AN OPTIMAL DOUBLE INEQUALITY AMONG THE ONE-PARAMETER, ARITHMETIC AND HARMONIC MEANS

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Abstract. For \( p \in \mathbb{R} \), the one-parameter mean \( J_p(a, b) \), arithmetic mean \( A(a, b) \), and harmonic mean \( H(a, b) \) of two positive real numbers \( a \) and \( b \) are defined by

\[
J_p(a, b) = \begin{cases} \frac{p(a^{p+1} - b^{p+1})}{(p+1)(a^p - b^p)}, & a \neq b, p \neq 0, -1, \\ \log a - \log b, & a \neq b, p = 0, \\ a^{-b}, & a \neq b, p = -1, \\ a, & a = b, \end{cases}
\]

\( A(a, b) = \frac{a+b}{2} \), and \( H(a, b) = \frac{2ab}{a+b} \), respectively.

In this paper, we answer the question: For \( \alpha \in (0,1) \), what are the greatest value \( r_1 \) and the least value \( r_2 \) such that the double inequality \( J_{r_1}(a, b) < \alpha A(a, b) + (1 - \alpha)H(a, b) < J_{r_2}(a, b) \) holds for all \( a, b > 0 \) with \( a \neq b \)?

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1. INTRODUCTION

For \( p \in \mathbb{R} \), the one-parameter mean \( J_p(a, b) \), arithmetic mean \( A(a, b) \), and harmonic mean \( H(a, b) \) of two positive real numbers \( a \) and \( b \) are defined by

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J_p(a, b) = \begin{cases} \frac{p(a^{p+1} - b^{p+1})}{(p+1)(a^p - b^p)}, & a \neq b, p \neq 0, -1, \\ \log a - \log b, & a \neq b, p = 0, \\ a^{-b}, & a \neq b, p = -1, \\ a, & a = b, \end{cases}
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\( A(a, b) = \frac{a+b}{2} \), and \( H(a, b) = \frac{2ab}{a+b} \), respectively.

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Recently, the one-parameter mean $J_p(a, b)$ has been the subject of intensive research. In particular, many remarkable inequalities and properties for the one-parameter mean $J_p$ can be found in the literature [1–7].

It is well-known that the one-parameter mean $J_p(a, b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$ [5]. Many mean values are special case of the one-parameter mean, for example

\[
J_1(a, b) = \frac{a+b}{2} = A(a, b), \quad \text{the arithmetic mean,}
\]

\[
J_{\frac{3}{2}}(a, b) = \frac{a+\sqrt{ab}+b}{3} = He(a, b), \quad \text{the Heronian mean,}
\]

\[
J_{-\frac{1}{2}}(a, b) = \sqrt{ab} = G(a, b), \quad \text{the geometric mean}
\]

and

\[
J_{-2}(a, b) = \frac{2ab}{a+b} = H(a, b), \quad \text{the harmonic mean.}
\]

For $r \in \mathbb{R}$, the power mean $M_r(a, b)$ of order $r$ of two positive numbers $a$ and $b$ is defined by

\[
M_r(a, b) = \begin{cases} 
\left(\frac{a^r+b^r}{2}\right)^{\frac{1}{r}}, & r \neq 0, \\
\sqrt{ab}, & r = 0.
\end{cases}
\]

The main properties of the power mean are given in [8]. In particular, $M_r(a, b)$ is continuous and strictly increasing with respect to $r \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$.

In [9], Alzer and Janous established the following sharp double inequality (see also [9, p. 350])

\[
M_{\log \frac{2}{3}}(a, b) < \frac{2}{3}J_1(a, b) + \frac{1}{3}J_{-\frac{1}{2}}(a, b) < M_{\frac{3}{2}}(a, b)
\]

for all $a, b > 0$ with $a \neq b$.

In [10], Mao proved

\[
M_{\frac{3}{2}}(a, b) \leq \frac{1}{3}J_1(a, b) + \frac{2}{3}J_{-\frac{1}{2}}(a, b) \leq M_1(a, b)
\]

for all $a, b > 0$, and $M_{\frac{3}{2}}(a, b)$ is the best possible lower power mean bound for the sum $\frac{1}{3}J_1(a, b) + \frac{2}{3}J_{-\frac{1}{2}}(a, b)$.

The purpose of this paper is to answer the question: For $\alpha \in (0, 1)$, what are the greatest value $r_1$ and the least value $r_2$ such that the double inequality

\[
J_{r_1}(a, b) < \alpha A(a, b) + (1 - \alpha)H(a, b) < J_{r_2}(a, b)
\]

holds for all $a, b > 0$ with $a \neq b$?

2. LEMMAS

In order to establish our main result we need two lemmas, which we present in this section.
Lemma 1. If \( t > 1 \), then
\[
- \log t + \frac{(t-1)(t^2+10t+1)}{6t(t+1)} > 0.
\]

Proof. Let
\[
h(t) = - \log t + \frac{(t-1)(t^2+10t+1)}{6t(t+1)}.
\]
Then simple computations lead to
\[
h(1) = 0,
\]
\[
h'(t) = \frac{(t-1)^4}{6t^2(t+1)^2} > 0
\]
for \( t > 1 \).

Therefore, Lemma 1 follows from (4)–(6). \( \square \)

Lemma 2. If \( t > 1 \), then
\[
\log t - \frac{3(t^2-1)}{t^2+4t+1} > 0.
\]

Proof. Let
\[
g(t) = \log t - \frac{3(t^2-1)}{t^2+4t+1}.
\]
Then simple computations lead to
\[
g(1) = 0,
\]
\[
g'(t) = \frac{(t-1)^4}{6t^2(t^2+4t+1)^2} > 0
\]
for \( t > 1 \).

Therefore, Lemma 2 follows from (8)–(10). \( \square \)

3. MAIN RESULT

Theorem 3. Inequality
\[
J_{3\alpha-2}(a, b) < \alpha A(a, b) + (1 - \alpha) H(a, b) < J_{\frac{\alpha}{2-\alpha}}(a, b)
\]
holds for all \( a, b > 0 \) with \( a \neq b \), and \( J_{3\alpha-2}(a, b) \) and \( J_{\frac{\alpha}{2-\alpha}}(a, b) \) are the best possible lower and upper one-parameter mean bounds for the sum \( \alpha A(a, b) + (1 - \alpha) H(a, b) \), respectively.

Proof. We first prove that
\[
\alpha A(a, b) + (1 - \alpha) H(a, b) > J_{3\alpha-2}(a, b)
\]
for \( \alpha \in (0, 1) \) and all \( a, b > 0 \) with \( a \neq b \).

Without loss of generality, we assume that \( a > b \) and take \( t = \frac{a}{b} > 1 \). We divide the proof into three cases.
Case 1. If \( \alpha = \frac{1}{3} \), then from (1) we have
\[
\alpha A(a, b) + (1 - \alpha) H(a, b) - J_{3\alpha - 2}(a, b) =
\]
\[
= b \left( \frac{1 + t}{6} + \frac{4t}{3(1 + t)} - \frac{t \log t}{t - 1} \right)
\]
\[
= \frac{bt}{t - 1} \left[ - \log t + \frac{(t - 1)(t^2 + 10t + 1)}{6t(1 + t)} \right].
\]
Therefore, inequality (11) follows from (12) and Lemma 1.

Case 2. If \( \alpha = \frac{2}{3} \), then (1) leads to
\[
\alpha A(a, b) + (1 - \alpha) H(a, b) - J_{3\alpha - 2}(a, b) =
\]
\[
= b \left( \frac{1 + t}{3} + \frac{2t}{3(1 + t)} - \frac{t - 1}{\log t} \right)
\]
\[
= \frac{bt}{3(t + 1)} \left[ \log t - \frac{3(t^2 - 1)}{t^2 + 4t + 1} \right].
\]
Therefore, inequality (11) follows from (13) and Lemma 2.

Case 3. If \( \alpha \in (0, \frac{1}{3}) \cup (\frac{1}{3}, \frac{2}{3}) \cup (\frac{2}{3}, 1) \), then (1) implies that
\[
\alpha A(a, b) + (1 - \alpha) H(a, b) - J_{3\alpha - 2}(a, b) =
\]
\[
= b \left[ a \frac{1 + t}{3} + (1 - \alpha) \frac{2t}{1 + t} - \frac{(3\alpha - 2)(t^{3\alpha - 1} - 1)}{(3\alpha - 1)(t^{3\alpha - 2} - 1)} \right].
\]
Let
\[
f(t) = \alpha \frac{1 + t}{2} + (1 - \alpha) \frac{2t}{1 + t} - \frac{(3\alpha - 2)(t^{3\alpha - 1} - 1)}{(3\alpha - 1)(t^{3\alpha - 2} - 1)},
\]
then \( f(t) \) can be rewritten as
\[
f(t) = \frac{f_1(t)}{2(3\alpha - 1)(t + 1)(t^{3\alpha - 2} - 1)},
\]
where
\[
f_1(t) = (1 - \alpha)(4 - 3\alpha)t^{3\alpha} + 2\alpha(4 - 3\alpha)t^{3\alpha - 1} + \alpha(3\alpha - 1)t^{3\alpha - 2}
\]
\[
- \alpha(3\alpha - 1)t^2 - 2\alpha(4 - 3\alpha)t - (1 - \alpha)(4 - 3\alpha).
\]
Note that
\[
f_1(1) = 0,
\]
\[
f_1'(t) = 3\alpha(1 - \alpha)(4 - 3\alpha)t^{3\alpha - 1} + 2\alpha(4 - 3\alpha)(3\alpha - 1)t^{3\alpha - 2}
\]
\[
+ \alpha(3\alpha - 1)(3\alpha - 2)t^{3\alpha - 3} - 2\alpha(3\alpha - 1)t
\]
\[
- 2\alpha(4 - 3\alpha),
\]
\[
f_1''(1) = 0,
\]
\[
f_1''(t) = 3\alpha(1 - \alpha)(3\alpha - 1)(4 - 3\alpha)t^{3\alpha - 2} + 2\alpha(4 - 3\alpha)(3\alpha - 1)
\]
\[
\times (3\alpha - 2)t^{3\alpha - 3} + 3\alpha(3\alpha - 1)(3\alpha - 2)(\alpha - 1)t^{3\alpha - 4}
\]
\[
- 2\alpha(3\alpha - 1),
\]
\[
f_1''(1) = 0.
\]
and

\[ \frac{d^3f_1}{dt^3}(t) = 3\alpha(1-\alpha)(4-3\alpha)(3\alpha-1)(3\alpha-2)t^{3\alpha-5}(t-1)^2. \]

We divide the proof into two subcases.

Subcase 1. If \( \alpha \in (0, \frac{1}{3}) \cup (\frac{2}{3}, 1) \), then from (23) we clearly see that \( \frac{d^3f_1}{dt^3}(t) > 0 \) for \( t \in (1, \infty) \). Then (18) and (20) together with (22) imply that \( f_1(t) > 0 \) for \( t \in (1, \infty) \). Note that

\[ (3\alpha-1)(t^{3\alpha-2} - 1) > 0 \]

for \( t > 1 \).

Therefore, inequality (11) follows from (14)–(16) and (24) together with the fact that \( f_1(t) > 0 \) for \( t \in (1, \infty) \).

Subcase 2. If \( \alpha \in (\frac{1}{3}, \frac{2}{3}) \), then from (23) we clearly see that \( \frac{d^3f_1}{dt^3}(t) < 0 \) for \( t \in (1, \infty) \). Then (18) and (20) together with (22) imply that \( f_1(t) < 0 \) for \( t \in (1, \infty) \). Note that

\[ (3\alpha-1)(t^{3\alpha-2} - 1) < 0 \]

for \( t > 1 \).

Therefore, inequality (11) follows from (14)–(16) and (25) together with the fact that \( f_1(t) < 0 \) for \( t \in (1, \infty) \).

Next, we prove that

\[ \alpha A(a,b) + (1-\alpha)H(a,b) < J_{\frac{\alpha}{2^\alpha}}(a,b) \]

for \( \alpha \in (0, 1) \) and all \( a, b > 0 \) with \( a \neq b \).

Without loss of generality, we assume that \( a > b \). Let \( t = \frac{a}{b} > 1 \), then from (1) we have

\[ \alpha A(a,b) + (1-\alpha)H(a,b) - J_{\frac{\alpha}{2^\alpha}}(a,b) = \]

\[ = b[a^{\frac{1+t}{2^\alpha}} + (1-a)^{\frac{2t}{1+t}} - \frac{a^{\frac{2}{(2^\alpha)-1}}}{2^{(2^\alpha)-1}}]. \]

Let

\[ F(t) = a^{\frac{1+t}{2^\alpha}} + (1-a)^{\frac{2t}{1+t}} - \frac{a^{\frac{2}{(2^\alpha)-1}}}{2^{(2^\alpha)-1}}, \]

then \( F(t) \) can be rewritten as

\[ F(t) = \frac{tF_1(t)}{2(t+1)(2^\alpha-1)}, \]

where

\[ F_1(t) = (4-3\alpha)t^{\frac{\alpha}{2^\alpha}} + \alpha t^{\frac{2(a-1)}{2^\alpha} - \alpha} - 4 + 3\alpha. \]

Note that

\[ F_1(1) = 0, \]
(32) \[ F_1'(t) = \frac{\alpha}{2-\alpha}[ (4-3\alpha)t^{\frac{2(\alpha-1)}{2-\alpha}} + 2(\alpha - 1)t^{\frac{3\alpha - 4}{2-\alpha}} - 2 + \alpha], \]

(33) \[ F_1'(1) = 0 \]

and

(34) \[ F_1''(t) = \frac{2\alpha(4-3\alpha)(1-\alpha)}{(2-\alpha)^2} t^{\frac{4\alpha - 6}{2-\alpha}} (1 - t) < 0 \]

for \( t \in (1, \infty) \) and \( \alpha \in (0,1) \).

Therefore, inequality (26) follows from (27)–(29), (31), (33) and (34).

At last, we prove that \( J_{3\alpha - 2}(a, b) \) and \( J_{\frac{2}{2-\alpha}}(a, b) \) are the best possible lower and upper one-parameter mean bounds for the sum \( \alpha A(a, b) + (1 - \alpha) H(a, b) \), respectively.

For any \( 0 < \varepsilon < \frac{\alpha}{2-\alpha} \) and \( x > 0 \), from (1) one has

(35) \[ \lim_{x \to \infty} \frac{\alpha A(x, 1) + (1 - \alpha) H(x, 1)}{J_{\frac{2}{2-\alpha}}(x, 1)} = \frac{2\alpha - (2-\alpha)\varepsilon}{2\alpha - 2(2-\alpha)\varepsilon} > 1. \]

For any \( \varepsilon > 0 \), \( \varepsilon \neq 1 - 3\alpha \), \( \varepsilon \neq 2 - 3\alpha \) and \( x > 0 \), let \( x \to 0 \), making use of (1) and the Taylor expansion one has

\[
J_{3\alpha - 2 + \varepsilon}(1 + x, 1) - \alpha A(1 + x, 1) - (1 - \alpha) H(1 + x, 1) = \\
= 1 + \frac{x}{2} + \frac{3\alpha - 3 + \varepsilon}{12} x^2 + o(x^2) - \alpha(1 + \frac{x}{2}) - (1 - \alpha) \\
\times (1 + \frac{x}{2} - \frac{1}{2} x^2 + o(x^2)) \\
= \frac{\varepsilon}{12} x^2 + o(x^2).
\]

Inequality (35) implies that for any \( 0 < \varepsilon < \frac{\alpha}{2-\alpha} \) there exists \( X = X(\alpha, \varepsilon) > 1 \) such that \( \alpha A(x, 1) + (1 - \alpha) H(x, 1) > J_{\frac{2}{2-\alpha}}(x, 1) \) for \( x \in (X, \infty) \), and inequality (36) implies that for any \( \varepsilon > 0 \), \( \varepsilon \neq 1 - 3\alpha \) and \( \varepsilon \neq 2 - 3\alpha \) there exists \( \delta = \delta(\alpha, \varepsilon) > 0 \) such that \( J_{3\alpha - 2 + \varepsilon}(1 + x, 1) > \alpha A(1 + x, 1) + (1 - \alpha) H(1 + x, 1) \) for \( x \in (0, \delta) \).

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REFERENCES

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