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INVERSE PROBLEMS VIA GENERALIZED CONTRACTIVE TYPE OPERATORS

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Abstract. We prove a "collage" theorem for a generalized contractive type operators.

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1. INTRODUCTION

Let X be a real Banach space, $T : X \to X$ be an operator. The following result of Barnsley, see [1], became "a classic".

THEOREM 1. ***(Collage Theorem)*** *Let $x \in X$ be given and $T : X \to X$ a contraction with contraction factor $L \in (0, 1)$, (i.e. $\|Tx - Ty\| \leq L \|x - y\|$, $\forall\ x,\ y \in X$), and fixed point x^*. Then*

$$\|x - x^*\| \leq \tfrac{1}{1-L} \|x - Tx\| .$$

Kunze *et. al.*, see [2], [3], [4], were able to apply the Collage Theorem to inverse problems in ODE , that is to reconstruct the field of an ODE, from a given "target" (trajectory). Our aim is to generalize the above Collage result to a larger operatorial class than contractions. Recently, similar results were introduced for other operatorial classes, see [5] and [6].

We shall consider the following class of operators: let T be such that there exist $\alpha, \beta, \gamma \in [0, 1)$, not simultaneous zero, satisfying,

$$0 < \alpha + \beta + \gamma < 1 \tag{1}$$

and for each $x, y \in X$,

$$\|Tx - Ty\| \leq \alpha \|x - y\| + \beta \|y - Tx\| + \gamma \|x - Ty\| . \tag{2}$$

For simplicity, let us denote this class by WR.

REMARK 2. Clearly, the contractions are included in this class Let $F(T)$ denote the fixed point set with respect to X for the operator T. □

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If an operator belongs to the above operatorial class and it has a fixed points then the successive approximation converges to the unique fixed point.

THEOREM 3. *Let X be a real Banach space and $\alpha, \beta, \gamma \in [0,1)$, not simultaneous zero, such that conditions* (1) *and* (2) *are satisfied, then the successive approximation iteration converges strongly to the unique fixed point of T.*

Proof. Let x^* be the fixed point.

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \|Tx_n - Tx^*\| \\ &\leq \alpha \|x_n - x^*\| + \beta \|x^* - Tx_n\| + \gamma \|Tx^* - x_n\| \\ &= (\alpha + \gamma) \|x_n - x^*\| + \beta \|Tx^* - Tx_n\| \\ &= (\alpha + \gamma) \|x_n - x^*\| + \beta \|x^* - x_{n+1}\|, \end{aligned}$$

$$\|x_{n+1} - x^*\| \leq \tfrac{\alpha+\gamma}{1-\beta} \|x_n - x^*\|.$$

$$\tfrac{\alpha+\gamma}{1-\beta} = 1 - \tfrac{1-\alpha-\beta-\gamma}{1-\beta}.$$

Therefore,

$$\|x_{n+1} - x^*\| \leq \left(1 - \tfrac{1-\alpha-\beta-\gamma}{1-\beta}\right)^n \|x_0 - x^*\| \to 0 \text{ as } n \to \infty,$$

since $\frac{1-\alpha-\beta-\gamma}{1-\beta} \in (0,1)$. Uniqueness easily results from condition (1) and (2). □

REMARK 4. It is straightforward to see that the above result holds in a complete metric space. □

2. THE INVERSE PROBLEM

Suppose that $x^* \in F(T)$. Kunze *et. al.*, see [2], [3], [4], have considered a class of inverse problems for ordinary differential equations and provided a mathematical basis for solving them within the framework of Banach spaces and contractions. We shall consider the same framework of Banach spaces and the larger class of operators which satisfy (1).

A typical inverse problem is the following, as formulated in [2]:

PROBLEM 5. *For given $\varepsilon > 0$ and a "target" $\bar{x}$, find $T_\varepsilon \in WR$ such that $\left\|\bar{x} - x^*_{T_\varepsilon}\right\| < \varepsilon$, where $x^*_{T_\varepsilon} = T_\varepsilon\left(x^*_{T_\varepsilon}\right)$ is the unique fixed point of the operator T_ε.*

Consider now the following problem which we shall fit in our framework and which is very useful for practitioners, see [2].

PROBLEM 6. *Let $\bar{x} \in X$ be a target and let $\delta > 0$ be given. Find $T_\delta \in WR$, such that $\|\bar{x} - T_\delta \bar{x}\| < \delta$.*

In other words, instead of searching for WR maps whose fixed points lie close to target $\bar{x}$, we search for WR maps that send $\bar{x}$ close to itself.

THEOREM 7. ***(Collage theorem for WR Operators)*** *Let X be a real Banach space and T an operator satisfying* (1) *with contraction factors $\alpha, \beta, \gamma \in [0,1)$, and fixed point $x^* \in X$. Then for any $x \in X$,*

$$\|x^* - x\| \le \tfrac{(\gamma+1)}{1-(\alpha+\beta+\gamma)} \|x - Tx\|.$$

Proof. The condition (1) assures that the fixed point x^* is unique. If $x = x^*$, the above inequality holds. If $x \neq x^*, \forall x \in X$, then one obtains

$$\begin{aligned}
\|x^* - x\| &\le \|Tx^* - Tx\| + \|Tx - x\| \\
&\le \alpha \|x^* - x\| + \beta \|x - x^*\| + \gamma \|x^* - Tx\| + \|Tx - x\| \\
&= \alpha \|x^* - x\| + \beta \|x - x^*\| + \gamma \|x^* - x\| + \gamma \|Tx - x\| + \|Tx - x\| \\
&= \alpha \|x^* - x\| + \beta \|x - x^*\| + \gamma \|x^* - x\| + (\gamma + 1) \|Tx - x\| \\
&= (\alpha + \beta + \gamma) \|x^* - x\| + (\gamma + 1) \|Tx - x\|.
\end{aligned}$$

From which one gets the conclusion by using (1). □

The above "Collage Theorem" allows us to reformulate the inverse Problem 5 in the particular and more convenient Problem 6.

THEOREM 8. *If Problem* 6 *has a solution, then Problem* 5 *has a solution too.*

Proof. Let $\varepsilon > 0$ and $\bar{x} \in X$ be given. For $\delta := ((1-(\alpha+\beta+\gamma)) / (\gamma + 1))\, \varepsilon$, let $T_\delta \in WR$ be such that $\|\bar{x} - T_\delta \bar{x}\| < \delta$. If $x^*_{T_\delta}$ is the unique fixed point of the CL mapping T_δ, then, by Theorem 7,

$$\left\|\bar{x} - x^*_{T_\delta}\right\| \le \tfrac{(\gamma+1)}{1-(\alpha+\beta+\gamma)} \|\bar{x} - T_\delta \bar{x}\| \le \tfrac{(\gamma+1)}{1-(\alpha+\beta+\gamma)} \delta = \varepsilon.$$

□

Note that shrinking the distance between two operators, one of them from WR, reduces the distance between their fixed points.

PROPOSITION 9. *Let X be a real Banach space and $T_1 \in WR$ with contraction factor $\alpha_1, \beta_1, \gamma_1 \in (0,1)$ and $T_2 : X \to X$ a map such that $x_1^*, x_2^* \in X$ are distinct fixed points for T_1 and T_2. Then,*

$$\|x_1^* - x_2^*\| \le \tfrac{(\gamma_1+1)}{1-(\alpha_1+\beta_1+\gamma_1)} \sup_{x \in X} \|T_1 x - T_2 x\|.$$

Proof. One obtains

$$\begin{aligned}
\|x_1^* - x_2^*\| &= \|T_1 x_1^* - T_2 x_2^*\| \\
&\le \|T_1 x_1^* - T_1 x_2^*\| + \|T_1 x_2^* - T_2 x_2^*\| \\
&\le \alpha_1 \|x_1^* - x_2^*\| + \beta_1 \|x_2^* - T_1 x_1^*\| + \gamma_1 \|x_1^* - T_1 x_2^*\| + \|T_1 x_2^* - T_2 x_2^*\| \\
&\le \alpha_1 \|x_1^* - x_2^*\| + \beta_1 \|x_2^* - x_1^*\| + \gamma_1 \|x_1^* - x_2^*\| + \\
&\quad + \gamma_1 \|x_2^* - T_1 x_2^*\| + \|T_1 x_2^* - T_2 x_2^*\| \\
&= \alpha_1 \|x_1^* - x_2^*\| + \beta_1 \|x_2^* - x_1^*\| + \gamma_1 \|x_1^* - x_2^*\| + \\
&\quad + (\gamma_1 + 1) \|T_1 x_2^* - T_2 x_2^*\| \\
&\le \alpha_1 \|x_1^* - x_2^*\| + \beta_1 \|x_2^* - x_1^*\| + \gamma_1 \|x_1^* - x_2^*\| + \\
&\quad + (\gamma_1 + 1) \sup_{x \in X} \|T_1 x T_2 x\|,
\end{aligned}$$

from which we get the conclusion. □

THEOREM 10. *Let X be a real Banach space, $T : X \to X$, $\bar{x} = T\bar{x}$ and suppose there exists $T_1 \in WR$ such that* $\sup_{x \in X} \|T_1 x - Tx\| \le \varepsilon$. *Then*

$$\|\bar{x} - T_1 \bar{x}\| \le \left(1 + \tfrac{\alpha+\beta}{1-\gamma}\right) \tfrac{(\gamma+1)}{1-(\alpha+\beta+\gamma)} \varepsilon.$$

Proof. Let $x^* = T_1 x^*$, and by use of Proposition 9 we obtain

$$\|\bar{x} - x^*\| \le \tfrac{(\gamma+1)}{1-(\alpha+\beta+\gamma)} \left(\sup_{x \in X} \|T_1 x - Tx\| \right).$$

We have

$$\begin{aligned}
\|\bar{x} - T_1 \bar{x}\| &\le \|\bar{x} - x^*\| + \|x^* - T_1 \bar{x}\| \\
&\le \|\bar{x} - x^*\| + \|T_1 x^* - T_1 \bar{x}\|,
\end{aligned}$$

and

$$\begin{aligned}
\|T_1 x^* - T_1 \bar{x}\| &\le \alpha \|\bar{x} - x^*\| + \beta \|\bar{x} - T_1 x^*\| + \gamma \|x^* - T_1 \bar{x}\| \\
&= \alpha \|\bar{x} - x^*\| + \beta \|\bar{x} - T_1 x^*\| + \gamma \|T_1 x^* - T_1 \bar{x}\|, \\
&\ i.e. \\
\|T_1 x^* - T_1 \bar{x}\| &\le \tfrac{\alpha+\beta}{1-\gamma} \|\bar{x} - x^*\|.
\end{aligned}$$

Thus

$$\begin{aligned}
\|\bar{x} - T_1 \bar{x}\| &\le \|\bar{x} - x^*\| + \tfrac{\alpha+\beta}{1-\gamma} \|\bar{x} - x^*\| \\
&\le \left(1 + \tfrac{\alpha+\beta}{1-\gamma}\right) \|\bar{x} - x^*\| \le \left(1 + \tfrac{\alpha+\beta}{1-\gamma}\right) \tfrac{(\gamma+1)}{1-(\alpha+\beta+\gamma)} \varepsilon.
\end{aligned}$$

□

REMARK 11. The quantity $\left(1 + \tfrac{\alpha+\beta}{1-\gamma}\right) \tfrac{(\gamma+1)}{1-(\alpha+\beta+\gamma)}$ can be written as $\tfrac{1+\gamma}{1-\gamma} \tfrac{(1+\gamma)+(\alpha+\beta)}{(1-\gamma)-(\alpha+\beta)}$. □

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