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DURRMEYER-STANCU TYPE OPERATORS

OVIDIU T. POP* and DAN BĂRBOSU†

Abstract. Considering two given real parameters α, β satisfying the conditions $0 \leq \alpha \leq \beta$. D. D. Stancu [7] constructed and studied the linear positive operators $P_m^{(\alpha,\beta)} : C([0,1]) \to C([0,1])$, defined for any $f \in C([0,1])$ and any positive integer m by (1). In this paper we are dealing with the Durrmeyer form of Stancu's operators. Some approximation properties of these Durrmeyer-Stancu operators are established. As a particular case, we retrieve approximation properties for the classical Durrmeyer operators [5].

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1. PRELIMINARIES

Let α, β be real parameters satisfying the conditions $0 \leq \alpha \leq \beta$.
In 1969 D. D. Stancu [7] constructed and studied the linear positive operators $P_m^{(\alpha,\beta)} : C([0,1]) \to C([0,1])$ defined for any $f \in C([0,1])$, any $x \in [0,1]$ and any positive integer m by:

$$\left(p_m^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\tfrac{k+\alpha}{m+\beta}\right) \tag{1}$$

where

$$p_{m,k}(x) = \tbinom{m}{k} x^k (1-x)^{m-k} \tag{2}$$

are the fundamental Bernstein polynomials [4].

In the monograph by F. Altomare and M. Campiti [2], the operators (1) are called "the Bernstein-Stancu operators".

In 1967, J. L. Durrmeyer [5] introduced the operators $D_m : L_1([0,1]) \to C([0,1])$, defined for any $f \in C([0,1])$, any $x \in [0,1]$ and any positive integer

*National College "Mihai Eminescu", 5 Mihai Eminescu Street, 440014 Satu Mare, Romania, e-mail: `ovidiutiberiu@yahoo.com`.

†North University of Baia Mare, Dept. of Math. and Comp. Science, Victoriei 76, 43012 Baia Mare, Romania, e-mail: `danbarbosu@yahoo.com`.

m by:

$$(D_m f)(x) = (m+1)\sum_{k=0}^{m} p_{m,k}(x)\int_0^1 p_{mk}(t)f(t)\mathrm{d}t. \tag{3}$$

Note that the operators (3) are known in the mathematical literature as the "Durrmeyer operators". Following the Durrmeyer ideas, the second author introduced and studied Durrmeyer-Schurer operators [3].

The aim of the present paper is to construct the Durrmeyer type operators associated to the operators (1), denoted Durrmeyer-Stancu operators and to establish some of them approximation properties.

In Section 2 we recall the well-known theorem due to O. Shisha and B. Mond [6] and some properties of Durrmeyer operators [5]. We also introduce here the Durrmeyer-Stancu operators, denoted $D_m^{(\alpha,\beta)}$ and we compute the images of test functions by these operators.

Section 3 contains the main results of the paper which are: a convergence theorem for the sequence $\{D_m^{(\alpha,\beta)} f\}_{m\in\mathbb{N}}$; the approximation order of f by $D_m^{(\alpha,\beta)} f$, under different assumption on the approximated function f.
As particular case, we get the classical Durrmeyer operator (3).

2. AUXILIARY RESULTS

As usually, $\omega_1(f;\delta)$ denotes the first order modulus of smoothness for the function f and $e_j(x) = x^j$ $(j = 0, 1, \dots)$ are the test functions.

First, we recall the following result, due to O. Shisha and B. Mond [6].

THEOREM 1. *Let I be a non-empty interval of real axis, let $L : C_B(I) \to B(I)$ be a linear and positive operator and, for $x \in I$, let $\varphi_x : I \to \mathbb{R}$ be defined by $\varphi_x(t) = |t - x|$, for any $t \in I$.*

a) Let $f \in C_B(I)$ be given; for any $\delta > 0$ and any $x \in I$ the following

$$\begin{aligned}|(Lf)(x) - f(x)| \le {} & |f(x)||(Le_0)(x) - 1| + \\ & + \left\{(Le_0)(x) + \delta^{-1}\sqrt{(Le_0)(x)(L\varphi_x^2)(x)}\right\}\omega_1(f;\delta)\end{aligned} \tag{4}$$

holds.

b) Let f be a given differentiable function such that $f' \in C_B(I)$; for any $\delta > 0$ and any $x \in I$, the following

$$\begin{aligned}|(Lf)(x) - f(x)| \le {} & |f(x)||(Le_0)(x) - 1| + \\ & + |f'(x)||(Le_1)(x) - x(Le_0)(x)| + \\ & + \sqrt{(L\varphi_x^2)(x)}\left\{\sqrt{Le_0)(x)} + \delta^{-1}\sqrt{(L\varphi_x^2)(x)}\right\}\omega_1(f';\delta)\end{aligned} \tag{5}$$

holds.

Next, let us to recall some properties of Durrmeyer operators.

Lemma 2. [5]. *For any positive integer m and any $x \in [0,1]$, the Durrmeyer operators* (3) *verify:*

$$(D_m e_0)(x) = 1; \tag{6}$$

$$(D_m e_1)(x) = \tfrac{mx+1}{m+2}\,; \tag{7}$$

$$(D_m e_2)(x) = \tfrac{m(m-1)x^2+4mx+2}{(m+2)(m+3)}\,; \tag{8}$$

$$(D_m \varphi_x^2)(x) = \tfrac{2(m-3)x(1-x)+2}{(m+2)(m+3)}\,. \tag{9}$$

Definition 3. *Let α, β be real parameters satisfying the conditions $0 \le \alpha \le \beta$. The Durrmeyer-Stancu operators $D_m^{(\alpha,\beta)} : L_1([0,1]) \to C([0,1])$ are defined for any $f \in L_1([0,1])$, any $x \in [0,1]$ and any positive integer m by:*

$$\left(D_m^{(\alpha,\beta)} f\right)(x) = (m+1)\sum_{k=0}^{m} p_{m,k}(x)\int_0^1 p_{m,k}(t) f\left(\tfrac{mt+\alpha}{m+\beta}\right) \mathrm{d}t. \tag{10}$$

Lemma 4. *The operators* (10) *are linear and positive.*

Proof. Directly, from Definition 3. □

Lemma 5. *For any $x \in [0,1]$ and any positive integer m the Durrmeyer-Stancu operators* (10) *verify:*

$$\left(D_m^{(\alpha,\beta)} e_0\right)(x) = 1; \tag{11}$$

$$\left(D_m^{(\alpha,\beta)} e_1\right)(x) = \tfrac{m^2}{(m+\beta)(m+2)}\, x + \tfrac{(\alpha+1)m+2\alpha}{(m+\beta)(m+2)}\,; \tag{12}$$

$$\begin{aligned}\left(D_m^{(\alpha,\beta)} e_2\right)(x) = {} & \tfrac{m^3(m-1)}{(m+\beta)^2(m+2)(m+3)}\, x^2 \\ & + \tfrac{4m^3+2\alpha m^2(m+3)}{(m+\beta)^2(m+2)(m+3)}\, x \\ & + \tfrac{2m^2+2\alpha m(m+3)+\alpha^2(m+2)(m+3)}{(m+\beta)^2(m+2)(m+1)}\,;\end{aligned} \tag{13}$$

$$\begin{aligned}\left(D_m^{(\alpha,\beta)} \varphi_x^2\right)(x) = {} & \left(\tfrac{m}{m+\beta}\right)^2 \tfrac{2(m-3)x(1-x)+2}{(m+2)(m+3)} + \\ & + \tfrac{\{\beta^2(m+2)+4\beta m\}x^2 - 2\{\alpha\beta(m+2)+\beta m+2\alpha m\}x + \alpha^2(m+2)+2m\alpha}{(m+\beta)^2(m+2)}.\end{aligned} \tag{14}$$

Proof. It is easy to observe the following identities:

$$\left(D_m^{(\alpha,\beta)} e_0\right)(x) = (D_m e_0)(x)$$

$$\left(D_m^{(\alpha,\beta)} e_1\right)(x) = \tfrac{m}{m+\beta}\,(D_m e_1)(x) + \tfrac{\alpha}{m+\beta}\,(D_m e_0)(x)$$

$$\begin{aligned}\left(D_m^{(\alpha,\beta)} e_2\right)(x) = {} & \left(\tfrac{m}{m+\beta}\right)^2 (D_m e_2)(x) + \tfrac{2\alpha m}{(m+\beta)^2}\,(D_m e_1)(x) + \\ & + \tfrac{\alpha^2}{(m+\beta)^2}\,(D_m e_0)(x)\end{aligned}$$

$$\left(D_m^{(\alpha,\beta)}\varphi_x^2\right)(x) = \left(D_m^{(\alpha,\beta)}e_2\right)(x) - 2x\left(D_m^{(\alpha,\beta)}e_1\right)(x) + x^2\left(D_m^{(\alpha,\beta)}e_0\right)(x).$$

Next, one applies Lemma 2. □

LEMMA 6. *For any positive integer m and any $x \in [0,1]$ the following*

$$\delta_m^{(\alpha,\beta)}(x) \leq \delta_m^{(\alpha,\beta)} \tag{15}$$

holds, where

$$\delta_m^{(\alpha,\beta)}(x) = \sqrt{\left(D_m^{(\alpha,\beta)}\varphi_m^2\right)(x)} \tag{16}$$

and

$$\delta_m^{(\alpha,\beta)} = \sqrt{\left(\frac{m}{m+\beta}\right)^2 \frac{m+1}{2(m+2)(m+3)} + \max\left\{\frac{\alpha^2(m+2)+2m\alpha}{(m+\beta)^2(m+2)}, \frac{(m+2)(\alpha-\beta)^2-2m(\alpha-\beta)}{(m+\beta)^2(m+2)}\right\}}. \tag{17}$$

Proof. Because $x(1-x) \leq \frac{1}{4}$ for any $x \in [0,1]$, we get

$$\frac{2(m-3)x(1-x)+2}{(m+2)(m+3)} \leq \frac{m+1}{2(m+2)(m+3)} \tag{18}$$

for any positive integer m.

For any positive integer m, let us to introduce the following notations:

$$\gamma_m^{(\alpha,\beta)} = \max\left\{\frac{\alpha^2(m+2)+2m\alpha}{(m+\beta)^2(m+2)}, \frac{(m+2)(\alpha-\beta)^2-2m(\alpha-\beta)}{(m+\beta)^2(m+2)}\right\};$$

$$a_m = \frac{\beta^2(m+2)+4\beta m}{(m+\beta)^2(m+2)};$$

$$b_m = -2\frac{\alpha\beta(m+2)+\beta m+2\alpha m}{(m+\beta)^2(m+2)};$$

$$c_m = \frac{\alpha^2(m+2)+2m\alpha}{(m+\beta)^2(m+2)}.$$

Let $f_m[0,1] \to \mathbb{R}$ be defined for any $x \in [0,1]$ by:

$$f_m(x) = a_m x^2 + b_m x + c_m.$$

If $\beta = 0$, follows $\alpha = 0$ and $f_m(x) = 0$, for any $x \in [0,1]$.

If $\beta \neq 0$ the function f is a polynomial function of second degree and because $a_m > 0$ follows that f_m has a maximum value in the point

$$x_m = -\frac{b_m}{2a_m} = \frac{\alpha\beta(m+2)+\beta m+2\alpha m}{\beta^2(m+2)+4\beta m}.$$

Because $\alpha \leq \beta$, we get

$$x_m \leq \frac{\beta^2(m+2)+3\beta m}{\beta^2(m+2)+4\beta m} < 1.$$

Taking the above inequality into account, yields:

$$f_m(x) \leq \max\left\{f_m(0), f_m(1)\right\},$$

for any $x \in [0,1]$. Follows that for any $x \in [0,1]$ the following

$$f_m(x) \leq \gamma_m^{(\alpha,\beta)}. \tag{19}$$

Applying (18) and (19) we get (15). □

3. MAIN RESULTS

THEOREM 7. *For any* $f \in L_1([0,1])$ *the sequence* $\left\{D_m^{(\alpha,\beta)} f\right\}$ *converges to* f*, uniformly on* $[0,1]$.

Proof. Lemma 5 (identity (14)) follows $\lim\limits_{m\to\infty} \left(D_m^{(\alpha,\beta)} \varphi_x^2\right)(x) = 0$, uniformly on $[0,1]$. Then one applies the well known Bohman-Korovkin theorem ([1] or [8]). □

THEOREM 8. (*i*) *For any* $f \in L_1([0,1])$*, any* $x \in [0,1]$*, any* $\delta > 0$ *and any positive integer* m*, the following*

$$\left|\left(D_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| \le \left(1 + \tfrac{1}{\delta}\, \delta_m^{(\alpha,\beta)}(x)\right) \omega_1(f;\delta) \tag{20}$$

holds.

(*ii*) *For any* $f \in C^1([0,1])$*, any* $x \in [0,1]$*, any* $\delta > 0$ *and any positive integer* m*, the following*

$$\begin{aligned}\left|\left(D_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| \le |f'(x)| \left| - \tfrac{(\beta+2)m+2\beta}{(m+\beta)(m+2)}\, x + \tfrac{(\alpha+1)m+2\alpha}{(m+\beta)(m+2)} \right| + \\ + \delta_m^{(\alpha,\beta)}(x) \left(1 + \tfrac{1}{\delta}\, \delta_m^{(\alpha,\beta)}(x)\right) \omega_1(f';\delta)\end{aligned} \tag{21}$$

holds.

In (20) *and* (21) $\delta_m^{(\alpha,\beta)}(x)$ *is defined at* (16).

Proof. One applies Theorem 1 and Lemma 5. □

THEOREM 9. (*i*) *For any* $f \in L_1([0,1])$*, any* $x \in [0,1]$ *and any positive integer* m*, the following*

$$\left|\left(D_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| \le 2\omega_1\left(f; \delta_m^{(\alpha,\beta)}(x)\right) \tag{22}$$

holds.

(*ii*) *For any* $f \in C^1([0,1])$*, any* $x \in [0,1]$ *and any positive integer* m *the following*

$$\begin{aligned}\left|\left(D_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| \le |f'(x)| \left| \tfrac{(\beta+2)m+2\beta}{(m+\beta)(m+2)}\, x + \tfrac{(\alpha+1)m+2\alpha}{(m+\beta)(m+2)} \right| + \\ + 2\delta_m^{(\alpha,\beta)}(x) \omega_1\left(f'; \delta_m^{(\alpha,\beta)}(x)\right).\end{aligned} \tag{23}$$

Proof. In Theorem 8 we choose $\delta = \delta_{m,p}^{(\alpha,\beta)}(x)$. □

REMARK 10. For any positive integer m, let $g_m : [0,1] \to \mathbb{R}$ be defined by

$$g_m(x) = -\tfrac{(\beta+2)m+2\beta}{(m+\beta)(m+2)}\, x + \tfrac{(\alpha+1)m+2\alpha}{(m+\beta)(m+2)}$$

for any $x \in [0,1]$. It is immediately that

$$|g_m(x)| \le \eta_m^{(\alpha,\beta)} \tag{24}$$

for any $x \in [0,1]$, where

$$\eta_m^{(\alpha,\beta)} = \max\left\{|g_m(0)|, g_m(1)\right\} = \tag{25}$$
$$= \max\left\{\frac{(\alpha+1)m+2\alpha}{(m+\beta)(m+2)}, \frac{|m(\alpha-\beta-1)+2(\alpha-\beta)|}{(m+\beta)(m+2)}\right\}.$$

□

Applying then Theorem 9, Lemma 6 and (24), we get:

Corollary 11. *(i) For any $f \in L_1([0,1])$, any $x \in [0,1]$ and any positive integer m, the following*

$$\left|\left(D_m^{(\alpha,\beta)} f\right)(x) - f(x)\right) \leq 2\omega_1\left(f; \delta_m^{(\alpha,\beta)}\right) \tag{26}$$

holds.

(ii) For any $f \in C^1([0,1])$, any $x \in [0,1]$ and any positive integer m, the following

$$\left|\left(D_m^{(\alpha,\beta)} f\right)(x) - f(x)\right| \leq M_1\eta_m^{(\alpha,\beta)} + 2\delta_m^{(\alpha,\beta)}\omega_1\left(f'; \delta_m^{(\alpha,\beta)}\right) \tag{27}$$

where $M_1 = \max\limits_{x\in[0,1]} |f'(x)|$.

Remark 12. For any $\alpha = \beta = 0$, the operators $D_m^{(0,0)}$ are the Durrmeyer operators. □

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