

Equations (3.12) and (3.13) imply that for any $0 < \varepsilon < 1$, there exists $0 < \delta = \delta(\varepsilon) < 1$, such that $I(1+x,1) > (\frac{5}{6}-\varepsilon)A(1+x,1)+(\frac{1}{6}+\varepsilon)H(1+x,1)$ for $x \in (0, \delta)$. Hence the parameter $\beta_2 \geq \frac{5}{6}$ cannot be improved. □

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