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THE FIRST ABSOLUTE MOMENT FOR SOME OPERATORS

OVIDIU T. POP† and PETRU I. BRAICA*

Abstract. In this paper we will determinate the first absolute moments for Bernstein, Szász-Mirakjan, Bleimann-Butzer-Hahn, Meyer-König and Zeller operators. For the Szász-Mirakjan operators we give some properties with the absolute moment of high order.

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1. PRELIMINARIES

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any $m \in \mathbb{N}$, let $B_m : C([0,1]) \to C([0,1])$ be the Bernstein operators, defined by

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\tfrac{k}{m}\right), \tag{1.1}$$

where $p_{m,k}(x)$ are the fundamental Bernstein's polynomials, given by

$$p_{m,k}(x) = \tbinom{m}{k} x^k (1-x)^{m-k}, \tag{1.2}$$

for any $x \in [0,1]$ and any $k \in \{0, 1, \ldots, m\}$ (see [3] or [11]).
For any $m \in \mathbb{N}$, let $S_m : C_2([0,+\infty)) \to C([0,+\infty))$ be the Szász-Mirakjan operators (see [4], [7], [11] or [12]), defined by

$$(S_m f)(x) = \mathrm{e}^{-mx} \sum_{k=0}^{\infty} \tfrac{(mx)^k}{k!} f\left(\tfrac{k}{m}\right), \tag{1.3}$$

for any $x \in [0,+\infty)$.
The operators $Z_m : B([0,1]) \to C([0,1])$ defined by

$$(Z_m f)(x) = (1-x)^{m+1} \sum_{k=0}^{\infty} \tbinom{m+k}{k} x^k f\left(\tfrac{k}{m+k}\right), \tag{1.4}$$

†National College "Mihai Eminescu", 5 Mihai Eminescu Street, 440014 Satu Mare, Romania, e-mail: ovidiutiberiu@yahoo.com.
* Secondary School "Grigore Moisil", 1 Mileniului Street, 440037 Satu Mare, Romania, e-mail: petrubr@yahoo.com.

for any $x \in [0,1)$ and $(Z_m f)(1) = f(1)$, $m \in \mathbb{N}$, are called Meyer-König and Zeller operators (see [6] or [11]).
For $m \in \mathbb{N}$, let the operators $L_m : C_B([0,+\infty)) \to C_B([0,+\infty))$, defined by

$$(L_m f)(x) = (1+x)^{-m} \sum_{k=0}^{m} \binom{m}{k} x^k f\left(\tfrac{k}{m+1-k}\right), \tag{1.5}$$

for any $x \in [0,\infty)$. These operators are called Bleimann-Butzer-Hahn operators (see [2] or [11]).
In what follows, let $I \subset \mathbb{R}$ be an interval. We recall that the functions $\varphi_x, \psi_x : I \to \mathbb{R}$ are defined by

$$\varphi_x(t) = |t-x|, \quad \psi_x(t) = t - x,$$

for any $(x,t) \in I \times I$. For a sequence of operators $(L_m)_{m\geq 1}$, define $T_{m,i}$ by

$$(T_{m,i} L_m)(x) = m^i \left(L_m \psi_x^i\right)(x), \tag{1.6}$$

for any $x \in I$, any $m \in \mathbb{N}$ and any $i \in \mathbb{N}_0$.

2. MAIN RESULTS

In [5] is proved the following result contained in Theorem 2.1.

THEOREM 2.1. *Let $m \in \mathbb{N}$ and $0 \leq x \leq 1$. If $i = [mx]$, then*

$$(B_m \varphi_x)(x) = 2x(1-x) p_{m-1,i}(x). \tag{2.1}$$

THEOREM 2.2. *Let $m \in \mathbb{N}$ and $0 \leq x$. If $i = [mx]$, then*

$$(S_m \varphi_x)(x) = 2x \mathrm{e}^{-mx} \tfrac{(mx)^i}{i!}. \tag{2.2}$$

Proof. From $i = [mx]$ it results $i \leq mx < i+1$, equivalent with $\dfrac{i}{m} \leq x < \dfrac{i+1}{m}$. We get

$$\begin{aligned}(S_m \varphi_x)(x) &= \mathrm{e}^{-mx} \sum_{k=0}^{\infty} \tfrac{(mx)^k}{k!} \left|\tfrac{k}{m} - x\right| \\ &= 2\mathrm{e}^{-mx} \sum_{k=0}^{i} \tfrac{(mx)^k}{k!} \left(x - \tfrac{k}{m}\right) + \mathrm{e}^{-mx} \sum_{k=0}^{\infty} \tfrac{(mx)^k}{k!} \left(\tfrac{k}{m} - x\right) \\ &= 2\mathrm{e}^{-mx} \left[x \sum_{k=0}^{i} \tfrac{(mx)^k}{k!} - x \sum_{k=1}^{i} \tfrac{(mx)^{k-1}}{(k-1)!} \right] + (S_m e_1)(x) - x (S_m e_0)(x) \\ &= 2x \mathrm{e}^{-mx} \tfrac{(mx)^i}{i!},\end{aligned}$$

so (2.2) is obtained. □

THEOREM 2.3. *Let* $m \in \mathbb{N}$ *and* $0 \le x < m$. *If* $i = \left[\frac{(m+1)x}{x+1}\right]$, *then*

$$(L_m \varphi_x)(x) = x(1+x)^{-m}\left(2\binom{m}{i}x^i - x^m\right). \tag{2.3}$$

In the case when $m \in \mathbb{N}$ and $x \ge m$, then

$$(L_m \varphi_x)(x) = x^{m+1}(1+x)^{-m}. \tag{2.4}$$

Proof. From $i = \left[\frac{(m+1)x}{x+1}\right]$ it results $i \le \frac{(m+1)x}{x+1} < i+1$, equivalent with $xi + i \le mx + x < xi + x + i + 1$, from where $\frac{i}{m-i+1} \le x < \frac{i+1}{m-i}$. Taking that $x < m$ into account, one obtains $i = \left[\frac{(m+1)x}{x+1}\right] \le \frac{(m+1)x}{x+1} < m$.

We get

$$\begin{aligned}
&(L_m \varphi_x)(x) = (1+x)^{-m}\sum_{k=0}^{m}\binom{m}{k}x^k\left|\frac{k}{m-k+1} - x\right| \\
&= 2(1+x)^{-m}\sum_{k=0}^{i}\binom{m}{k}x^k\left(x - \frac{k}{m-k+1}\right) \\
&\quad + (1+x)^{-m}\sum_{k=0}^{m}\binom{m}{k}x^k\left(\frac{k}{m-k+1} - x\right) \\
&= 2(1+x)^{-m}\left(x\sum_{k=1}^{i}\binom{m}{k}x^k - \sum_{k=1}^{i}\binom{m}{k}x^k\frac{k}{m-k+1}\right) + (L_m e_1)(x) - x(L_m e_0)(x) \\
&= 2(1+x)^{-m}\left(x\sum_{k=1}^{i}\binom{m}{k}x^k - x\sum_{k=1}^{i}\binom{m}{k-1}x^{k-1}\right) - x\left(\frac{x}{1+x}\right)^m \\
&= 2x(1+x)^{-m}\binom{m}{i}x^i - x\left(\frac{x}{1+x}\right)^m = x(1+x)^{-m}\left(2\binom{m}{i}x^i - x^m\right),
\end{aligned}$$

so (2.3) is obtained. If $x \ge m$, we have that $\frac{k}{m-k+1} \le x$, for any $k \in \{0, 1, \dots, m\}$ and then

$$\begin{aligned}
(L_m \varphi_x)(x) &= (1+x)^{-m}\sum_{k=0}^{m}\binom{m}{k}x^k\left(x - \frac{k}{m-k+1}\right) \\
&= x(L_m e_0)(x) - (L_m e_1)(x) = x^{m+1}(1+x)^{-m},
\end{aligned}$$

so (2.4) holds. □

THEOREM 2.4. *Let* $m \in \mathbb{N}$ *and* $0 \le x < 1$. *If* $i = \left[\frac{mx}{1-x}\right]$, *then*

$$(Z_m \varphi_x)(x) = 2x^{i+1}(1-x)^{m+1}\binom{m+i}{i}. \tag{2.5}$$

Proof. From $i = \left[\frac{mx}{1-x}\right]$ it results $i \le \frac{mx}{1-x} < i+1$, equivalent with $\frac{i}{m+i} \le x < \frac{i+1}{m+i+1}$. We get

$$\begin{aligned}(Z_m\varphi_x)(x) &= (1-x)^{m+1}\sum_{k=0}^{i}\tbinom{m+k}{k}x^k\left|x-\tfrac{k}{m+k}\right| \\ &= 2(1-x)^{m+1}\sum_{k=0}^{i}\tbinom{m+k}{k}x^k\left(x-\tfrac{k}{m+k}\right) \\ &\quad + (1-x)^{m+1}\sum_{k=0}^{\infty}\tbinom{m+k}{k}x^k\left(\tfrac{k}{m+k}-x\right) \\ &= 2(1-x)^{m+1}\left(x\sum_{k=0}^{i}\tbinom{m+k}{k}x^k-\sum_{k=1}^{i}\tbinom{m+k}{k}x^k\tfrac{k}{m+k}\right)+ \\ &\quad + (Z_me_1)(x)-x(Z_me_0)(x) \\ &= 2x(1-x)^{m+1}\left(1+\sum_{k=1}^{i}\left(\tbinom{m+k}{k}x^k-\tbinom{m+k-1}{k-1}x^{k-1}\right)\right) \\ &= 2x^{i+1}(1-x)^{m+1}\tbinom{m+i}{i},\end{aligned}$$

so (2.5) is obtained. □

It is known the result contained in Lemma 2.5.

LEMMA 2.5. *If $m,n \in \mathbb{N}_0, m \ne 0$, then $(B_m\psi_x^n)(x)$, with $x \in [0,1]$ is a polynomial in variable x.*

LEMMA 2.6. *If $m,n \in \mathbb{N}$ and n is even, then*

$$(B_m\varphi_x^n)(x) = x(1-x)q_{m,n}(x), \tag{2.6}$$

for any $x \in [0,1]$, where $q_{m,n}(x)$ is a polynomial in variable x and $q_{m,n}(x) > 0$, for any $x \in [0,1]$.

Proof. For $m = 1$, we get

$$\begin{aligned}(B_1\varphi_x^n)(x) &= \sum_{k=0}^{1}(k-x)^n p_{1,k}(x) \\ &= x^n p_{1,0}(x) + (1-x)^n p_{1,1}(x) = x(1-x)q_{1,n}(x),\end{aligned}$$

where $q_{1,n}(x) = x^{n-1} + (1-x)^{n-1}$.
For $m \geq 2$, we get

$$\begin{aligned}
(B_m \varphi_x^n)(x) &= \sum_{k=0}^{m} \left(\tfrac{k}{m} - x\right)^n p_{m,k}(x) \\
&= x^n (1-x)^m + \sum_{k=1}^{m-1} \left(\tfrac{k}{m} - x\right)^n p_{m,k}(x) + (1-x)^n x^m \\
&= x(1-x) q_{m,n},
\end{aligned}$$

where

$$\begin{aligned}
q_{m,n}(x) &= x^{n-1}(1-x)^{m-1} \\
&\quad + \sum_{k=1}^{m-1} \left(\tfrac{k}{m} - x\right)^n \tbinom{m}{k} x^{k-1} (1-x)^{m-k-1} + (1-x)^{n-1} x^{m-1}.
\end{aligned}$$

For any $x \in [0,1]$, every summand from $q_{m,n}(x)$ is positive. But for $k = 1$ and $k = m - 1$, which means $\left(\frac{1}{m} - x\right)^n \binom{m}{1}(1-x)^{m-2}$, respectively $\left(\frac{m-1}{m} - x\right)^n \binom{m}{m-1} x^{m-2}$ cannot be simultaneously null.
Taking the above remark into account, it results

$$(B_m \varphi_x^n)(x) = x(1-x) q_{m,n}(x)$$

is a polynomial in variable x and $q_{m,n}(x) > 0$, for any $x \in [0,1]$. □

Corollary 2.7. *If $m, n \in \mathbb{N}$, with n even, then*

(2.7)
$$(T_{m,n} B_m)(x) = m^n (B_m \psi_x^n)(x) = m^n (B_m \varphi_x^n)(x) = m^n x(1-x) q_{m,n}(x).$$

For the Szász-Mirakjan operators, we get

$$(T_{m,s} S_m)(x) = m^s (S_m \psi_x^s)(x) = m^s e^{-mx} \sum_{k=0}^{\infty} \frac{(mx)^k}{k!} \left(\frac{k}{m} - x\right)^s, \tag{2.8}$$

where $m \in \mathbb{N}$, $s \in \mathbb{N}_0$ and $x \in [0, +\infty)$.

Lemma 2.8. *If $m, s \in N$, with $s \geq 2$, then the following*

$$((T_{m,s} S_m)(x))' = m \sum_{i=0}^{s-2} \binom{s}{i} (T_{m,i} S_m)(x) \tag{2.9}$$

holds, for any $x \in [0, +\infty)$.

Proof. From (2.8), we obtain

$$((T_{m,s}S_m)(x))' = \left(m^s \mathrm{e}^{-mx} \sum_{k=0}^{\infty} \tfrac{(mx)^k}{k!} \left(\tfrac{k}{m} - x\right)^s\right)'$$

$$= -m^{s+1}\mathrm{e}^{-mx} \sum_{k=0}^{\infty} \tfrac{(mx)^k}{k!} \left(\tfrac{k}{m} - x\right)^s + m^{s+1}\mathrm{e}^{-mx} \sum_{k=0}^{\infty} \tfrac{(mx)^{k-1}}{(k-1)!} \left(\tfrac{k}{m} - x\right)^s -$$

$$- sm^s \mathrm{e}^{-mx} \sum_{k=0}^{\infty} \tfrac{(mx)^k}{k!} \left(\tfrac{k}{m} - x\right)^{s-1}$$

$$= -m\left(T_{m,s}S_m\right)(x) - ms\left(T_{m,s-1}S_m\right)(x) + m^{s+1}\mathrm{e}^{-mx} \sum_{k=0}^{\infty} \tfrac{(mx)^k}{k!} \left(\tfrac{k+1}{m} - x\right)^s$$

$$= -m\left(T_{m,s}S_m\right)(x) - ms\left(T_{m,s-1}S_m\right)(x) + m^{s+1}\mathrm{e}^{-mx} \sum_{k=0}^{\infty} \tfrac{(mx)^k}{k!} \left(\left(\tfrac{k}{m} - x\right) + \tfrac{1}{m}\right)^s$$

$$= -m\left(T_{m,s}S_m\right)(x) - ms\left(T_{m,s-1}S_m\right)(x)$$

$$+ m^{s+1}\mathrm{e}^{-mx} \sum_{k=0}^{\infty} \tfrac{(mx)^k}{k!} \sum_{i=0}^{s} \tbinom{s}{i} \left(\tfrac{1}{m}\right)^{s-i} \left(\tfrac{k}{m} - x\right)^i$$

$$= -m\left(T_{m,s}S_m\right)(x) - ms\left(T_{m,s-1}S_m\right)(x) + \sum_{i=0}^{s} \tbinom{s}{i} m^{i+1}\mathrm{e}^{-mx} \sum_{k=0}^{\infty} \tfrac{(mx)^k}{k!} \left(\tfrac{k}{m} - x\right)^i$$

$$= -m\left(T_{m,s}S_m\right)(x) - ms\left(T_{m,s-1}S_m\right)(x) + m \sum_{i=0}^{s} \tbinom{s}{i} \left(T_{m,i}S_m\right)(x)$$

from where, the relation (2.9) follows. □

LEMMA 2.9. *If $m, s \in \mathbb{N}$, with $s \geq 2$, then*

$$(T_{m,s}S_m)(x) = m \sum_{i=0}^{s-2} \tbinom{s}{i} \int_0^x (T_{m,i}S_m)(t)\mathrm{d}t \tag{2.10}$$

holds, for any $x \in [0, \infty)$.

Proof. It yields immediately, taking that $(T_{m,s}S_m)(0) = 0$ into account and integrate the relation (2.9). □

For any $m \in \mathbb{N}$ and any $x \in [0, \infty)$, it is well known that $(T_{m,0}S_m)(x) = 1$, $(T_{m,1}S_m)(x) = 0$, then the following holds:

COROLLARY 2.10. *If $m, s \in N$, with $s \geq 4$, then*

$$(T_{m,s}S_m)(x) = mx + m \sum_{i=2}^{s-2} \tbinom{s}{i} \int_0^x (T_{m,i}S_m)(t)\mathrm{d}t \tag{2.11}$$

for any $x \in [0, \infty)$.

Let $m \in \mathbb{N}$ and $x \in [0,\infty)$. We obtain

$$\begin{aligned}
(T_{m,0}S_m)(x) =&(S_m e_0)(x) = 1,\\
(T_{m,1}S_m)(x) =&m(S_m\psi_x)(x) = m((S_m e_1)(x) - x(S_m e_0)(x)) = 0,\\
(T_{m,2}S_m)(x) =&m\tbinom{2}{0}\int_0^x (T_{m,0}S_m)(t)\mathrm{d}t = mx,\\
(T_{m,3}S_m)(x) =&m\left(\tbinom{3}{0}\int_0^x (T_{m,0}S_m)(t)\mathrm{d}t + \tbinom{3}{1}\int_0^x (T_{m,1}S_m)(t)\mathrm{d}t\right) = mx,\\
(T_{m,4}S_m)(x) =&mx + m\tbinom{4}{2}\int_0^x (T_{m,2}S_m)(t)\mathrm{d}t = mx + 3m^2x^2.
\end{aligned}$$

Lemma 2.11. *If $m, s \in \mathbb{N}$, with $s > 3$, then exist $a_2^{(s)}, a_3^{(s)}, \ldots, a_{[s/2]}^{(s)} \geq 0$, $a_{[s/2]}^{(s)} \neq 0$, such that*

$$(T_{m,s}S_m)(x) = mx + m\sum_{k=2}^{[s/2]} a_k^{(s)} x^k, \tag{2.12}$$

for any $x \in [0,\infty)$, where $a_k^{(s)}$ depends of m and $k \in \left\{2,3,\ldots,\left[\frac{s}{2}\right]\right\}$.

Proof. We prove by mathematical induction.

For $s = 4$ we get $(T_{m,4}S_m)(x) = mx + 3m^2x^2$, so that $a_2^{(4)} = 3m > 0$. We assume that

$$(T_{m,j}S_m)(x) = mx + m\sum_{k=2}^{[j/2]} a_k^{(j)} x^k,$$

for $a_2^{(j)}, a_3^{j}, \ldots, a_{[j/2]}^{(j)} \geq 0$, $a_{[j/2]}^{(j)} \neq 0$, for any $j \in \{4,5,\ldots,s\}$. Taking relation (2.11) into account, we obtain

$$\begin{aligned}
&(T_{m,s+1}S_m)(x) =\\
&= mx + m\Bigg(\tbinom{s+1}{2}\int_0^x (T_{m,2}S_m)(t)\mathrm{d}t + \tbinom{s+1}{3}\int_0^x (T_{m,3}S_m)(t)\mathrm{d}t\\
&\quad + \sum_{i=4}^{s-1}\tbinom{s+1}{i}\int_0^x \left(mt + m\sum_{k=2}^{[i/2]} a_k^{(i)} t^k\right)\mathrm{d}t\Bigg)\\
&= mx + m\left(\tbinom{s+1}{2}\tfrac{mx^2}{2} + \tbinom{s+1}{3}\tfrac{mx^2}{2} + \sum_{i=4}^{s-1}\tbinom{s+1}{i}\left(\tfrac{mx^2}{2} + m\sum_{k=2}^{[i/2]}\tfrac{a_k^{(i)}}{k+1}x^{k+1}\right)\right)\\
&= mx + \tfrac{1}{2}\left(m\tbinom{s+1}{2} + m\tbinom{s+1}{3} + \ldots + m\tbinom{s+1}{s-1}\right)mx^2\\
&\quad + \left(m\tbinom{s+1}{4}\tfrac{a_2^{(4)}}{3} + m\tbinom{s+1}{5}\tfrac{a_2^{(5)}}{3} + \ldots + m\tbinom{s+1}{s-1}\tfrac{a_2^{(s-1)}}{3}\right)mx^3 +
\end{aligned}$$

$$+\ldots+m\cdot a^{(s+1)}_{\left[\frac{s+1}{2}\right]}x^{\left[\frac{s+1}{2}\right]}$$

where:

- if s is even,

then for $k\in\left\{2,3,\ldots,\left[\frac{s-3}{2}\right]\right\}$ we get $k+1<\left[\frac{s-2}{2}\right]+1=\left[\frac{s-1}{2}\right]+1=\left[\frac{s+1}{2}\right]$ and

$$a^{(s+1)}_{\left[\frac{s+1}{2}\right]}=m\binom{s+1}{s-2}\frac{a^{(s-2)}_{\left[\frac{s-2}{2}\right]}}{\left[\frac{s-2}{2}\right]+1}+m\binom{s+1}{s-1}\frac{a^{(s-1)}_{\left[\frac{s-1}{2}\right]}}{\left[\frac{s-1}{2}\right]+1}>0;$$

- if s is odd,

then for $k\in\{2,3\ldots,\left[\frac{s-2}{2}\right]\}$ we get $k+1<\left[\frac{s-1}{2}\right]+1=\left[\frac{s+1}{2}\right]$ and

$$a^{(s+1)}_{\left[\frac{s+1}{2}\right]}=m\binom{s+1}{s-1}\frac{a^{(s-1)}_{\left[\frac{s-1}{2}\right]}}{\left[\frac{s-1}{2}\right]+1}>0.$$

And now, the proof is done. □

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