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AN OPTIMAL DOUBLE INEQUALITY AMONG THE ONE-PARAMETER, ARITHMETIC AND HARMONIC MEANS*

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Abstract. For $p \in \mathbb{R}$, the one-parameter mean $J_p(a,b)$, arithmetic mean $A(a,b)$, and harmonic mean $H(a,b)$ of two positive real numbers a and b are defined by

$$J_p(a,b) = \begin{cases} \frac{p(a^{p+1}-b^{p+1})}{(p+1)(a^p-b^p)}, & a \neq b, p \neq 0, -1, \\ \frac{a-b}{\log a - \log b}, & a \neq b, p = 0, \\ \frac{ab(\log a - \log b)}{a-b}, & a \neq b, p = -1, \\ a, & a = b, \end{cases}$$

$A(a,b) = \frac{a+b}{2}$, and $H(a,b) = \frac{2ab}{a+b}$, respectively.

In this paper, we answer the question: For $\alpha \in (0,1)$, what are the greatest value r_1 and the least value r_2 such that the double inequality $J_{r_1}(a,b) < \alpha A(a,b) + (1-\alpha)H(a,b) < J_{r_2}(a,b)$ holds for all $a, b > 0$ with $a \neq b$?

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1. INTRODUCTION

For $p \in \mathbb{R}$, the one-parameter mean $J_p(a,b)$, arithmetic mean $A(a,b)$, and harmonic mean $H(a,b)$ of two positive real numbers a and b are defined by

$$(1) \qquad J_p(a,b) = \begin{cases} \frac{p(a^{p+1}-b^{p+1})}{(p+1)(a^p-b^p)}, & a \neq b, p \neq 0, -1, \\ \frac{a-b}{\log a - \log b}, & a \neq b, p = 0, \\ \frac{ab(\log a - \log b)}{a-b}, & a \neq b, p = -1, \\ a, & a = b, \end{cases}$$

$A(a,b) = \frac{a+b}{2}$, and $H(a,b) = \frac{2ab}{a+b}$, respectively.

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Recently, the one-parameter mean $J_p(a,b)$ has been the subject of intensive research. In particular, many remarkable inequalities and properties for the one-parameter mean J_p can be found in the literature [1–7].

It is well-known that the one-parameter mean $J_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$ [5]. Many mean values are special case of the one-parameter mean, for example

$$
\begin{aligned}
&J_1(a,b) = \tfrac{a+b}{2} = A(a,b), && \text{the arithmetic mean,}\\
&J_{\frac{1}{2}}(a,b) = \tfrac{a+\sqrt{ab}+b}{3} = He(a,b), && \text{the Heronian mean,}\\
&J_{-\frac{1}{2}}(a,b) = \sqrt{ab} = G(a,b), && \text{the geometric mean}
\end{aligned}
$$

and

$$J_{-2}(a,b) = \tfrac{2ab}{a+b} = H(a,b), \qquad \text{the harmonic mean.}$$

For $r \in \mathbb{R}$, the power mean $M_r(a,b)$ of order r of two positive numbers a and b is defined by

$$M_r(a,b) = \begin{cases} \left(\frac{a^r+b^r}{2}\right)^{\frac{1}{r}}, & r \neq 0,\\ \sqrt{ab}, & r = 0. \end{cases} \tag{2}$$

The main properties of the power mean are given in [8]. In particular, $M_r(a,b)$ is continuous and strictly increasing with respect to $r \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$.

In [9], Alzer and Janous established the following sharp double inequality (see also [9, p. 350])

$$M_{\frac{\log 2}{\log 3}}(a,b) < \tfrac{2}{3}J_1(a,b) + \tfrac{1}{3}J_{-\frac{1}{2}}(a,b) < M_{\frac{2}{3}}(a,b)$$

for all $a, b > 0$ with $a \neq b$.

In [10], Mao proved

$$M_{\frac{1}{3}}(a,b) \leq \tfrac{1}{3}J_1(a,b) + \tfrac{2}{3}J_{-\frac{1}{2}}(a,b) \leq M_{\frac{1}{2}}(a,b)$$

for all $a, b > 0$, and $M_{\frac{1}{3}}(a,b)$ is the best possible lower power mean bound for the sum $\frac{1}{3}J_1(a,b) + \frac{2}{3}J_{-\frac{1}{2}}(a,b)$.

The purpose of this paper is to answer the question: For $\alpha \in (0,1)$, what are the greatest value r_1 and the least value r_2 such that the double inequality

$$J_{r_1}(a,b) < \alpha A(a,b) + (1-\alpha)H(a,b) < J_{r_2}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$?

2. LEMMAS

In order to establish our main result we need two lemmas, which we present in this section.

LEMMA 1. *If $t > 1$, then*

$$-\log t + \frac{(t-1)(t^2+10t+1)}{6t(t+1)} > 0. \tag{3}$$

Proof. Let

$$h(t) = -\log t + \frac{(t-1)(t^2+10t+1)}{6t(t+1)}. \tag{4}$$

Then simple computations lead to

$$h(1) = 0, \tag{5}$$

$$h'(t) = \frac{(t-1)^4}{6t^2(t+1)^2} > 0 \tag{6}$$

for $t > 1$.

Therefore, Lemma 1 follows from (4)–(6). □

LEMMA 2. *If $t > 1$, then*

$$\log t - \frac{3(t^2-1)}{t^2+4t+1} > 0. \tag{7}$$

Proof. Let

$$g(t) = \log t - \frac{3(t^2-1)}{t^2+4t+1}. \tag{8}$$

Then simple computations lead to

$$g(1) = 0, \tag{9}$$

$$g'(t) = \frac{(t-1)^4}{t(t^2+4t+1)^2} > 0. \tag{10}$$

for $t > 1$.

Therefore, Lemma 2 follows from (8)–(10). □

3. MAIN RESULT

THEOREM 3. *Inequality*

$$J_{3\alpha-2}(a,b) < \alpha A(a,b) + (1-\alpha)H(a,b) < J_{\frac{\alpha}{2-\alpha}}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$, and $J_{3\alpha-2}(a,b)$ and $J_{\frac{\alpha}{2-\alpha}}(a,b)$ are the best possible lower and upper one-parameter mean bounds for the sum $\alpha A(a,b) + (1-\alpha)H(a,b)$, respectively.

Proof. We first prove that

$$\alpha A(a,b) + (1-\alpha)H(a,b) > J_{3\alpha-2}(a,b) \tag{11}$$

for $\alpha \in (0,1)$ and all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$ and take $t = \frac{a}{b} > 1$. We divide the proof into three cases.

Case 1. If $\alpha = \frac{1}{3}$, then from (1) we have

$$\begin{aligned} &\alpha A(a,b) + (1-\alpha)H(a,b) - J_{3\alpha-2}(a,b) = \\ &= b[\tfrac{1+t}{6} + \tfrac{4t}{3(1+t)} - \tfrac{t\log t}{t-1}] \\ &= \tfrac{bt}{t-1}[-\log t + \tfrac{(t-1)(t^2+10t+1)}{6t(1+t)}]. \end{aligned} \tag{12}$$

Therefore, inequality (11) follows from (12) and Lemma 1.

Case 2. If $\alpha = \frac{2}{3}$, then (1) leads to

$$\begin{aligned} &\alpha A(a,b) + (1-\alpha)H(a,b) - J_{3\alpha-2}(a,b) = \\ &= b[\tfrac{1+t}{3} + \tfrac{2t}{3(1+t)} - \tfrac{t-1}{\log t}] \\ &= \tfrac{b(t^2+4t+1)}{3(1+t)\log t}[\log t - \tfrac{3(t^2-1)}{t^2+4t+1}]. \end{aligned} \tag{13}$$

Therefore, inequality (11) follows from (13) and Lemma 2.

Case 3. If $\alpha \in (0, \frac{1}{3}) \cup (\frac{1}{3}, \frac{2}{3}) \cup (\frac{2}{3}, 1)$, then (1) implies that

$$\begin{aligned} &\alpha A(a,b) + (1-\alpha)H(a,b) - J_{3\alpha-2}(a,b) = \\ &= b[\alpha\tfrac{1+t}{2} + (1-\alpha)\tfrac{2t}{1+t} - \tfrac{(3\alpha-2)(t^{3\alpha-1}-1)}{(3\alpha-1)(t^{3\alpha-2}-1)}]. \end{aligned} \tag{14}$$

Let

$$f(t) = \alpha\tfrac{1+t}{2} + (1-\alpha)\tfrac{2t}{1+t} - \tfrac{(3\alpha-2)(t^{3\alpha-1}-1)}{(3\alpha-1)(t^{3\alpha-2}-1)}, \tag{15}$$

then $f(t)$ can be rewritten as

$$f(t) = \tfrac{f_1(t)}{2(3\alpha-1)(t+1)(t^{3\alpha-2}-1)}, \tag{16}$$

where

$$\begin{aligned} f_1(t) = {} & (1-\alpha)(4-3\alpha)t^{3\alpha} + 2\alpha(4-3\alpha)t^{3\alpha-1} + \alpha(3\alpha-1)t^{3\alpha-2} \\ & -\alpha(3\alpha-1)t^2 - 2\alpha(4-3\alpha)t - (1-\alpha)(4-3\alpha). \end{aligned} \tag{17}$$

Note that

$$f_1(1) = 0, \tag{18}$$

$$\begin{aligned} {f_1}'(t) = {} & 3\alpha(1-\alpha)(4-3\alpha)t^{3\alpha-1} + 2\alpha(4-3\alpha)(3\alpha-1)t^{3\alpha-2} \\ & +\alpha(3\alpha-1)(3\alpha-2)t^{3\alpha-3} - 2\alpha(3\alpha-1)t \\ & -2\alpha(4-3\alpha), \end{aligned} \tag{19}$$

$${f_1}'(1) = 0, \tag{20}$$

$$\begin{aligned} {f_1}''(t) = {} & 3\alpha(1-\alpha)(3\alpha-1)(4-3\alpha)t^{3\alpha-2} + 2\alpha(4-3\alpha)(3\alpha-1) \\ & \times(3\alpha-2)t^{3\alpha-3} + 3\alpha(3\alpha-1)(3\alpha-2)(\alpha-1)t^{3\alpha-4} \\ & -2\alpha(3\alpha-1), \end{aligned} \tag{21}$$

$${f_1}''(1) = 0 \tag{22}$$

and

$$f_1'''(t) = 3\alpha(1-\alpha)(4-3\alpha)(3\alpha-1)(3\alpha-2)t^{3\alpha-5}(t-1)^2. \tag{23}$$

We divide the proof into two subcases.

Subcase 1. If $\alpha \in (0, \frac{1}{3}) \cup (\frac{2}{3}, 1)$, then from (23) we clearly see that $f_1'''(t) > 0$ for $t \in (1, \infty)$. Then (18) and (20) together with (22) imply that $f_1(t) > 0$ for $t \in (1, \infty)$. Note that

$$(3\alpha - 1)(t^{3\alpha-2} - 1) > 0 \tag{24}$$

for $t > 1$.

Therefore, inequality (11) follows from (14)–(16) and (24) together with the fact that $f_1(t) > 0$ for $t \in (1, \infty)$.

Subcase 2. If $\alpha \in (\frac{1}{3}, \frac{2}{3})$, then from (23) we clearly see that $f_1'''(t) < 0$ for $t \in (1, \infty)$. Then (18) and (20) together with (22) imply that $f_1(t) < 0$ for $t \in (1, \infty)$. Note that

$$(3\alpha - 1)(t^{3\alpha-2} - 1) < 0 \tag{25}$$

for $t > 1$.

Therefore, inequality (11) follows from (14)–(16) and (25) together with the fact that $f_1(t) < 0$ for $t \in (1, \infty)$.

Next, we prove that

$$\alpha A(a,b) + (1-\alpha)H(a,b) < J_{\frac{\alpha}{2-\alpha}}(a,b) \tag{26}$$

for $\alpha \in (0, 1)$ and all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $t = \frac{a}{b} > 1$, then from (1) we have

$$\alpha A(a,b) + (1-\alpha)H(a,b) - J_{\frac{\alpha}{2-\alpha}}(a,b) = \tag{27}$$

$$= b\Big[\alpha\tfrac{1+t}{2} + (1-\alpha)\tfrac{2t}{1+t} - \frac{\alpha(t^{\frac{2}{2-\alpha}}-1)}{2(t^{\frac{\alpha}{2-\alpha}}-1)}\Big].$$

Let

$$F(t) = \alpha\tfrac{1+t}{2} + (1-\alpha)\tfrac{2t}{1+t} - \frac{\alpha(t^{\frac{2}{2-\alpha}}-1)}{2(t^{\frac{\alpha}{2-\alpha}}-1)}, \tag{28}$$

then $F(t)$ can be rewritten as

$$F(t) = \frac{tF_1(t)}{2(t+1)(t^{\frac{\alpha}{2-\alpha}}-1)}, \tag{29}$$

where

$$F_1(t) = (4-3\alpha)t^{\frac{\alpha}{2-\alpha}} + \alpha t^{\frac{2(\alpha-1)}{2-\alpha}} - \alpha t - 4 + 3\alpha. \tag{30}$$

Note that

$$F_1(1) = 0, \tag{31}$$

$$F_1{}'(t) = \tfrac{\alpha}{2-\alpha}[(4-3\alpha)t^{\frac{2(\alpha-1)}{2-\alpha}} + 2(\alpha-1)t^{\frac{3\alpha-4}{2-\alpha}} - 2 + \alpha], \tag{32}$$

$$F_1{}'(1) = 0 \tag{33}$$

and

$$F_1{}''(t) = \tfrac{2\alpha(4-3\alpha)(1-\alpha)}{(2-\alpha)^2}t^{\frac{4\alpha-6}{2-\alpha}}(1-t) < 0 \tag{34}$$

for $t \in (1, \infty)$ and $\alpha \in (0, 1)$.

Therefore, inequality (26) follows from (27)–(29), (31), (33) and (34).

At last, we prove that $J_{3\alpha-2}(a, b)$ and $J_{\frac{\alpha}{2-\alpha}}(a, b)$ are the best possible lower and upper one-parameter mean bounds for the sum $\alpha A(a, b) + (1-\alpha)H(a, b)$, respectively.

For any $0 < \varepsilon < \frac{\alpha}{2-\alpha}$ and $x > 0$, from (1) one has

$$\lim_{x\to\infty} \frac{\alpha A(x,1)+(1-\alpha)H(x,1)}{J_{\frac{\alpha}{2-\alpha}-\varepsilon}(x,1)} = \frac{2\alpha-\alpha(2-\alpha)\varepsilon}{2\alpha-2(2-\alpha)\varepsilon} > 1. \tag{35}$$

For any $\varepsilon > 0$, $\varepsilon \neq 1-3\alpha$, $\varepsilon \neq 2-3\alpha$ and $x > 0$, let $x \to 0$, making use of (1) and the Taylor expansion one has

$$\begin{aligned} &J_{3\alpha-2+\varepsilon}(1+x, 1) - \alpha A(1+x, 1) - (1-\alpha)H(1+x, 1) = \\ &\quad = 1 + \tfrac{x}{2} + \tfrac{3\alpha-3+\varepsilon}{12}x^2 + o(x^2) - \alpha(1+\tfrac{x}{2}) - (1-\alpha) \\ &\qquad \times (1 + \tfrac{x}{2} - \tfrac{1}{4}x^2 + o(x^2)) \\ &\quad = \tfrac{\varepsilon}{12}x^2 + o(x^2). \end{aligned} \tag{36}$$

Inequality (35) implies that for any $0 < \varepsilon < \frac{\alpha}{2-\alpha}$ there exists $X = X(\alpha, \varepsilon) > 1$ such that $\alpha A(x, 1) + (1-\alpha)H(x, 1) > J_{\frac{\alpha}{2-\alpha}-\varepsilon}(x, 1)$ for $x \in (X, \infty)$, and inequality (36) implies that for any $\varepsilon > 0$, $\varepsilon \neq 1-3\alpha$ and $\varepsilon \neq 2-3\alpha$ there exists $\delta = \delta(\alpha, \varepsilon) > 0$ such that $J_{3\alpha-2+\varepsilon}(1+x, 1) > \alpha A(1+x, 1) + (1-\alpha)H(1+x, 1)$ for $x \in (0, \delta)$. □

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