REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 39 (2010) no. 2, pp. 134–140
ictp.acad.ro/jnaat

ON THE EXISTENCE AND UNIQUENESS OF EXTENSIONS OF SEMI-HÖLDER REAL-VALUED FUNCTIONS

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Abstract. Let (X,d) be a quasi-metric space, $y_0 \in X$ a fixed element and Y a subset of X such that $y_0 \in Y$. Denote by $(\Lambda_{\alpha,0}(Y,d), \|\cdot|^{\alpha}_{Y,d})$ the asymmetric normed cone of real-valued d-semi-Hölder functions defined on Y of exponent $\alpha \in (0,1]$, vanishing in y_0, and by $(\Lambda_{\alpha,0}(Y,\bar{d}), \|\cdot|^{\alpha}_{Y,\bar{d}})$ the similar cone if d is replaced by conjugate $\bar{d}$ of d.

One considers the following claims:

(a) For every f in the linear space $\Lambda_{\alpha,0}(Y) = \Lambda_{\alpha,0}(Y,d) \cap \Lambda_{\alpha,0}(Y,\bar{d})$ there exist $F \in \Lambda_{\alpha,0}(X,d)$ such that $F|_Y = f$ and $\|F|^{\alpha}_{Y,d} = \|f|^{\alpha}_{Y,d}$;

(b) For every $f \in \Lambda_{\alpha,0}(Y)$ there exists $\bar{F} \in \Lambda_{\alpha,0}(X,\bar{d})$ such that $\bar{F}|_Y = f$ and $\|\bar{F}|^{\alpha}_{Y,\bar{d}} = \|f|^{\alpha}_{Y,\bar{d}}$;

(c) The extension F in (a) is unique;

(d) The extension $\bar{F}$ in (b) is unique;

(e) The annihilator $Y^{\perp}_{\bar{d}}$ of Y in $\Lambda_{\alpha,0}(X,\bar{d})$ is proximinal for the elements of $\Lambda_{\alpha,0}(X)$ with respect to the distance generated by $\|\cdot|^{\alpha}_{Y,d}$;

(f) The annihilator $Y^{\perp}_{d}$ of Y in $\Lambda_{\alpha,0}(X,d)$ is proximinal for the elements of $\Lambda_{\alpha,0}(X)$ with respect to the distance generated by $\|\cdot|^{\alpha}_{Y,\bar{d}}$;

(g) $Y^{\perp}_{\bar{d}}$ in the claim (e) is Chebyshevian;

(h) $Y^{\perp}_{d}$ in the claim (f) is Chebyshevian.

Then the following equivalences hold:
$(a) \Leftrightarrow (e)$; $(b) \Leftrightarrow (f)$; $(c) \Leftrightarrow (g)$; $(d) \Leftrightarrow (h)$.

MSC 2000. 46A22, 41A50, 41A52.
Keywords. Extensions, semi-Lipschitz functions, semi-Hölder functions, best approximation, quasi-metric spaces.

1. INTRODUCTION

Let X be a nonempty set and $d : X \times X \to [0,\infty)$ a function with the properties:

(QM_1) $d(x,y) = d(y,x) = 0$ iff $x = y$,
(QM_2) $d(x,y) \le d(x,z) + d(z,y)$,
for all $x,y,z \in X$.

Then the function d is called a *quasi-metric* on X and the pair (X,d) is called *quasi-metric space* ([13]).

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Because, in general, $d(x,y) \neq d(y,x)$, for $x, y \in X$ one defines the conjugate quasi-metric $\overline{d}$ of d, by the equality $\bar{d}(x,y) = d(y,x)$, for all $x, y \in X$.

Let Y be a nonvoid subset of (X,d) and $\alpha \in (0,1]$ a fixed number.

Definition 1. a) *A function $f : Y \to \mathbb{R}$ is called d-semi-Hölder (of exponent α) if there exists a constant $K_Y(f) \geq 0$ such that*

$$f(x) - f(y) \leq K_Y(f) d^\alpha(x,y), \tag{1}$$

for all $x, y \in Y$.

b) *$f : Y \to \mathbb{R}$ is called $\overline{d}$-semi-Hölder (of exponent α) if there exists a constant $\overline{K}_Y(f) \geq 0$ such that*

$$f(x) - f(y) \leq \overline{K}_Y(f) \cdot d^\alpha(y,x), \tag{2}$$

for all $x, y \in Y$.

The smallest constant $K_Y(f)$ in (1) is denoted by $\|f|^\alpha_{Y,d}$ and one shows that

$$\|f|^\alpha_{Y,d} := \sup\left\{ \tfrac{(f(x)-f(y))\vee 0}{d^\alpha(x,y)} : d(x,y) > 0; x, y \in Y \right\}. \tag{3}$$

Analogously one defines $\|f|^\alpha_{Y,\overline{d}}$.

Observe that the function f is d-semi-Hölder on Y iff $-f$ is $\overline{d}$-semi-Hölder on Y. Moreover

$$\|f|^\alpha_{Y,d} = \|-f|^\alpha_{Y,\overline{d}}. \tag{4}$$

Definition 2. *([14]). Let (X,d) be a quasi-metric space and $Y \subseteq X$ a nonempty set. The function $f : Y \to \mathbb{R}$ is called $\leq_d$-increasing on Y if $f(x) \leq f(y)$ whenever $d(x,y) = 0$, $x, y \in Y$.*

The set of all $\leq_d$-increasing functions on Y is denoted by $\mathbb{R}^Y_{\leq_d}$ and it is a cone in the linear space $\mathbb{R}^Y$ of all real-valued functions on Y.

The set

$$\Lambda_\alpha(Y,d) := \{f \in \mathbb{R}^Y_{\leq_d}; f \text{ is } d\text{-semi-Hölder and } \|f|^\alpha_{Y,d} < \infty\} \tag{5}$$

is also a cone, called the cone of d-semi-Hölder functions on Y.

If $y_0 \in Y$ is arbitrary, but fixed, one considers the cone

$$\Lambda_{\alpha,0}(Y,d) := \{f \in \Lambda_\alpha(Y,d) : f(y_0) = 0\}. \tag{6}$$

Then the functional $\| \,|^\alpha_{Y,d} : \Lambda_{\alpha,0}(Y,d) \to [0,\infty)$ is subadditive, positively homogeneous and the equality $\|f|^\alpha_{Y,d} = \|-f|^\alpha_{Y,d} = 0$ implies $f \equiv 0$. This means that $\|\cdot|^\alpha_{Y,d}$ is an asymmetric norm (see [13], [14]), on the cone $\Lambda_{\alpha,0}(Y,d)$.

The pair $\left(\Lambda_{\alpha,0}(Y,d), \| \,|^\alpha_{Y,d}\right)$ is called the asymmetric normed cone of d-semi-Hölder real-valued function on Y (compare with [14]).

Analogously, one defines the asymmetric normed cone $(\Lambda_{\alpha,0}(Y,\overline{d}), \|\cdot|^\alpha_{Y,\overline{d}})$. of all $\overline{d}$-semi-Hölder real-valued functions on Y, vanishing at the fixed point $y_0 \in Y$.

By the above definitions it follows that

$$f \in (\Lambda_{\alpha,0}(Y,d), \| \ |_{Y,d}^{\alpha}) \text{ iff } -f \in (\Lambda_{\alpha,0}(Y,\overline{d}), \| \ |_{Y,\overline{d}}^{\alpha})$$

and, moreover, $\|f|_{Y,d}^{\alpha} = \|-f|_{Y,\overline{d}}^{\alpha}$.

Defining $\Lambda_{\alpha,0}(Y)$ by

$$\Lambda_{\alpha,0}(Y) = \Lambda_{\alpha,0}(Y,d) \cap \Lambda_{\alpha,0}(Y,\overline{d}), \tag{7}$$

It follows that $\Lambda_{\alpha,0}(Y)$ is a linear subspace. The following, theorem holds.

THEOREM 3. *For every $f \in \Lambda_{\alpha,0}(Y)$ there exist at least one function $F \in \Lambda_{\alpha,0}(Y,d)$ and at least one function $\overline{F} \in \Lambda_{\alpha,0}(Y,\overline{d})$ such that*

a) $F|_Y = \overline{F}\big|_Y = f$.

b) $\|F|_{Y,d}^{\alpha} = \|f|_{Y,d}^{\alpha}$ *and* $\|\overline{F}|_{Y,\overline{d}}^{\alpha} = \|f|_{Y,\overline{d}}^{\alpha}$.

Proof. By Theorem 2 and Remark 3 in [11] it follows that the functions defined by the formulae:

$$F(f)(x) = \inf_{y\in Y}\{f(y) + \|f|_{Y,d}^{\alpha}\, d^{\alpha}(x,y)\},\ x \in X, \tag{8}$$
$$G(f)(x) = \sup_{y\in Y}\{f(y) - \|f|_{Y,d}^{\alpha}\, d^{\alpha}(y,x\},\ x \in X,$$

are elements of $\Lambda_{\alpha,0}(X,d)$ and, respectively, the functions given by

$$\overline{F}(f)(x) = \inf_{y\in Y}\{f(y) + \|f|_{Y,\overline{d}}^{\alpha}\, d^{\alpha}(y,x)\},\ x \in X, \tag{9}$$
$$\overline{G}(f)(x) = \sup_{y\in Y}\{f(y) - \|f|_{Y,\overline{d}}^{\alpha}\, d^{\alpha}(x,y)\},\ x \in X$$

are elements of $\Lambda_{\alpha,0}(X,\overline{d})$ such that

$$F(f)|_Y = G(f)|_Y = f \text{ and } \|F(f)|_{Y,d}^{\alpha} = \|G(f)|_{Y,d}^{\alpha} = \|f|_{Y,d}^{\alpha}, \tag{10}$$

respectively

$$\overline{F}(f)\big|_Y = \overline{G}(f)\big|_Y = f \text{ and } \big\|\overline{F}(f)\big|_{Y,\overline{d}}^{\alpha} = \big\|\overline{G}(f)\big|_{Y,\overline{d}}^{\alpha} = \|f|_{Y,\overline{d}}^{\alpha}. \tag{11}$$

□

For $f \in \Lambda_{\alpha,0}(Y)$ let us consider the following (nonempty) sets of extensions:

$$\mathcal{E}_d(f) := \{H \in \Lambda_{\alpha,0}(X,d) : H|_Y = f \text{ and } \|H|_{Y,d}^{\alpha} = \|f|_{Y,d}^{\alpha}\} \tag{12}$$

and

$$\mathcal{E}_{\overline{d}}(f) := \{\overline{H} \in \wedge_{\alpha,0}(X,\overline{d}) : \overline{H}\big|_Y = f \text{ and } \big\|\overline{H}\big|_{Y,\overline{d}}^{\alpha} = \|f|_{Y,\overline{d}}^{\alpha}\}. \tag{13}$$

The sets $\mathcal{E}_d(f)$ and $\mathcal{E}_{\overline{d}}(f)$ are convex and

$$F(f)(x) \geq H(x) \geq G(f)(x),\ x \in X \tag{14}$$

for all $H \in \mathcal{E}_d(f)$;

$$\overline{F}(f)(x) \geq \overline{H}(x) \geq \overline{G}(f)(x),\ x \in H, \tag{15}$$

for all $\overline{H} \in \mathcal{E}_{\overline{d}}(f)$.

Also, for $F \in \Lambda_{\alpha,0}(X),\ F|_Y \in \Lambda_{\alpha,0}(Y)$ and

$$F - H \in \Lambda_{\alpha,0}(X, \overline{d}),\ \text{for all } H \in \mathcal{E}_d(F|_Y),$$
$$F - \overline{H} \in \Lambda_{\alpha,0}(X, d) \text{ for all } \overline{H} \in \mathcal{E}_{\overline{d}}(F|_Y).$$

Let (X, d) be a quasi-metric space, $y_0 \in X$ fixed and $Y \subseteq X$ such that $y_0 \in Y$. Let

$$Y_d^{\perp} := \{G \in \Lambda_{\alpha,0}(X, d) : G|_Y = 0\} \tag{16}$$

and

$$Y_{\overline{d}}^{\perp} := \{\overline{G} \in \wedge_{\alpha,d}(X, \overline{d}) : \overline{G}\big|_Y = 0\}. \tag{17}$$

Obviously, for $F \in \Lambda_{\alpha,0}(X)$

$$F - \mathcal{E}_d(F|_Y) \subset \Lambda_{\alpha,0}(X, \overline{d}) \tag{18}$$

and

$$F - \mathcal{E}_{\overline{d}}(F|_Y) \subset \Lambda_{\alpha,0}(X, d). \tag{19}$$

In the sequel we prove a result of Phelps type ([1], [10], [12]) concerning the existence and uniqueness of the extensions preserving the smallest semi-Hölder constants and a problem of best approximation by elements of $Y_d^{\perp}$ and $Y_{\overline{d}}^{\perp}$, respectively.

Let $(X, \|\ |)$ be an asymmetric norm (see [13], [14]) and let M be a nonempty set of X. The set M is called *proximinal for* $x \in X$ iff there exists at least one element $m_0 \in M$ such that

$$\|x - m_0| = \inf\{\|x - m| : m \in M\} = \rho(x, M).$$

If M is proximinal for x, then the set $P_M(x) = \{m_0 \in M : \|x - m_0| = \rho(x, M)\}$ is called the set of *elements of best approximations* for x in M. If card $P_M(x) = 1$ then the set M is called Chebyshevian for x.

The set M is called proximinal if M is proximinal for every $x \in X$, and Chebyshevian if M is Chebyshevian for every $x \in X$.

Now, consider the following two problems of best approximation:

$\mathbf{P}_{\overline{\mathrm{d}}}\mathbf{(F)}$**.** For $F \in \Lambda_{\alpha,0}(X)$ find $G_0 \in Y_d^{\perp}$ such that

$$\|F - G_0|_{Y,\overline{d}}^{\alpha} = \inf\{\|F - G|_{Y,\overline{d}}^{\alpha} : G \in Y_d^{\perp}\} = \rho_{\overline{d}}(F, Y_d^{\perp}) \tag{20}$$

and

$\mathbf{P}_{\mathrm{d}}\mathbf{(F)}$**.** For $F \in \Lambda_{\alpha,0}(X)$ find $\overline{G}_0 \in Y_{\overline{d}}^{\perp}$ such that

$$\big\|F - \overline{G}_0\big|_{Y,d}^{\alpha} = \inf\{\big\|F - \overline{G}\big|_{X,d}^{\alpha} : \overline{G} \in Y_{\overline{d}}^{\perp}\} = \rho_d(F, Y_{\overline{d}}^{\perp}). \tag{21}$$

Let

$$P_{Y_{\overline{d}}^{\perp}}(F) := \{\overline{G}_0 \in Y_{\overline{d}}^{\perp} : \big\|F - \overline{G}_0\big|_{X,d}^{\alpha} = \rho_d(F, Y_{\overline{d}}^{\perp})\} \tag{22}$$

and

$$P_{Y_d^{\perp}}(F) := \{G_0 \in Y_d^{\perp} : \|F - G_0|_{X,\overline{d}}^{\alpha} = \rho_{\overline{d}}(F, Y_d^{\perp})\}. \tag{23}$$

The following theorem holds.

THEOREM 4. *If* $F \in \Lambda_{\alpha,0}(X)$ *then*

$$P_{Y_{\overline{d}}^{\perp}}(F) = F - \mathcal{E}_d(F|_Y), \tag{24}$$

$$P_{Y_d^{\perp}}(F) = F - \mathcal{E}_{\overline{d}}(F|_Y) \tag{25}$$

and

$$\rho_d(F, Y_{\overline{d}}^{\perp}) = \|F|_Y\|_{Y,d}^{\alpha}, \tag{26}$$

$$\rho_{\overline{d}}(F, Y_d^{\perp}) = \|F|_Y\|_{Y,\overline{d}}^{\alpha}. \tag{27}$$

Proof. Let $F \in \Lambda_{\alpha,0}(X)(= \Lambda_{\alpha,0}(X,d) \cap \Lambda_{\alpha,0}(X,\overline{d}))$

Then, for every $\overline{G} \in Y_{\overline{d}}^{\perp}$,

$$\|F|_Y\|_{Y,d}^{\alpha} = \|F|_Y - \overline{G}|_Y\|_{Y,d}^{\alpha} \leq \|F - \overline{G}\|_{X,d}^{\alpha}.$$

Taking the infimum with respect to $\overline{G} \in Y_{\overline{d}}^{\perp}$, one obtains $\|F|_Y\|_{Y,d}^{\alpha} \leq \rho_d(F, Y_{\overline{d}}^{\perp})$. On the other hand, for every $H \in \mathcal{E}_d(F|_Y)$,

$$\|F|_Y\|_{Y,d}^{\alpha} = \|H\|_{X,d}^{\alpha} = \|F - (F - H)\|_{X,d}^{\alpha}.$$

Because $F - H \in Y_{\overline{d}}^{\perp}$, it follows $\|F|_Y\|_{Y,d}^{\alpha} \geq \rho_d(F, Y_{\overline{d}}^{\perp})$.

Consequently, $Y_{\overline{d}}^{\perp}$ is proximinal with respect to the distance ρ_d (ρ_d-proximinal in short) and

$$\rho_d(F, Y_{\overline{d}}^{\perp}) = \|F|_Y\|_{Y,d}^{\alpha}.$$

Now, let $\overline{G}_0 \in P_{Y_{\overline{d}}^{\perp}}(F)$. Then $(F - \overline{G}_0)|_Y = F|_Y$ and $\|F - \overline{G}_0\|_{X,d}^{\alpha} = \|F|_Y\|_{Y,d}^{\alpha}$. This means that $F - \overline{G}_0 \in \mathcal{E}_d(F|_Y)$, i.e., $\overline{G}_0 \in F - \mathcal{E}_d(F|_Y)$. Consequently, $\overline{G}_0 \in P_{Y_{\overline{d}}^{\perp}}(F)$ implies $\overline{G}_0 \in F - \mathcal{E}_d(F|_Y)$.

Taking into account the first part of the proof it follows $P_{Y_{\overline{d}}^{\perp}}(F) = F - \mathcal{E}_d(F|_Y)$.

Analogously, one obtains $\rho_{\overline{d}}(F, Y_d^{\perp}) = \|F|_Y\|_{Y,\overline{d}}^{\alpha}$ and $P_{Y_d^{\perp}}(F) = F - \mathcal{E}_{\overline{d}}(F|_Y)$. □

By the equalities (22) and (23) it follows.

COROLLARY 5. *Let* $F \in \Lambda_{\alpha,0}(X)$ *and* $Y \subset X$ *such that* $y_0 \in Y$. *Then*

a) card $\mathcal{E}_d(F|_Y) = 1$ *iff* $Y_{\overline{d}}^{\perp}$ *is* ρ_d*-Chebyshevian;*

b) card $\mathcal{E}_{\overline{d}}(F|_Y) = 1$ *iff* $Y_d^{\perp}$ *is* $\rho_{\overline{d}}$*-Chebyshevian.*

REMARK 6. Observe that the linear space $\Lambda_{\alpha,0}(X) = \Lambda_{\alpha,0}(X,d) \cap \Lambda_{\alpha,0}(X,\overline{d})$ is a Banach space with respect to the norm

$$\|F\|_X^{\alpha} = \max\{\|F\|_{X,d}^{\alpha}, \|F\|_{X,\overline{d}}^{\alpha}\}. \tag{28}$$

In fact this space in the space of all real-valued Lipschitz functions defined on the quasi-metric space (X, d^{α}), vanishing at a fixed point $y_0 \in X$. Obviously,

$$\|F\|_X^{\alpha} = \sup\left\{\frac{|F(x)-F(y)|}{d^{\alpha}(x,y)} : d(x,y) > 0;\ x, y \in X\right\} \tag{29}$$

is a norm on $\Lambda_{\alpha,0}(X)$. □

Corollary 7. *For every element f in the space $\Lambda_{\alpha,0}(Y) = \Lambda_{\alpha,0}(Y,d) \cap \Lambda_{\alpha,0}(Y,\overline{d})$ there exists $F \in \Lambda_{\alpha,0}(X)$ such that*

$$F|_Y = f \ \textit{and}\ \|F\|_X^{\alpha} = \|f\|_Y^{\alpha}.$$

The set of all extensions of $f \in \Lambda_{\alpha,0}(Y)$ preserving the norm $\|f\|_Y^{\alpha}$ (of the form (29), is denoted by $\mathcal{E}(f)$, i.e.,

$$\mathcal{E}(f) := \{F \in \Lambda_{\alpha,0}(X) : F|_Y = f \text{ and } \|F\|_X^{\alpha} = \|f\|_Y^{\alpha}\}. \tag{30}$$

Denote by

$$Y^{\perp} := \{G \in \Lambda_{\alpha,0}(X) : G|_Y = 0\}. \tag{31}$$

the annihilator of the set Y in Banach space $\Lambda_{\alpha,0}(X)$, and one considers the following problem of best approximation:

P. For $F \in \Lambda_{\alpha,0}(X)$ find $G_0 \in Y^{\perp}$ such that

$$\|F - G_0\|_X^{\alpha} = \inf\{\|F - G\|_X^{\alpha} : G \in Y^{\perp}\} = \rho(F, Y^{\perp}).$$

Corollary 8. *The subspace $Y^{\perp}$is proximinal in $\Lambda_{\alpha,0}(X)$ and the set of elements of best approximation for $F \in \Lambda_{\alpha,0}(X)$ is*

$$P_{Y^{\perp}}(F) = F - \mathcal{E}(\,F|_Y).$$

The distance of F to $Y^{\perp}$ is given by

$$\rho(F, Y^{\perp}) = \|F|_Y\|_Y^{\alpha}.$$

The subspace $Y^{\perp}$ is Chebyshevian for F iff card $\mathcal{E}(\,F|_Y) = 1$.

For f in the linear space $\Lambda_{\alpha,0}(Y)$, the equalities $F(f)(x) = \overline{F}(f)(x)$, $x \in X$ and $G(f)(x) = \overline{G}(f)(x)$, $x \in X$ are verified iff $\|f|_{Y,d}^{\alpha} = \|f|_{Y,\overline{d}}^{\alpha}$. This means that $\|f|_{Y,d}^{\alpha} = \|-f|_{Y,d}^{\alpha}$ and, consequently,

$$\|f\|_Y^{\alpha} = \max\{\|f|_{Y,d}^{\alpha}; \|f|_{Y,\overline{d}}^{\alpha}\} = \|f|_{Y,d}^{\alpha}.$$

By Theorem 3 in [14], it follows that $\Lambda_{a,0}(Y)$ is a Banach space and (Y, d^{α}) is a metric space.

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Received by the editors: April 13, 2010.