DURRMEYER-STANCU TYPE OPERATORS

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Abstract. Considering two given real parameters \( \alpha, \beta \) satisfying the conditions \( 0 \leq \alpha \leq \beta \). D. D. Stancu [7] constructed and studied the linear positive operators \( P^m_{\alpha, \beta} : C([0, 1]) \to C([0, 1]) \), defined for any \( f \in C([0, 1]) \) and any positive integer \( m \) by (1). In this paper we are dealing with the Durrmeyer form of Stancu’s operators. Some approximation properties of these Durrmeyer-Stancu operators are established. As a particular case, we retrieve approximation properties for the classical Durrmeyer operators [5].


Keywords. Linear positive operators, Durrmeyer operators, first order modulus of smoothness, Shisha-Mond theorem.

1. PRELIMINARIES

Let \( \alpha, \beta \) be real parameters satisfying the conditions \( 0 \leq \alpha \leq \beta \). In 1969 D. D. Stancu [7] constructed and studied the linear positive operators \( P^m_{\alpha, \beta} : C([0, 1]) \to C([0, 1]) \) defined for any \( f \in C([0, 1]) \), any \( x \in [0, 1] \) and any positive integer \( m \) by:

\[
\left( P^m_{\alpha, \beta} f \right)(x) = \sum_{k=0}^{m} p_{m,k}(x)f\left( \frac{k+\alpha}{m+\beta} \right)
\]

where

\[
p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}
\]

are the fundamental Bernstein polynomials [4].

In the monograph by F. Altomare and M. Campiti [2], the operators (1) are called “the Bernstein-Stancu operators”.

In 1967, J. L. Durrmeyer [5] introduced the operators \( D_m : L_1([0, 1]) \to C([0, 1]) \), defined for any \( f \in C([0, 1]) \), any \( x \in [0, 1] \) and any positive integer

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m by:

\[(D_m f)(x) = (m + 1) \sum_{k=0}^{m} p_{m,k}(x) \int_0^1 p_{mk}(t)f(t)dt.\]

Note that the operators (3) are known in the mathematical literature as the “Durrmeyer operators”. Following the Durrmeyer ideas, the second author introduced and studied Durrmeyer-Schurer operators [3].

The aim of the present paper is to construct the Durrmeyer type operators associated to the operators (1), denoted Durrmeyer-Stancu operators and to establish some of them approximation properties.

In Section 2 we recall the well-known theorem due to O. Shisha and B. Mond [6] and some properties of Durrmeyer operators [5]. We also introduce here the Durrmeyer-Stancu operators, denoted \(D^{(\alpha,\beta)}_m\) and we compute the images of test functions by these operators.

Section 3 contains the main results of the paper which are: a convergence theorem for the sequence \(\{D^{(\alpha,\beta)}_m f\}_{m \in \mathbb{N}}\); the approximation order of \(f\) by \(D^{(\alpha,\beta)}_m f\), under different assumption on the approximated function \(f\).

As particular case, we get the classical Durrmeyer operator (3).

2. AUXILIARY RESULTS

As usually, \(\omega_1(f;\delta)\) denotes the first order modulus of smoothness for the function \(f\) and \(e_j(x) = x^j\) \((j = 0, 1, \ldots)\) are the test functions.

First, we recall the following result, due to O. Shisha and B. Mond [6].

**Theorem 1.** Let \(I\) be a non-empty interval of real axis, let \(L : C_B(I) \to B(I)\) be a linear and positive operator and, for \(x \in I\), let \(\varphi_x : I \to \mathbb{R}\) be defined by \(\varphi_x(t) = |t - x|\), for any \(t \in I\).

a) Let \(f \in C_B(I)\) be given; for any \(\delta > 0\) and any \(x \in I\) the following

\[|(Lf)(x) - f(x)| \leq |f(x)||(|L\varphi_0)(x)| - 1| + \left\{(|L\varphi_0)(x) + \delta^{-1} \sqrt{(|L\varphi_0)(x)(L\varphi^2_0)(x)}\right\} \omega_1(f;\delta)\]

holds.

b) Let \(f\) be a given differentiable function such that \(f' \in C_B(I)\); for any \(\delta > 0\) and any \(x \in I\), the following

\[|(Lf)(x) - f(x)| \leq |f(x)||(|L\varphi_0)(x)| - 1| + \left|f'(x)||(|L\varphi_0)(x)| - x(L\varphi_0)(x)| + \sqrt{(|L\varphi^2_0)(x)}\left\{|L\varphi_0)(x) + \delta^{-1} \sqrt{(|L\varphi^2_0)(x)}\right\} \omega_1(f';\delta)\]

holds.

Next, let us to recall some properties of Durrmeyer operators.
Lemma 2. [5]. For any positive integer $m$ and any $x \in [0, 1]$, the Durrmeyer operators [3] verify:

(6) \[ (D_m e_0)(x) = 1; \]

(7) \[ (D_m e_1)(x) = \frac{mx+1}{m+2}; \]

(8) \[ (D_m e_2)(x) = \frac{m(m-1)x^2+4mx+2}{(m+2)(m+3)}; \]

(9) \[ (D_m \varphi^2_2)(x) = \frac{2(m-3)x(1-x)+2}{(m+2)(m+3)}. \]

Definition 3. Let $\alpha, \beta$ be real parameters satisfying the conditions $0 \leq \alpha \leq \beta$. The Durrmeyer-Stancu operators $D_m^{(\alpha, \beta)} : L_1([0, 1]) \rightarrow C([0, 1])$ are defined for any $f \in L_1([0, 1])$, any $x \in [0, 1]$ and any positive integer $m$ by:

(10) \[ (D_m^{(\alpha, \beta)} f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_{0}^{1} p_{m,k}(t) f \left( \frac{mt+\alpha}{m+\beta} \right) dt. \]

Lemma 4. The operators (10) are linear and positive.

Proof. Directly, from Definition 3.

Lemma 5. For any $x \in [0, 1]$ and any positive integer $m$ the Durrmeyer-Stancu operators (10) verify:

(11) \[ (D_m^{(\alpha, \beta)} e_0)(x) = 1; \]

(12) \[ (D_m^{(\alpha, \beta)} e_1)(x) = \frac{m^2}{(m+\beta)(m+\beta)} x^{(\alpha+1)m+2\alpha}{(m+\beta)(m+\beta)}; \]

(13) \[ (D_m^{(\alpha, \beta)} e_2)(x) = \frac{m^3(m-1)}{(m+\beta)^4(m+2)(m+3)} x^{2} + \frac{4m^3+2\alpha m^2+2m+1}{(m+\beta)^4(m+2)(m+3)} x + \frac{2m+2\alpha m(m+3)+\alpha^2(m+2)(m+3)}{(m+\beta)^4(m+2)(m+1)}; \]

(14) \[ (D_m^{(\alpha, \beta)} \varphi^2_2)(x) = \left( \frac{m}{m+\beta} \right)^2 \frac{2(m-3)x(1-x)+2}{(m+2)(m+3)} + \left\{ \frac{\beta^2(m+2)+4\beta m}{m+2(m+2)(m+2)(m+3)} \right\} x^{2-2\alpha \beta(m+2)+\alpha^2 (m+2)+2m+\alpha}. \]

Proof. It is easy to observe the following identities:

\[ (D_m^{(\alpha, \beta)} e_0)(x) = (D_m e_0)(x) \]
\[ (D_m^{(\alpha, \beta)} e_1)(x) = \frac{m}{m+\beta} (D_m e_1)(x) + \frac{\alpha}{m+\beta} (D_m e_0)(x) \]
\[ (D_m^{(\alpha, \beta)} e_2)(x) = \left( \frac{m}{m+\beta} \right)^2 (D_m e_2)(x) + \frac{2\alpha m}{(m+\beta)^2} (D_m e_1)(x) + \frac{\alpha^2}{(m+\beta)^2} (D_m e_0)(x) \]
Next, one applies Lemma 2.

**Lemma 6.** For any positive integer \( m \) and any \( x \in [0, 1] \) the following
\[
\delta_m^{(\alpha, \beta)}(x) \leq \delta_m^{(\alpha, \beta)}
\]
holds, where
\[
\delta_m^{(\alpha, \beta)}(x) = \sqrt{\left(D_m^{(\alpha, \beta)} \varphi_m^2\right)(x)}
\]
and
\[
\delta_m^{(\alpha, \beta)} = \sqrt{\left(\frac{m}{m+1}\right)^2 \frac{m+1}{2(m+2)(m+3)} + \max \left\{ \frac{\alpha^2(m+2)+2m(\alpha-\beta)(m+\beta)^2(m+2)}{(m+\beta)^2(m+2)}, \frac{\alpha^2(\alpha+2m(m+\alpha-\beta)^2-2m(m+\alpha-\beta)}{(m+\beta)^2(m+2)} \right\}}.
\]

**Proof.** Because \( x(1-x) \leq \frac{1}{4} \) for any \( x \in [0, 1] \), we get
\[
\frac{2(m-3)x(1-x)+2}{(m+2)(m+3)} \leq \frac{m+1}{2(m+2)(m+3)}
\]
for any positive integer \( m \).

For any positive integer \( m \), let us to introduce the following notations:
\[
\gamma_m^{(\alpha, \beta)} = \max \left\{ \frac{\alpha^2(m+2)+2m(\alpha-\beta)(m+\beta)^2(m+2)}{(m+\beta)^2(m+2)}, \frac{\alpha^2(\alpha+2m(m+\alpha-\beta)^2-2m(m+\alpha-\beta)}{(m+\beta)^2(m+2)} \right\} ;
\]
\[
a_m = \frac{\beta^2(m+2)+4\beta m}{(m+\beta)^2(m+2)} ;
\]
\[
b_m = -2 \frac{\alpha \beta(m+2)+\beta m+2am}{(m+\beta)^2(m+2)} ;
\]
\[
c_m = \frac{\alpha^2(m+2)+2m(\alpha-\beta)(m+\beta)^2(m+2)}{(m+\beta)^2(m+2)} .
\]

Let \( f_m[0, 1] \to \mathbb{R} \) be defined for any \( x \in [0, 1] \) by:
\[
f_m(x) = a_m x^2 + b_m x + c_m.
\]

If \( \beta = 0 \), follows \( \alpha = 0 \) and \( f_m(x) = 0 \), for any \( x \in [0, 1] \).

If \( \beta \neq 0 \) the function \( f \) is a polynomial function of second degree and because \( a_m > 0 \) follows that \( f_m \) has a maximum value in the point
\[
x_m = -\frac{b_m}{2a_m} = \frac{\alpha \beta(m+2)+\beta m+2am}{\beta^2(m+2)+4\beta m} .
\]

Because \( \alpha \leq \beta \), we get
\[
x_m \leq \frac{\beta^2(m+2)+3\beta m}{\beta^2(m+2)+4\beta m} < 1.
\]

Taking the above inequality into account, yields:
\[
f_m(x) \leq \max \{ f_m(0), f_m(1) \} ,
\]
for any \( x \in [0, 1] \). Follows that for any \( x \in [0, 1] \) the following
\[
f_m(x) \leq \gamma_m^{(\alpha, \beta)} .
\]
Applying (18) and (19) we get (15). \qed
3. MAIN RESULTS

**Theorem 7.** For any \( f \in L_1([0, 1]) \) the sequence \( \{D_m^{(\alpha, \beta)} f\} \) converges to \( f \), uniformly on \([0, 1] \).

**Proof.** Lemma 5 (identity (14)) follows \( \lim_{m \to \infty} (D_m^{(\alpha, \beta)} \varphi_2^2)(x) = 0 \), uniformly on \([0, 1] \). Then one applies the well known Bohman-Korovkin theorem ([1] or [8]). \( \square \)

**Theorem 8.**

(i) For any \( f \in L_1([0, 1]) \), any \( x \in [0, 1] \), any \( \delta > 0 \) and any positive integer \( m \), the following

\[
\left| \left( D_m^{(\alpha, \beta)} f \right)(x) - f(x) \right| \leq \left( 1 + \frac{1}{2} \delta_m^{(\alpha, \beta)}(x) \right) \omega_1(f; \delta) \tag{20}
\]

holds.

(ii) For any \( f \in C^1([0, 1]) \), any \( x \in [0, 1] \), any \( \delta > 0 \) and any positive integer \( m \), the following

\[
\left| \left( D_m^{(\alpha, \beta)} f \right)(x) - f(x) \right| \leq |f'(x)| \left| \frac{(\beta+2)m+2\beta}{(m+\beta)(m+2)} x + \frac{(\alpha+1)m+2\alpha}{(m+\beta)(m+2)} \right| + \delta_m^{(\alpha, \beta)}(x) \left( 1 + \frac{1}{2} \delta_m^{(\alpha, \beta)}(x) \right) \omega_1(f'; \delta) \tag{21}
\]

holds.

In (20) and (21) \( \delta_m^{(\alpha, \beta)}(x) \) is defined at (16).

**Proof.** One applies Theorem 1 and Lemma 5. \( \square \)

**Theorem 9.**

(i) For any \( f \in L_1([0, 1]) \), any \( x \in [0, 1] \) and any positive integer \( m \), the following

\[
\left| \left( D_m^{(\alpha, \beta)} f \right)(x) - f(x) \right| \leq 2\omega_1 \left( f; \delta_m^{(\alpha, \beta)}(x) \right) \tag{22}
\]

holds.

(ii) For any \( f \in C^1([0, 1]) \), any \( x \in [0, 1] \) and any positive integer \( m \) the following

\[
\left| \left( D_m^{(\alpha, \beta)} f \right)(x) - f(x) \right| \leq |f'(x)| \left| \frac{(\beta+2)m+2\beta}{(m+\beta)(m+2)} x + \frac{(\alpha+1)m+2\alpha}{(m+\beta)(m+2)} \right| + 2\delta_m^{(\alpha, \beta)}(x) \omega_1 \left( f'; \delta_m^{(\alpha, \beta)}(x) \right) \tag{23}
\]

holds.

**Proof.** In Theorem 8 we choose \( \delta = \delta_m^{(\alpha, \beta)}(x) \). \( \square \)

**Remark 10.** For any positive integer \( m \), let \( g_m : [0, 1] \to \mathbb{R} \) be defined by

\[
g_m(x) = \frac{(\beta+2)m+2\beta}{(m+\beta)(m+2)} x + \frac{(\alpha+1)m+2\alpha}{(m+\beta)(m+2)}
\]

for any \( x \in [0, 1] \). It is immediately that

\[
|g_m(x)| \leq \eta_m^{(\alpha, \beta)} \tag{24}
\]
for any $x \in [0, 1]$, where

$$
\eta_m^{(\alpha, \beta)} = \max \{|g_m(0)|, g_m(1)| = \\
= \max \left\{ \frac{(\alpha+1)m+2\alpha}{(m+\beta)(m+2)}, \frac{|m(\alpha-\beta-1)+2(\alpha-\beta)|}{(m+\beta)(m+2)} \right\}.
$$

Applying then Theorem 9, Lemma 6 and (24), we get:

**Corollary 11.**

(i) For any $f \in L_1([0, 1])$, any $x \in [0, 1]$ and any positive integer $m$, the following

$$
\left| D_m^{(\alpha, \beta)} f(x) - f(x) \right| \leq 2\omega_1 \left( f; \delta_m^{(\alpha, \beta)} \right)
$$

holds.

(ii) For any $f \in C^1([0, 1])$, any $x \in [0, 1]$ and any positive integer $m$, the following

$$
\left| D_m^{(\alpha, \beta)} f(x) - f(x) \right| \leq M_1 \eta_m^{(\alpha, \beta)} + 2\delta_m^{(\alpha, \beta)} \omega_1 \left( f'; \delta_m^{(\alpha, \beta)} \right)
$$

where $M_1 = \max_{x \in [0, 1]} |f'(x)|$.

**Remark 12.** For any $\alpha = \beta = 0$, the operators $D_m^{(0, 0)}$ are the Durrmeyer operators.

**REFERENCES**


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