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OPTIMIZATION PROBLEMS AND SECOND ORDER APPROXIMATED OPTIMIZATION PROBLEMS

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Abstract. In this paper, a so-called second order approximated optimization problem associated to an optimization problem is considered. The equivalence between the saddle points of the lagrangian of the second order approximated optimization problem and optimal solutions of the original optimization problem is established.

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1. INTRODUCTION

We consider the optimization problem

$$(P) \qquad \begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in X \\ & g(x) \leqq 0, \end{array}$$

where X is a subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}$ and $g = (g_1, ..., g_m) : X \to \mathbb{R}^m$ are two functions.

Let

$$\mathfrak{F}(P) := \{x \in X : \; g(x) \leqq 0\}$$

denote the set of all feasible solutions of Problem (P).

For solving optimization problem (P), there are various manners to approach. One of these manners is that for Problem (P) one attached to another optimization problem, problem whose solution gives us the (information about) optimal solution of the initial problem (P).

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Assuming that X is open, and that f and g are differentiable on X, Mangasarian [12] attached to Problem (P) and the point $x^0 \in X$, the problem

$$\begin{array}{ll} \min & f\left(x^0\right) + \left\langle u, \nabla f\left(x^0\right)\right\rangle \\ \text{s.t.} & u \in \mathbb{R}^n \\ & g\left(x^0\right) + \left[\nabla g\left(x^0\right)\right](u) \leqq 0. \end{array}$$

He took the dual of this linear optimization problem, and then considered x^0 to be a variable. This last problem is precisely the classical dual of the nonlinear optimization problem, introduced in a different way by Wolfe [16] and investigated extensively (see, for example [11]). Connections between optimal solutions of the dual and the primal are known (see, for example [11]).

The above process is repeated but taking nonlinear instead of linear approximation of f and g around some fixed $x^0 \in X$ and taking the dual of the resulting optimization problem. One takes the dual of this optimization problem and then one considers x^0 to be a variable in X. One obtains the so called higher-order dual problem of Problem (P). In [12], there are given connections between the optimal solutions of higher-order dual and initial problem (P). D.I. Duca [7], [8] used this idea for optimization problems in complex space.

Another idea came from Antczak [4], [3], [2], who attached to Problem (P) the following problem

$$(AP1) \qquad \begin{array}{ll} \min & f\left(x^0\right) + \left\langle \nabla f\left(x^0\right), \eta\left(x, x^0\right)\right\rangle \\ \text{s.t.} & x \in X \\ & g\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x, x^0\right)\right) \leqq 0, \end{array}$$

where $x^0 \in X$ is an interior point of X, $\eta : X \times X \to \mathbb{R}^n$ is a function, and $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}^m$ are differentiable at x^0. He studied the connections between the saddle points of Problem $(AP1)$ and optimal solutions of Problem (P).

In [1], [14], [15], [17] the another problems are attached to Problem (P).

In this paper, we attached to Problem (P), the problem

$$\begin{array}{ll} \min & F(x) \\ \text{s.t.} & x \in X \\ & G(x) \leqq 0, \end{array}$$

where $F : X \to \mathbb{R}$ and $G : X \to \mathbb{R}^m$ are the functions defined by

$$F(x) := f\left(x^0\right) + \left\langle \nabla f\left(x^0\right), \eta\left(x, x^0\right)\right\rangle + \tfrac{1}{2}\left\langle \eta\left(x, x^0\right), \left[\nabla^2 f\left(x^0\right)\right]\left(\eta\left(x, x^0\right)\right)\right\rangle,$$

$$G(x) := g\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x, x^0\right)\right),$$

for all $x \in X$, and we study the connections between saddle points of this Problem and the optimal solutions of Problem (P).

2. NOTIONS AND PRELIMINARY RESULTS

In the last few years, attempts have been made to weaken the convexity hypotheses and thus to explore the existence of optimality conditions applicability. Various classes of generalized convex functions have been suggested for the purpose of weakening the convexity limitation in this result. Among these, the concept of an invex function proposed by Hanson ([10]) has received more attention. The name of invex (invariant convex) function was given by Craven ([6])

Definition 2.1. *Let* X *be a nonempty subset of* $\mathbb{R}^n$, x^0 *be an interior point of* X, $f : X \to \mathbb{R}$ *be a differentiable function at* x^0, *and* $\eta : X \times X \to \mathbb{R}^n$ *be a function. We say that the function* f *is invex at* x^0 *with respect to* (*w.r.t.*) η *if*

$$f(x) - f(x^0) \geqq \langle \nabla f(x^0), \eta(x, x^0) \rangle, \text{ for all } x \in X. \tag{2.1}$$

Hanson defined invex functions which allow the use of the Kuhn-Tucker conditions as sufficient conditions for optimality in constrained optimization problems. Later, Martin ([13]) proved that invexity hypotheses are not only sufficient but also necessary when using the Kuhn-Tucker optimality conditions for unconstrained optimization problems.

After the works of Hanson and Craven, other types of differentiable functions have appeared with the intent of generalizing invex function from different points of view.

Ben-Israel and Mond [5] defined the so-called pseudoinvex functions, generalizing pseudoconvex functions in the same way that invex functions generalize convex functions.

Definition 2.2. *Let* X *be a nonempty subset of* $\mathbb{R}^n$, x^0 *be an interior point of* X, $f : X \to \mathbb{R}$ *be a differentiable function at* x^0, *and* $\eta : X \times X \to \mathbb{R}^n$ *be a function. We say that* f *is pseudoinvex at* x^0 *w.r.t.* η *if, for each* $x \in X$ *with the property that*

$$\langle \nabla f(x^0), \eta(x, x^0) \rangle \geqq 0,$$

we have

$$f(x) \geqq f(x^0).$$

Definition 2.3. *Let* X *be a nonempty subset of* $\mathbb{R}^n$, x^0 *be an interior point of* X, $f : X \to \mathbb{R}$ *be a differentiable function at* x^0, *and* $\eta : X \times X \to \mathbb{R}^n$ *be a function. We say that* f *is quasiinvex at* x^0 *w.r.t.* η *if, for each* $x \in X$ *with the property that*

$$f(x) \leqq f(x^0),$$

we have

$$\langle \nabla f(x^0), \eta(x, x^0) \rangle \leqq 0.$$

Remark 2.4. Note that, in general, there exists no unique function $\eta = \eta_{x^0}$ such that the function f is invex, respectively pseudoinvex and quasiinvex at the point $x^0 \in X$.

Indeed, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \exp x, \text{ for all } x \in \mathbb{R},$$

is invex at $x^0 = 0$ w.r.t. the function $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$\eta(x, u) = x - u, \text{ for all } (x, u) \in \mathbb{R} \times \mathbb{R}.$$

Also, the function f is invex at $x^0 = 0$ w.r.t. the function $\eta : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\eta(x, u) = x + \tfrac{x^2}{2} + \tfrac{x^3}{6}, \text{ for all } (x, u) \in \mathbb{R}^2.$$

And also, the function f is invex at x^0 w.r.t. the function $\eta : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\eta(x, u) = x - 2, \text{ for all } (x, u) \in \mathbb{R}^2.$$

$\square$

Definition 2.5. *Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $f : X \to \mathbb{R}$ be a twice differentiable function at x^0 and $\eta : X \times X \to \mathbb{R}^n$ be a function. We say that the function f is second order invex at x^0 w.r.t. η if*

$$f(x) - f(x^0) \geqq \left\langle \nabla f(x^0), \eta(x, x^0) \right\rangle + \left\langle \left[\nabla^2 f(x^0)\right](y), \eta(x, x^0) \right\rangle - \tfrac{1}{2} \left\langle y, \left[\nabla^2 f(x^0)\right](y) \right\rangle, \tag{2.2}$$

for all $x \in X$ and $y \in \mathbb{R}^n$.

Remark 2.6. If f is a second order invexity at x^0 w.r.t. η, then (2.2) is also satisfied for $y = \eta(x, x^0)$. Then (2.2) gives

$$f(x) - f(x^0) \geqq \left\langle \nabla f(x^0), \eta(x, x^0) \right\rangle + \tfrac{1}{2} \left\langle \left[\nabla^2 f(x^0)\right](\eta(x, x^0)), \eta(x, x^0) \right\rangle,$$

for all $x \in X$. $\square$

Definition 2.7. *Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to \mathbb{R}^n$ be a function, and $f : X \to \mathbb{R}$ be a twice differentiable function at x^0. We say that f is second order pseudoinvex at x^0 with respect to (w.r.t.) η if, for each $x \in X$ with the property that*

$$\left\langle \nabla f(x^0), \eta(x, x^0) \right\rangle + \tfrac{1}{2} \left\langle \left[\nabla^2 f(x^0)\right](\eta(x, x^0)), \eta(x, x^0) \right\rangle \geqq 0,$$

we have

$$f(x) \geqq f(x^0).$$

Remark 2.8. Obviously, if the function f is second order invex at x^0 w.r.t. η, then the function f is second order pseudoinvex at x^0 w.r.t. η. $\square$

Now let us attach to Problem (P) its lagrangian $L : X \times \mathbb{R}^m_+ \to \mathbb{R}$ defined by

$$L(x,v) := f(x) + \langle v, g(x) \rangle, \text{ for all } (x,v) \in X \times \mathbb{R}^m_+$$

Then we have the following theorem (see, for example [11]):

THEOREM 2.9. *If $(x^0, v^0) \in X \times \mathbb{R}^m_+$ is a saddle point of the lagrangian L of Problem (P), i.e. we have*

$$L(x^0, v) \leqq L(x^0, v^0) \leqq L(x, v^0), \text{ for all } (x,v) \in X \times \mathbb{R}^m_+,$$

then x^0 is an optimal solution of Problem (P).

3. η-APPROXIMATED OPTIMIZATION PROBLEM

In what follows, X is a nonempty subset of $\mathbb{R}^n$, x^0 is an interior point of X, and $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}^m$ are two differentiable functions at x^0.

For $\eta : X \times X \to \mathbb{R}^n$, Antczak ([4]) attaches to Problem (P) the problem $(P_\eta(x^0))$, called η-approximated at x^0 of Problem (P).

In [4] and [9] one establishes the equivalence between saddle points of η-approximated problem $(P_\eta(x^0))$ and of the original problem (P).

If x^0 is a feasible solution of Problem (P), then

$$I(x^0) = \{i \in \{1, ..., m\} : \ g_i(x^0) = 0\}$$

denote the indices of the active restrictions at x^0.

In Ref. [9], generalizing a result from [2], one proves the following statement:

THEOREM 3.1. *Let $\eta : X \times X \to \mathbb{R}^n$ such that $\eta(x^0, x^0) = 0$, $f : X \to \mathbb{R}$ be pseudoinvex at x^0 w.r.t. η and $g = (g_1, ..., g_m) : X \to \mathbb{R}^m$ such that g_i, $i \in I(x^0)$ are quasiinvex at x^0 w.r.t. η.*

If $(x^0, v^0) \in X \times \mathbb{R}^m_+$ is a saddle point of the lagrangian L_η of Problem $(P_\eta(x^0))$, then x^0 is an optimal solution of the original problem (P).

Also, in [9], generalizing another result from [2], we showed that, if x^0 is an optimal solution of the original problem (P), then under certain conditions, there exists a point $v^0 \in \mathbb{R}^m_+$ such that (x^0, v^0) is a saddle point of the η-approximated problem $(\mathrm{P}_\eta(x^0))$.

More exactly, the following statement is true

THEOREM 3.2. *Let x^0 be an optimal solution of the original problem (P) and assume that a suitable constraint qualification is satisfied at x^0 (CQ in [11]). If*

(i) $\langle \nabla f(x^0), \eta(x^0, x^0) \rangle \leqq 0$;

(ii) $g(x^0) + [\nabla g(x^0)](\eta(x^0, x^0)) \leqq 0$ (*i.e.* $x^0 \in \mathfrak{F}(P_\eta(x^0))$),

then there exists a point $v^0 \in \mathbb{R}^m_+$ such that (x^0, v^0) is a saddle point of the lagrangian L_η of the η-approximated problem $(P_\eta(x^0))$.

Remark 3.3. If $\eta\left(x^0,x^0\right)=0$, then the hypotheses (i) and (ii) from Theorem 3.2 are satisfied. □

Remark 3.4. If f and g are invex at x^0 w.r.t. η, then the hypotheses (i) and (ii) from Theorem 3.2 are satisfied. □

4. $(2,1)$-η-APPROXIMATED OPTIMIZATION PROBLEM

In this section, X is a subset of $\mathbb{R}^n$, x^0 is an interior point of X, $f:X\to\mathbb{R}$ is a twice continuously differentiable function at x^0, and $g:X\to\mathbb{R}^m$ is a differentiable function at x^0.

For $\eta:X\times X\to\mathbb{R}^n$, we attach to Problem (P) the following optimization problem

$$(AP2)\qquad \begin{array}{ll}\min & F(x)\\ \text{s.t.} & x\in X\\ & G(x)\leqq 0,\end{array}$$

where $F:X\to\mathbb{R}$ and $G:X\to\mathbb{R}^m$ are the functions defined by

$$F(x):=f\left(x^0\right)+\left\langle\nabla f\left(x^0\right),\eta\left(x,x^0\right)\right\rangle+\tfrac{1}{2}\left\langle\eta\left(x,x^0\right),\left[\nabla^2 f\left(x^0\right)\right]\left(\eta\left(x,x^0\right)\right)\right\rangle,$$

$$G(x):=g\left(x^0\right)+\left[\nabla g\left(x^0\right)\right]\left(\eta\left(x,x^0\right)\right),$$

for all $x\in X$.

Problem $(AP2)$ is called $(2,1)$-η-approximated at x^0 of Problem (P).

Remark 4.1. If $X=\mathbb{R}^n$ and $\eta\left(x,x^0\right)=x-x_0$, for all $x\in X$, then Problem $(AP2)$ is quadratic. □

Let

$$\begin{aligned}\mathfrak{F}(AP2)&:=\{x\in X:\ g\left(x^0\right)+\left[\nabla g\left(x^0\right)\right]\left(\eta\left(x,x^0\right)\right)\leqq 0\}\\ &=\{x\in X:G(x)\leqq 0\},\end{aligned}$$

denote the set of all feasible solutions of Problem $(AP2)$.

Theorem 4.2. *Assume that $g:X\to\mathbb{R}^m$ is invex at x^0 w.r.t. η. If x is a feasible solution of Problem (P), then x is a feasible solution of Problem $(AP2)$, i.e.*

$$\mathfrak{F}(P)\subseteq\mathfrak{F}(AP2).\tag{4.1}$$

Proof. Let $x\in X$ be a feasible solution of Problem (P), i.e. $g(x)\leqq 0$.

Since g is invex at x^0 w.r.t. η, we have

$$\left[\nabla g\left(x^0\right)\right]\left(\eta\left(x,x^0\right)\right)\leqq g(x)-g\left(x^0\right),$$

i.e.

$$\left[\nabla g\left(x^0\right)\right]\left(\eta\left(x,x^0\right)\right)+g\left(x^0\right)\leqq g(x).$$

But $g(x)\leqq 0$ and then

$$\left[\nabla g\left(x^0\right)\right]\left(\eta\left(x,x^0\right)\right)+g\left(x^0\right)\leqq 0,$$

hence x is a feasible solution of Problem $(AP2)$. □

Example 4.3. For Problem

$$(\widetilde{P}) \qquad \begin{array}{ll} \min & f(x) = x^2 \\ \text{s.t.} & x \in X = \mathbb{R} \\ & g(x) = x^2 - x \leqq 0, \end{array}$$

we have $\mathfrak{F}\left(\widetilde{P}\right) = [0,1]$.

The function g is invex at $x^0 = 0$ w.r.t. the function $\eta : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\eta(x,u) = x - u, \text{ for all } (x,u) \in \mathbb{R}^2.$$

On the other hand, the $(2,1)$-η-approximated optimization problem $\left(\widetilde{P}\right)$ is

$$(A\widetilde{P}2) \qquad \begin{array}{ll} \min & x^2 \\ \text{s.t.} & x \in X = \mathbb{R} \\ & -x \leqq 0, \end{array}$$

which has $\mathfrak{F}\left(A\widetilde{P}2\right) = [0, +\infty[$. □

Theorem 4.4. *Assume that $f : X \to \mathbb{R}$ is a second order invex at x^0 w.r.t. η, and $g : X \to \mathbb{R}^m$ is invex at x^0 w.r.t. η. If x^0 is an optimal solution of Problem (P), then*

$$\min\{f(x) : x \in \mathfrak{F}(P)\} \geqq \inf\{F(x) : x \in \mathfrak{F}(AP2)\}.$$

Proof. The function f is second order invexity at x^0 w.r.t. η, then

$$f(x) \geqq f\left(x^0\right) + \left\langle \nabla f\left(x^0\right), \eta\left(x,x^0\right)\right\rangle + \tfrac{1}{2}\left\langle \eta\left(x,x^0\right), \left[\nabla^2 f\left(x^0\right)\right]\left(\eta\left(x,x^0\right)\right)\right\rangle,$$

for all $x \in X$. It follows that

$$\begin{aligned} &\min\{f(x) : x \in \mathfrak{F}(P)\} = f\left(x^0\right) \geqq \\ &\geqq f\left(x^0\right) + \left\langle \nabla f\left(x^0\right), \eta\left(x^0,x^0\right)\right\rangle + \tfrac{1}{2}\left\langle \eta\left(x^0,x^0\right), \left[\nabla^2 f\left(x^0\right)\right]\left(\eta\left(x^0,x^0\right)\right)\right\rangle \geqq \\ &\geqq \inf\left\{f\left(x^0\right) + \left\langle \nabla f\left(x^0\right), \eta\left(x,x^0\right)\right\rangle + \right. \\ &\quad \left. + \tfrac{1}{2}\left\langle \eta\left(x,x^0\right), \left[\nabla^2 f\left(x^0\right)\right]\left(\eta\left(x,x^0\right)\right)\right\rangle : x \in \mathfrak{F}(P)\right\} \geqq \text{ (from (4.1))} \\ &\geqq \inf\left\{f\left(x^0\right) + \left\langle \nabla f\left(x^0\right), \eta\left(x,x^0\right)\right\rangle + \right. \\ &\quad \left. + \tfrac{1}{2}\left\langle \eta\left(x,x^0\right), \left[\nabla^2 f\left(x^0\right)\right]\left(\eta\left(x,x^0\right)\right)\right\rangle : x \in \mathfrak{F}(AP2)\right\} = \\ &= \inf\{F(x) : x \in \mathfrak{F}(AP2)\}. \end{aligned}$$

□

Theorem 4.5. *If $\eta\left(x^0,x^0\right) = 0$, $g : X \to \mathbb{R}^m$ is invex at x^0 w.r.t. η, and x^0 is an optimal solution of Problem (P), then*

$$\min\{f(x) : x \in \mathfrak{F}(P)\} \geqq \inf\{F(x) : x \in \mathfrak{F}(AP2)\}.$$

Proof. We have

$$\begin{aligned}
&\min\{f(x) : x \in \mathfrak{F}(P)\} = f\left(x^0\right) = \\
&= f\left(x^0\right) + \left\langle \nabla f\left(x^0\right), \eta\left(x^0, x^0\right)\right\rangle + \tfrac{1}{2}\left\langle \eta\left(x^0, x^0\right), \left[\nabla^2 f\left(x^0\right)\right]\left(\eta\left(x^0, x^0\right)\right)\right\rangle \geqq \\
&\geqq \inf\left\{ f\left(x^0\right) + \left\langle \nabla f\left(x^0\right), \eta\left(x, x^0\right)\right\rangle + \right. \\
&\quad \left. + \tfrac{1}{2}\left\langle \eta\left(x, x^0\right), \left[\nabla^2 f\left(x^0\right)\right]\left(\eta\left(x, x^0\right)\right)\right\rangle : x \in \mathfrak{F}(P)\right\} \geqq \ \text{(from (4.1))} \\
&\geqq \inf\left\{ f\left(x^0\right) + \left\langle \nabla f\left(x^0\right), \eta\left(x, x^0\right)\right\rangle + \right. \\
&\quad \left. + \tfrac{1}{2}\left\langle \eta\left(x, x^0\right), \left[\nabla^2 f\left(x^0\right)\right]\left(\eta\left(x, x^0\right)\right)\right\rangle : x \in \mathfrak{F}(AP2)\right\} = \\
&= \inf\{F(x) : x \in \mathfrak{F}(AP2)\}.
\end{aligned}$$

□

The lagrangian of Problem $(AP2)$ will be denoted by $L_\eta^{(2,1)}$, i.e. $L_\eta^{(2,1)} : X \times \mathbb{R}_+^m \to \mathbb{R}$ is defined by

$$\begin{aligned}
L_\eta^{(2,1)}(x, v) &:= F(x) + \langle v, G(x)\rangle = \\
&= f\left(x^0\right) + \left\langle \nabla f\left(x^0\right), \eta\left(x, x^0\right)\right\rangle + \\
&\quad + \tfrac{1}{2}\left\langle \eta\left(x, x^0\right), \left[\nabla^2 f\left(x^0\right)\right]\left(\eta\left(x, x^0\right)\right)\right\rangle + \\
&\quad + \left\langle v, g\left(x^0\right)\right\rangle + \left\langle v, \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x, x^0\right)\right)\right\rangle,
\end{aligned}$$

for all $(x, v) \in X \times \mathbb{R}_+^m$.

EXAMPLE 4.6. Let us consider the optimization problem

$$(\overline{P}) \qquad \begin{array}{ll} \min & f(x) = \exp x \\ \text{s.t.} & x \in X = \mathbb{R} \\ & g(x) = x^2 - x \leqq 0. \end{array}$$

We have that $\mathfrak{F}\left(\overline{P}\right) = [0, 1]$ and $x^0 = 0$ is the unique optimal solution of Problem $\left(\overline{P}\right)$.

The functions f and g are invex at $x^0 = 0$ w.r.t. the function $\eta : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\eta(x, u) = x - u, \text{ for all } (x, u) \in \mathbb{R}^2.$$

Then the $(2, 1)$-η-approximated optimization problem is

$$(A\overline{P}2) \qquad \begin{array}{ll} \min & \left(1 + x + \frac{1}{2}x^2\right) \\ \text{s.t.} & x \in X = \mathbb{R} \\ & -x \leqq 0, \end{array}$$

which has the optimal solution $x^0 = 0$.

On the other hand, the lagrangian $\overline{L}_\eta^{(2,1)}$ of Problem $\left(A\overline{P}2\right)$ is defined by

$$\overline{L}_\eta^{(2,1)}(x, v) = 1 + x + \tfrac{1}{2}x^2 - vx, \text{ for all } (x, v) \in \mathbb{R} \times \mathbb{R}_+.$$

Obviously, $\left(x^0, v^0\right) = (0,1)$ is a saddle point of the lagrangian $\overline{L}_\eta^{(2,1)}$ of Problem $\left(A\overline{P}2\right)$. □

In this section we show the equivalence between saddle points of the lagrangian $L_\eta^{(2,1)}$, of Problem $(AP2)$, and optimal solutions of Problem $\left(P_\eta^{(2,1)}\left(x^0\right)\right)$.

By Theorem 2.9, the following saddle point theorem follows:

THEOREM 4.7. *If $\left(x^0, v^0\right) \in X \times \mathbb{R}_+^m$ is a saddle point of the lagrangian $L_\eta^{(2,1)}$ of Problem $(AP2)$, then x^0 is an optimal solution of Problem $(AP2)$.*

REMARK 4.8. We established Theorem 4.7, without any assumption about the function involved in Problem $(AP2)$. □

Now, we can state the converse theorem of Theorem 4.7.

THEOREM 4.9. *Let $x^0 \in X$ be an optimal solution of Problem $(AP2)$, $\mu : X \times X \to \mathbb{R}^n$ be a function. Assume that $\eta\left(\cdot, x^0\right) : X \to \mathbb{R}^n$ is differentiable at x^0, the functions F, $G = (G_1, ..., G_m) : X \to \mathbb{R}^m$ are invex at x^0 w.r.t. μ and a suitable constraint qualification $(CQ$, [11]$)$ is satisfied at x^0. Then there exists a point $v^0 \in \mathbb{R}_+^m$ such that $\left(x^0, v^0\right)$ is a saddle point of Problem $(AP2)$.*

Proof. In view of Karush-Kuhn-Tucker theorem, there exists a point $v^0 \in \mathbb{R}_+^m$ such that

$$\nabla F\left(x^0\right) + \left[\nabla G\left(x^0\right)\right]^{\mathrm{T}}\left(v^0\right) = 0, \tag{4.2}$$

$$\left\langle v^0, G\left(x^0\right)\right\rangle = 0. \tag{4.3}$$

The functions F and G are invex at x^0 w.r.t. μ, then, for each $x \in X$, we have

$$F(x) - F\left(x^0\right) \geqq \left\langle \nabla F\left(x^0\right), \mu\left(x, x^0\right)\right\rangle, \tag{4.4}$$

$$G(x) - G\left(x^0\right) \geqq \left[\nabla G\left(x^0\right)\right]\left(\mu\left(x, x^0\right)\right). \tag{4.5}$$

Since $v^0 \in \mathbb{R}_+^m$, by (4.5), we obtain

$$\left\langle v^0, G(x) - G\left(x^0\right)\right\rangle \geqq \left\langle v^0, \left[\nabla G\left(x^0\right)\right]\left(\mu\left(x, x^0\right)\right)\right\rangle = \tag{4.6}$$

$$= \left\langle \left[\nabla G\left(x^0\right)\right]^{\mathrm{T}}\left(v^0\right), \mu\left(x, x^0\right)\right\rangle, \text{ for all } x \in X.$$

Then, for each $x \in X$

$$L_\eta^{(2,1)}\left(x, v^0\right) - L_\eta^{(2,1)}\left(x^0, v^0\right) =$$
$$= F(x) - F\left(x^0\right) + \left\langle v^0, G(x) - G\left(x^0\right)\right\rangle \geqq \text{ (by (4.4), and (4.6))}$$
$$\geqq \left\langle \nabla F\left(x^0\right) + \left[\nabla G\left(x^0\right)\right]^{\mathrm{T}}\left(v^0\right), \mu\left(x, x^0\right)\right\rangle = \text{ (by (4.2))} = 0.$$

Consequently, the second inequality of the definition of saddle point is satisfied.

In order to prove the first inequality of the definition of saddle point, let $v \in \mathbb{R}^m_+$. Then

$$L^{(2,1)}_\eta\left(x^0, v^0\right) - L^{(2,1)}_\eta\left(x^0, v\right) =$$
$$= \left\langle v^0, G\left(x^0\right)\right\rangle - \left\langle v, G\left(x^0\right)\right\rangle = \text{ (by (4.3))}$$
$$= -\left\langle v, G\left(x^0\right)\right\rangle \geqq 0,$$

because $G\left(x^0\right) \leqq 0$ and $v \in \mathbb{R}^m_+$. □

5. EQUIVALENCE BETWEEN SADDLE POINTS OF $(2,1)$-η-APPROXIMATED PROBLEM AND OF THE ORIGINAL PROBLEM

In this section, X is a subset of $\mathbb{R}^n$, x^0 is an interior point of X, $f : X \to \mathbb{R}$ is a twice continuously differentiable function at x^0, and $g : X \to \mathbb{R}^m$ is a differentiable function at x^0.

We will prove the equivalence between the original optimization problem (P) and its associated $(2,1)$-η-approximated optimization problem $(AP2)$. We establish the results where one assumes that the function η satisfies only the condition $\eta\left(x^0, x^0\right) = 0$.

The following statement is true

Theorem 5.1. *Let $\eta : X \times X \to \mathbb{R}^n$ such that $\eta\left(x^0, x^0\right) = 0$, $f : X \to \mathbb{R}$ be second order pseudoinvex function at x^0 w.r.t. η and $g = (g_1, ..., g_m) : X \to \mathbb{R}^m$ such that g_i, $i \in I\left(x^0\right)$ are quasiinvex functions at x^0 w.r.t. η.*

If $\left(x^0, v^0\right) \in X \times \mathbb{R}^m_+$ is a saddle point of the lagrangian $L^{(2,1)}_\eta$ of Problem $(AP2)$, then x^0 is an optimal solution of the original problem (P).

Proof. The point $\left(x^0, v^0\right) \in X \times \mathbb{R}^m_+$ is a saddle point of the lagrangian $L^{(2,1)}_\eta$ of Problem $(AP2)$; then

$$L^{(2,1)}_\eta\left(x^0, v\right) \leqq L^{(2,1)}_\eta\left(x^0, v^0\right), \text{ for all } v \in \mathbb{R}^m_+,$$

i.e.

$$\left\langle v - v^0, g\left(x^0\right)\right\rangle \leqq 0, \text{ for all } v \in \mathbb{R}^m_+, \tag{5.1}$$

because $\eta\left(x^0, x^0\right) = 0$.

Let $i \in \{1, ..., m\}$, and $e^i = (0, ..., 1, ..., 0) \in \mathbb{R}^m$ be the i-th unit point of $\mathbb{R}^m$. Then, for $v = e^i + v^0 \in \mathbb{R}^m_+$, relation (5.1) becomes $g_i\left(x^0\right) \leqq 0$. Hence

$$g_i\left(x^0\right) \leqq 0, \text{ for all } i \in \{1, ..., m\}.$$

Consequently,

$$x^0 \in \mathfrak{F}(P).$$

If follows that

$$\left\langle v^0, g\left(x^0\right)\right\rangle \leqq 0, \tag{5.2}$$

because $v^0 \in \mathbb{R}^m_+$. But, from (5.1) we deduce

$$\left\langle v^0, g\left(x^0\right)\right\rangle \geqq 0, \tag{5.3}$$

because $v = 0 \in \mathbb{R}^m_+$.

Thus, by (5.2) and (5.3)

$$\left\langle v^0, g\left(x^0\right)\right\rangle = 0. \tag{5.4}$$

From (5.4) it follows that

$$v_i^0 = 0, \text{ for all } i \in \{1, ..., m\} \backslash I\left(x^0\right). \tag{5.5}$$

On the other hand, from

$$L_\eta^{(2,1)}\left(x^0, v^0\right) \leqq L_\eta^{(2,1)}\left(x, v^0\right), \text{ for all } x \in X,$$

we deduce that

(5.6)

$$\left\langle \nabla f\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]^T\left(v^0\right) + \frac{1}{2}\left[\nabla^2 f\left(x^0\right)\right]\left(\eta\left(x, x^0\right)\right), \eta\left(x, x^0\right)\right\rangle \geqq 0,$$

for all $x \in X$.

In order to prove that x^0 is an optimal solution of Problem (P), let $x \in \mathfrak{F}(P)$. Then

$$g_i(x) \leqq 0, \text{ for all } i \in \{1, ..., m\}.$$

Let $i \in I\left(x^0\right)$. Since

$$g_i(x) - g_i\left(x^0\right) = g_i(x) \leqq 0,$$

and g_i is quasiinvex at x^0 w.r.t. η, we have

$$\left\langle \nabla g_i\left(x^0\right), \eta\left(x, x^0\right)\right\rangle \leqq 0,$$

hence

$$\left\langle v_i^0 \nabla g_i\left(x^0\right), \eta\left(x, x^0\right)\right\rangle \leqq 0,$$

because $v_i^0 \geqq 0$. Then

$$\left\langle \left[\nabla g\left(x^0\right)\right]^{\mathrm{T}}\left(v^0\right), \eta\left(x, x^0\right)\right\rangle \leqq 0, \tag{5.7}$$

because $v_i^0 = 0$, for all $i \in \{1, ..., m\} \backslash I\left(x^0\right)$.

From (5.6) and (5.7) it follows that

$$\left\langle \nabla f\left(x^0\right) + \frac{1}{2}\left[\nabla^2 f\left(x^0\right)\right]\left(\eta\left(x, x^0\right)\right), \eta\left(x, x^0\right)\right\rangle \geqq 0. \tag{5.8}$$

But, the function f is second order pseudoinvex at x^0 w.r.t. η, and then, by (5.8), we deduce that

$$f(x) \geqq f\left(x^0\right).$$

Consequently, x^0 is an optimal solution of the original problem (P). The theorem is proved. □

Remark 5.2. If the function f is second order invex at x^0 w.r.t. η, and $g_1, ..., g_m$ are invex at x^0 with respect to η, then the hypotheses that f is second order pseudoinvex at x^0 w.r.t. η and g_i, $i \in I(x^0)$ are quasiinvex at x^0 w.r.t. η are satisfied. $\square$

Remark 5.3. The assumption that the function η satisfies the condition $\eta(x^0, x^0) = 0$ is essential in order to have the equivalence between the saddle points of the lagrangian $L_\eta^{(2,1)}$ of Problem $(AP2)$, and the optimal solutions of the original problem (P). (see Example 3.4 from [2]) $\square$

Now, we show that, if x^0 is an optimal solution of the original problem (P), then under certain conditions, there exists a point $v^0 \in \mathbb{R}_+^m$ such that (x^0, v^0) is a saddle point of the η-approximated problem $(AP2)$.

More exactly, the following statement is true:

Theorem 5.4. *Let $x^0 \in X$ be an optimal solution of the original problem (P) and assume that a suitable constraint qualification is satisfied at x^0 (CQ in Ref.* [11]*). If the function $\eta : X \times X \to \mathbb{R}^m$ satisfies*

(i) $\langle \nabla f(x^0), \eta(x^0, x^0) \rangle \leqq 0$;

(ii) $g(x^0) + [\nabla g(x^0)](\eta(x^0, x^0)) \leqq 0$ $(i.e.\ x^0 \in \mathfrak{F}(AP2))$,

(iii) x^0 *is an optimal solution of the problem*

$$\begin{array}{ll} \min & \langle \eta(x, x^0), [\nabla^2 f(x^0)](\eta(x, x^0)) \rangle \\ s.t. & x \in X, \end{array}$$

then there exists a point $v^0 \in \mathbb{R}_+^m$ such that (x^0, v^0) is a saddle point of the lagrangian $L_\eta^{(2,1)}$ of the $(2,1)$-η-approximated problem $(AP2)$.

Proof. Since x^0 is an optimal solution of Problem (P), and a suitable constraint qualification at x^0 is satisfied, by Karush-Kuhn-Tucker's Theorem, there exists a point $v^0 \in \mathbb{R}_+^m$ such that

$$\nabla f(x^0) + [\nabla g(x^0)]^{\mathrm{T}}(v^0) = 0, \tag{5.9}$$

$$\langle v^0, g(x^0) \rangle = 0. \tag{5.10}$$

Let $x \in X$. Then, from (5.9) and hypothesis (iii), we have

$$\begin{aligned} & L_\eta^{(2,1)}(x, v^0) - L_\eta^{(2,1)}(x^0, v^0) = \\ & = f(x^0) + \langle \nabla f(x^0), \eta(x, x^0) \rangle + \tfrac{1}{2} \langle \eta(x, x^0), [\nabla^2 f(x^0)](\eta(x, x^0)) \rangle + \\ & \quad + \langle v^0, g(x^0) \rangle + \langle v^0, [\nabla g(x^0)](\eta(x, x^0)) \rangle - \\ & \quad - f(x^0) - \langle \nabla f(x^0), \eta(x^0, x^0) \rangle - \tfrac{1}{2} \langle \eta(x^0, x^0), [\nabla^2 f(x^0)](\eta(x^0, x^0)) \rangle - \\ & \quad - \langle v^0, g(x^0) \rangle - \langle v^0, [\nabla g(x^0)](\eta(x^0, x^0)) \rangle = \end{aligned}$$

$$= \left\langle \nabla f\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]^{\mathrm{T}}\left(v^0\right), \eta\left(x, x^0\right)\right\rangle -$$
$$- \left\langle \nabla f\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]^{\mathrm{T}}\left(v^0\right), \eta\left(x^0, x^0\right)\right\rangle +$$
$$+ \tfrac{1}{2}\left[\left\langle \eta\left(x, x^0\right), \left[\nabla^2 f\left(x^0\right)\right]\left(\eta\left(x, x^0\right)\right)\right\rangle -\right.$$
$$\left. - \left\langle \eta\left(x^0, x^0\right), \left[\nabla^2 f\left(x^0\right)\right]\left(\eta\left(x^0, x^0\right)\right)\right\rangle\right] =$$
$$= \tfrac{1}{2}\left[\left\langle \eta\left(x, x^0\right), \left[\nabla^2 f\left(x^0\right)\right]\left(\eta\left(x, x^0\right)\right)\right\rangle -\right.$$
$$\left. - \left\langle \eta\left(x^0, x^0\right), \left[\nabla^2 f\left(x^0\right)\right]\left(\eta\left(x^0, x^0\right)\right)\right\rangle\right] \geqq 0.$$

Consequently, the second inequality from the saddle point definition is true.

In order to prove the first inequality from the saddle point definition, let $v \in \mathbb{R}^m_+$. Then

$$L^{(2,1)}_{\eta}\left(x^0, v^0\right) - L^{(2,1)}_{\eta}\left(x^0, v\right) =$$
$$= f\left(x^0\right) + \left\langle \nabla f\left(x^0\right), \eta\left(x^0, x^0\right)\right\rangle + \tfrac{1}{2}\left\langle \eta\left(x^0, x^0\right), \left[\nabla^2 f\left(x^0\right)\right]\left(\eta\left(x^0, x^0\right)\right)\right\rangle +$$
$$+ \left\langle v^0, g\left(x^0\right)\right\rangle + \left\langle v^0, \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x^0, x^0\right)\right)\right\rangle -$$
$$- f\left(x^0\right) - \left\langle \nabla f\left(x^0\right), \eta\left(x^0, x^0\right)\right\rangle - \tfrac{1}{2}\left\langle \eta\left(x^0, x^0\right), \left[\nabla^2 f\left(x^0\right)\right]\left(\eta\left(x^0, x^0\right)\right)\right\rangle -$$
$$- \left\langle v, g\left(x^0\right)\right\rangle - \left\langle v, \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x^0, x^0\right)\right)\right\rangle =$$
$$= \left\langle \nabla f\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]^{\mathrm{T}}\left(v^0\right), \eta\left(x^0, x^0\right)\right\rangle - \left\langle \nabla f\left(x^0\right), \eta\left(x^0, x^0\right)\right\rangle -$$
$$- \left\langle g\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]^{\mathrm{T}}\left(\eta\left(x^0, x^0\right)\right), v\right\rangle = \text{ (by (5.9) and (5.10))}$$
$$= - \left\langle \nabla f\left(x^0\right), \eta\left(x^0, x^0\right)\right\rangle - \left\langle g\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x^0, x^0\right)\right), v\right\rangle \geqq$$
$$\geqq \text{ (by } (i) \text{ and } (ii)) \geqq 0.$$

Consequently, $\left(x^0, v^0\right)$ is a saddle point of the lagrangian of Problem $(AP2)$.

□

REMARK 5.5. If $\eta\left(x^0, x^0\right) = 0$, then the hypotheses (i) and (ii) from Theorem 5.4 are satisfied. □

REMARK 5.6. If f and $g = (g_1, ..., g_m)$ are invex at x^0 w.r.t. η, then the hypotheses (i) and (ii) from Theorem 5.4 are satisfied. □

REMARK 5.7. The hypothesis that the original problem (P) satisfies a suitable constraint qualification at x^0 is essential. Indeed, for the problem

$$(\widehat{P}) \qquad \begin{array}{ll} \min & f(x) = x_2 \\ \text{s.t.} & x \in X = \mathbb{R}^2 \\ & g_1(x) = x_1 + x_2^2 \leqq 0, \\ & g_2(x) = -x_1 + x_2^2 \leqq 0, \end{array}$$

we have the set of all feasible solutions $\mathfrak{F}(\widehat{P}) = \{(0,0)\}$, and hence $x^0 = (0,0)$ is the unique optimal solution. Let us remark that Problem $(\widehat{P})$ is convex, and then the functions f, g_1, g_2 are invex w.r.t. $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\eta(x,u) = x - u, \text{ for all } (x,u) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

In this case, the $(2,1)$-η-approximated optimization problem is

$$(A\widehat{P}2) \qquad \begin{array}{ll} \min & x_2 \\ \text{s.t.} & (x_1,x_2) \in \mathbb{R}^2 \\ & -x_1 \leqq 0 \\ & x_1 \leqq 0. \end{array}$$

Thus, $\widehat{L}_\eta^{(2,1)} : \mathbb{R}^2 \times \mathbb{R}^2_+ \to \mathbb{R}$ is defined by

$$\widehat{L}_\eta^{(2,1)}(x,v) = x_2 - v_1 x_1 + v_2 x_1,$$

for all $(x,v) = ((x_1,x_2),(v_1,v_2)) \in \mathbb{R}^2 \times \mathbb{R}^2_+$. □

Easy to show that, for each $v^0 = \left(v_1^0, v_2^0\right) \in \mathbb{R}^2_+$, the point $\left(x^0, v^0\right)$ is not a saddle point of the lagrangian of Problem $\left(A\widehat{P}2\right)$.

6. CONCLUSIONS

This paper shows how, under some hypotheses, to solve an optimization problem is equivalent with finding the saddle points $\left(x^0, v^0\right)$ of the so called $(2,1)$-η- approximated problem at x^0 of the original problem.

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