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NEWTON'S METHOD AND REGULARLY SMOOTH OPERATORS

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Abstract. A semilocal convergence analysis for Newton's method in a Banach space setting is provided in this study. Using a combination of regularly smooth and center regularly smooth conditions on the operator involved, we obtain more precise majorizing sequences than in [7]. It then follows that under the same computational cost and the same or weaker hypotheses than in [7] the following benefits are obtained: larger convergence domain; finer estimates on the distances involved, and an at least as precise information on the location of the solution of the corresponding equation.

Numerical examples are given to further validate the results obtained in this study.

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1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1}$$

where F is a Fréchet-differentiable operator defined on an open convex subset D of a Banach space X with values in a Banach space Y.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $\dot{x} = Q(x)$, for some suitable operator Q, where x is the state. Then the equilibrium states are determined by solving equation 1. Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers

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(single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative - when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of theses methods have the same recursive structure, they can be introduced and discussed in a general framework.

Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \ (n \geq 0), \ (x_0 \in D) \tag{2}$$

is undoubtedly the most popular iterative procedure for generating a sequence $\{x_n\}$ $(n \geq 0)$ approximating x^*. Here, $F'(x) \in L(X, Y)$ the space of bounded linear operators from X into Y.

There is an extensive literature on local as well as semilocal convergence results for Newton's method under various assumptions [1]–[5], [6]–[10], [12].

The hypothesis of w-smoothness:

$$\left\| F'(x) - F'(y) \right\| \leq w(\|x - y\|) \text{ for all } x, y \in D, \tag{3}$$

has been used to provide a semilocal convergence analysis for Newton's method, where $w : [0, \infty) \to [0, \infty)$ is a continuous non-decreasing function which vanishes at zero, and it is positive elsewhere [1], [4], [5], [7], [12]. In the case: $w(r) = cr$, condition (3) reduces to the common Lipschitz hypothesis whereas, when $w(r) = cr^p$ $p \in [0, 1)$ we obtain the Hölder assumption.

Recently in [3]–[5] we introduced the center w-smoothness condition:

$$\left\| F'(x) - F'(x_0) \right\| \leq w_0(\|x - x_0\|) \tag{4}$$

where w_0 is a function with the same for all $x \in D$, properties as w.

Note that condition (3) implies (4). Using weaker (4) (which is what is really needed for finding bounds on $\left\| F'(x_n)^{-1}F'(x_0) \right\|$) instead of condition (3), leads to more precise majorizing sequences, which in turn are used to provide under the same hypotheses a finer semilocal convergence analysis with the following advantages over the earlier mentioned works: larger convergence domain; finer error bounds on the distances involved, and an at least as precise information on the location of the solution x^*. Note that the above advantages are obtained under the same computational cost, since in practice finding function w requires that of w_0. In this study we show that the above advantages hold true if operator F is w-regularly smooth on D [7] (to be precised in Definition 1).

Numerical examples are also provided to further validate the results obtained in this study.

2. SEMILOCAL CONVERGENCE ANALYSIS OF NEWTON'S METHOD (??)

Let N denote the class of non-decreasing continuous functions $w : [0, \infty) \to [0, \infty)$ that are concave. That is they have convex subgraphs $\{(s, t) : s \geq 0$, and $t \leq w(s)\}$, and vanish at zero [11].

We need the definition:

Definition 1. [7] *Let $F : D \subseteq X \to Y$ be a Fréchet-differentiable operator. Denote by $\underline{h}(F)$ the quantity* $\inf \|F'(x)\|$, $x \in D$. *Given $w \in N$, we say that F is w-regularly smooth on D, if there exists an $\underline{h} \in [0, \underline{h}(F)]$ such that for all $x, y \in D$:*

$$w^{-1}(h_F(x, y) + \|F'(y) - F'(x)\|) - w^{-1}(h_F(x, y)) \leq \|y - x\|, \tag{5}$$

where,

$$h_F(x, y) = \min\{\|F'(x)\|, \|F'(y)\|\} - \underline{h}. \tag{6}$$

The operator is regularly smooth on D, if it is w-regularly smooth there for some $w \in N$.

Throughout this study w^{-1} denotes the function whose closed epigraph $cl\{(s, t) : s \geq 0$, and $t \geq w^{-1}(s)\}$ is symmetrical to closed of the subgraph of w with respect to the axis $t = s$. Due to the convexity of w^{-1}, each w-regularly smooth operator is also w-smooth but not necessarily vice versa [7]. Several properties for operators F that are w-regularly smooth can be found in [7].

It follows from condition (5) that for $x = \overline{x}$ fixed there exists a function w_0 with the same properties as w such that for all $y \in D$:

$$w_0^{-1}(h_F(\overline{x}, y) + \|F'(y) - F'(\overline{x})\|) - w^{-1}(h_F(\overline{x}, y)) \leq \|y - \overline{x}\|. \tag{7}$$

Clearly,

$$w_0(s) \leq w(s) \text{ for all } s \in [0, \infty) \tag{8}$$

holds in general, and $\frac{w(s)}{w_0(s)}$ can be arbitrarily large [3]–[5].

It is convenient for us to introduce suitable notations. The superscript t means the non-negative part of a real number: $a^+ := \max\{a, 0\}$.

Denote:

$$Z(p, q) := \{q, p - q\},$$
$$m(p, q, r) := \min\{p, (c - z(p - q, w))^+\},$$

and

$$g(p, q, r) := \int_0^r [w(m(p, q, \theta) + \theta) - w(m(p, q, \theta))]\mathrm{d}\theta$$
$$\text{for all } p \geq 0, \text{ and } r \geq 0.$$

In order for us to apply Newton's method (2) to equation (1), choose $x_0 \in D$ such that $F'(x_0)^{-1} \in L(Y, X)$, and set $F_0 := F'(x_0)^{-1}F$. Clearly, equation (1) is equivalent to the equation $F_0(x) = 0$, and the Newton iterations for F and F_0 starting at x_0 are identical.

Let $\underline{h}$ be a lower bound for $\underline{h}\ (F_0)$:

(9) $$0 \leq \underline{h} \leq \underline{h}(F_0),$$

and let $w \in N$, $w_0 \in N$ satisfy (5) and (6), respectively with F_0 instead of F and $\overline{x} = x_0$. Then F_0 is w-regularly smooth, and center-w_0-smooth on D.

Let us define constants

(10) $$k_0 = W_0^{-1}(1 - \underline{h}),$$

(11) $$k = w^{-1}(1 - \underline{h}),$$

and denote by a an upper bound on $\|F_0(x_0\|$:

(12) $$\|F_0(x_0)\| \leq a.$$

Define scalar sequences $\{\alpha_n\},\ \{\gamma_n\},\ \{\gamma_n\}\ (n \geq 0)$ as follows:

(13) $$\begin{aligned} &\alpha_0 := k,\ \gamma_0 := 1,\ \delta_0 := a, \\ &\alpha_n := (\alpha_{n-1} - \delta_{n-1})^+,\ \gamma_n := 1 - w_0(\alpha_n + t_n) + w_0(\alpha_n), \\ &\delta_n := \gamma_n^{-1} g(\alpha_{n-1}, \alpha_{n-1} - \delta_{n-1}, \delta_{n-1}), \end{aligned}$$

where,

$$t_n := \sum_{i=0}^{n-1} \delta_i,$$

function $g_{\underline{h}}(t)$ on $[0, \infty)$ by:

(14) $$g_{\underline{h}}(t) := a - t + g(k, (k-t)^+ - t, t) \text{ for all } t \geq 0,$$

and set

$$t^* = \lim_{n\to\infty} t_n.$$

The triple $(\alpha_n, \gamma_n,\, \delta_n)$ is well defined for all $n \geq 0$ provided that

(15) $$\gamma_n > 0 \text{ for all } n \geq 0.$$

Condition (15) can be replaced by

(16) $$t_n < w_0^{-1}(1) \text{ for all } n \geq 0.$$

We can now state the main semilocal convergence theorem for Newton's method (2) for operator F_0 that are w-regularly smooth on D.

THEOREM 2. *Let the operator F_0 be w-regularly smooth, and center- w_0-regularly smooth on D. Assume*
$x_0 \in D$ satisfies condition

(17) $$t_n \leq w^{-1}(1),$$

and

(18) $$\overline{U}(x_0, t^*) = \{x \in X : \|x - x_0\| \leq t^*\} \subseteq D,$$

then, sequence $\{x_n\}$ generated by Newton's method (2) *(with F_0 replacing F) is well defined, remains in $U(x_0\ ,t^*)$ for all $n \geq 0$, and converges to a solution x^* of equation $F(x) = 0$ in $\overline{U}(x_0, t^*)$.*

Moreover the following estimates hold true for all $n \geq 0$:

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \tag{19}$$

$$\|x_n - x^*\| \leq t^* - t_n, \tag{20}$$

$$\left\|F_0'(x_n)^{-1}\right\| \leq \gamma_n^{-1}, \tag{21}$$

and

$$\|F_0(x_n)\| \leq g(\alpha_{n-1}, \alpha_n - \delta_{n-1}, \delta_{n-1}). \tag{22}$$

Furthermore, if a is such that $t^* \leq k_0$, then the solution x^* is unique in $\overline{U}(x_0, g_{\underline{h},2}^{-1}(0)$, where

$g_{\underline{h},2}^{-1}$ stands for the inverse of the restriction of function $g_{\underline{h}}$ on the interval $[w^{-1}(1), \infty)$.

Proof. The proof is similar to the corresponding one given in Theorem 4.3 in [7], p. 83)]. Simply use (6) instead of (5) (used in [7]), in the derivation of the estimate (21). To avoid duplications we will only sketch the above mentioned differences in the proofs.

It is convenient to set:

$$\begin{aligned} \overline{\alpha}_n &:= w^{-1}(\left\|F_0'(x_n)\right\| - \underline{h}),\ \overline{\gamma}_n := \left\|F_0'(x_n)^{-1}\right\|^{-1}, \\ \overline{\delta}_n &:= \|x_{n+1} - x_n\|\ \ (n \geq 0). \end{aligned} \tag{23}$$

For $n = 0$, we have

$$\overline{\alpha}_0 = k = \alpha_0,\ \overline{\gamma}_0 = 1 = \gamma_0,\ \text{and } \overline{\delta}_0 \leq a = \delta_0,$$

hold true.

Assume, that for all $i = 0, 1, ..., n-1$ $(n \geq 1)$:

$$F_0'(x_i) \text{ exists so that } F_0'(x_i)^{-1} \in L(Y, X),$$

and

$$\overline{\alpha}_i \geq \alpha_i,\ \overline{\gamma}_i \geq \gamma_i,\ \overline{\delta}_i \leq \delta_i. \tag{24}$$

It follows by the induction hypothesis that

$$\|x_i - x_0\| \leq \sum_{j=0}^{i-1} \overline{\delta}_j \leq \sum_{j=0}^{i-1} \delta_j = t_i. \tag{25}$$

Hence, $F_0'(x_i)$ exists. Using (6) on $U(x_0, t^*) \subseteq D$ we obtain in turn (for $\overline{x} = x_0$) :

(26)
$$\begin{aligned}&\left\|F_0'(x_i) - F_0'(x_0)\right\| \\ &\leq w_0^{-1}(w_0^{-1}(\min\{1, \left\|F_0'(x_i)\right\|\} - \underline{h}) + \|x_i - x_0\| - \\ &\quad - \min\{1, \left\|F_0'(x_i)\right\|\} + \underline{h} \\ &= w_0(\min\{k, \overline{\alpha}_i\} + \|x_i - x_0\|) - w_0(\min\{k, \overline{\alpha}_i\})\end{aligned}$$

By Lemma 2.1 in [7], we have:

$$\overline{\alpha}_i \geq (k - \|x_i - x_0\|)^+ \geq (k - t_i)^+ = \alpha_i,$$

which together with (26) gives

(27)
$$\begin{aligned}\left\|F_0'(x_i) - F_0'(x_0)\right\| &\leq w_0(\min\{k, \alpha_i\} + \|x_i - x_0\|) - w_0(\min\{k, \alpha_i\} \\ &\leq w_0(\alpha_i + t_i) - w_0(\alpha_i).\end{aligned}$$

In view of (8), (17), (23)-(27) we obtain

(28)
$$\overline{\gamma}_i \geq 1 - w_0(\alpha_i + t_i) + w_0(\alpha_i) \geq 1 - w_0(t_i) \geq 1 - w(t_i).$$

It follows from (28), and the Banach Lemma on invertible operators [5], [9], that $F_0'(x_i)^{-1}$ exists, so that (21) is satisfied and

$$\gamma_i \leq \overline{\gamma}_i.$$

Using (2) (for F_0 replacing F) we obtain the approximation

(29)
$$\begin{aligned}x_{2+1} - x_i &= -F_0'(x_i)^{-1}[F_0(x_i) - F_0(x_{i-1}) - F_o'(x_{i-1})(x_i - x_{i-1})] \\ &= -F_0'(x_i)^{-1}\int_0^1 [F_0'(x_{i-1} + \theta(x_i - x_{i-1})) - F_0'(x_{i-1})](x_i - x_{i-1}d\theta.\end{aligned}$$

It follows by Lemma 2.2 in [7] that:

(30)
$$\begin{aligned}r(x_{i-1}, x_i) &:= \left\|\int_0^1 [F_0'(x_{i-1} + \theta(x_i - x_{i-1})) - F_0'(x_{i-1})](x_i - x_{i-1})d\theta\right\| \\ &\leq g(\overline{\alpha}_{i-1}, \overline{\alpha}_i - \overline{\delta}_{i-1}, \overline{\delta}_{i-1}) \leq g(\alpha_{i-1}, \alpha_i - \delta_{i-1}, \delta_{i-1}).\end{aligned}$$

Hence, we obtain by (21), (29) and (30) that:

(31)
$$\overline{\delta}_i \leq \gamma_i^{-1} r(x_{i-1}, x_i) g(\alpha_{i-1}, \alpha_i - \delta_{i-1}, \delta_{i-1}) =: \delta_i,$$

which completes the induction for (24). It now follows:

(32)
$$\|x_{n+i} - x_n\| \leq \sum_{j=n}^{n+i-1} \overline{\delta}_j \leq \sum_{j=n}^{n+i-1} \delta_j < \sum_{j=n}^{\infty} \delta_j = t^* - t_n$$

It also follows from (32) that sequence $\{x_n\}$ is Cauchy (since $\{t_n\}$ is a convergent sequence) in a Banach space X, and as such it converges to some

$x^* \in \overline{U}(x_0, t^*)$ (since $\overline{U}(x_0, t^*)$ is a closed set). By letting $i \to \infty$ in (30) we obtain

$$F_0(x^*) = 0 \Rightarrow F(x^*) = 0.$$

Estimate (20) follows from (32) (i.e. (19)) by using standard majorization techniques [4], [9]. Moreover, estimate (22) is simply (30) for

$$r(x_{n-1}, x_n) = \|F_0(x_n)\|.$$

The uniqueness part of the proof as identical to the corresponding one in [7] is omitted.

That completes the proof of the Theorem. □

Remark 3. In order for us to compare our Theorem 2 with the related Theorem 4.3 in [7], let us define sequences $\{\alpha_n^1\}, \{\gamma_n^1\}, \{\delta_n^1\}, \{t_n^1\}$ as $\{\alpha_n\}, \{\gamma_n\}, \{\delta_n^1\}, \{t_n^1\}$ respectively by simply setting $w_0 = w$ in (13). Clearly, if $w_0 = w$, then our Theorem 2 reduces to Theorem 4.3 in [7]. Otherwise (i.e. if (8) holds as a strict inequality), then with the exception of the convergence domain, the rest of the advantages of our approach over the corresponding ones in [7] (as already stated in the Introduction of this study) hold true.

Remark 4. The convergence domain can also be extended as follows: Define sequences $\{\alpha_n^2\}$, $\{\gamma_n^2\}$, $\{\delta_n^2\}$, $\{t_n^2\}$ as $\{\alpha_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{t_n\}$ respectively by with k_0 (given by (10)) replacing k (given by (11)) in (13). Moreover, replace condition (17) by weaker (16) (with t_n^2 replacing t_n in (16)). It then follows from the proof of Theorem 2 that with the above changes the conclusions of this theorem hold true with the exception of the uniqueness part which holds true on $\overline{U}(x_0, t_2^*)$, $t_2^* = \lim_{n\to\infty} t_n^2$. That is we arrived at:

Theorem 5. *Let the operator F_0 be w-regularly smooth, and center- w_0-regularly smooth on D. Assume: $x_0 \in D$ satisfies:*

$$t_n^2 \le w_0^{-1}(1), \tag{33}$$

$$\alpha_0 := k_0,$$

and

$$\overline{U}(x_0, t_2^*) \subseteq D,$$

then sequence $\{x_n\}$ generated by Newton's method (2) (with F_0 replacing F) is well defined, remains in $U(x_0, t_2^*)$, and converges to a solution x of equation $F(x) = 0$ in $\overline{U}(x_0, t_n^*)$.

Moreover the following estimates hold true for all $n \ge 0$:

$$\|x_{n+1} - x_n\| \le t_{n+1}^2 - t_n^2, \tag{34}$$

$$\|x_n - x^*\| \le t_2^* - t_n^2, \tag{35}$$

$$\left\|F_0'(x_n)^{-1}\right\| \le \tfrac{1}{\gamma_n^2}, \tag{36}$$

and

(37) $$\|F_0(x_n)\| \leq g(\alpha_{n-1}^2, \alpha_n^2 - \delta_{n-1}^2, \delta_{n-1}^2).$$

Furthermore the solution x^* is unique in $U(x_0, w_0^{-1}(1))$.

Proof. In view of the proof of Theorem 2, we only need to show the uniqueness part. Let y^* be a solution of equation

$$F_0(x) = 0 \text{ in } \ U(x_0, w_0^{-1}(1)).$$

Let us, denote

$$L := \int_0^1 F_0'(y^* + \theta(x^* - y^*))\mathrm{d}\theta$$

Using (6) for $\overline{x} = x_0$, $y = y^* + \theta(x^* - y^*), \theta \in [0,1]$, and F replaced by F_0, as in (27) we obtain

$$\left\|F_0'(x_0) - L\right\| < 1 - w_0(t_2^*) < 1.$$

Hence, the linear operator L is invertible on $U(x_0, w_0^{-1}(1)$. Moreover using the identity

$$0 = F(x^*) - F(y^*) = L\ (x^* - y^*),$$

we obtain $x^* = y^*$.

That completes the proof of the theorem. □

In the next result we compare majorizing sequences $\{t_n\},\ \{t_n^1\},\ \{t_n^2\}$:

PROPOSITION 6. *Under the hypotheses of Theorem 2, the conclusions of our Theorem 2.4, and Theorem 4.3 in [7] hold true.*

Moreover the following estimates hold true for all $n \geq 0$:

(38) $$0 \leq t_{n+1}^2 - t_n^2 \leq t_{n+1} - t_n \leq t_{n+1}^1 - t_n^1,$$

(39) $$0 \leq t_2^* \leq t^* \leq t_1^* - \lim_{n \Rightarrow \infty} t_n^1,$$

and

(40) $$t_2^* - t_n^2 \leq t^* - t_n \leq t_1^* - t_n.$$

Proof. The proof follows immediately by induction on $n \geq 0$, the definition of the "t" sequences and (8).

That completes the proof of the Proposition. □

REMARK 7. (a) If strict inequality holds in (8), so does in (38).

(b) If equality holds in (8), then Theorem 2.4 reduces to Theorem 4.3 in [7]. Otherwise it is an improvement with advantages as stated in the Introduction of this study (see also (38)-(40), and compare (16), (17), (33) in this case). Note also that sufficient convergence conditions other than the ones given here for the satisfaction of conditions (16), (17), (33) (or (15)), weaker (in general), have been given by us in [4], [5] for operators F that are w-smooth. Clearly, those conditions can replace (16), (17), (33) in the above results provided that operator F is w-regularly smooth.

As an application, we compare the "t" iterations in the interesting case, when

$$w_0(s) = c_0 s \text{ and } w(s) = cs. \tag{41}$$

Using (41), and the definitions of the "t" iterations, we obtain:

$$t_0 = 0,\ t_1 = a, t_{n+1} + \tfrac{f(t_n)}{2(1-c_0 t_n)},$$

$$t_0^1 = 0,\ t_1^1 = a,\ t_{n+1}^1 = t_n + \tfrac{f(t_n^1)}{2(1-ct_n^1)}, \tag{42}$$

and

$$t_0^2 = 0,\ t_2^1 = a,\ t_{n+1}^2 = t_n^2 + \tfrac{c(t_n^2 - t_{n-1}^2)^2}{2(1-c_0 t_n^2)},$$

where,

$$f(s) = \tfrac{c}{2}s^2 - s + a. \tag{43}$$

Condition (17) is satisfied provided that the famous Newton-Kantorovich hypothesis:

$$K = ca \leq \tfrac{1}{2} \tag{44}$$

holds true [3], [5], [9]. It then also follows that sequences $\{t_n\}$, $\{t_n^1\}$ given by (42), and (43) converge to t^*, t_1^* respectively with

$$t^* \leq t_1^* = \tfrac{1-\sqrt{1-2ca}}{c_0},$$

so that the other conclusions of Theorem 2 also hold true.

It was shown by us in [3] (see also [4], [5]) that finer sequence $\{t_n^2\}$ converges to t_2^* provided that

$$K_\beta = (c + \beta c_0)a \leq \beta \text{ for some } \beta \in [0, 1] \tag{45}$$

or (45),

$$\tfrac{2c_0 a}{2-\beta} \leq 1, \tag{46}$$

and

$$\tfrac{c_0 \beta^2}{2-\beta} \leq c \text{ for some } \beta \in [0, 2), \tag{47}$$

or

$$c_0 a \leq 1 - \tfrac{1}{2}\beta \text{ for } \beta \in [\beta_0, 2), \tag{48}$$

where

$$\beta_0 = \tfrac{-\beta_1 + \sqrt{\beta_1^2 + 8\beta_1}}{2},\ \beta_1 = \tfrac{c}{c_0}.$$

or

$$K_2 = c_2 a \leq \tfrac{1}{2}, \tag{49}$$

where,

$$c_2 = \tfrac{1}{2}\left(c + 4c_0 + \sqrt{c^3 + 8c_0 c}\right).$$

The simplest from conditions (45)–(49) is K_1:

(50) $$K_1 = c_1 a \leq \tfrac{1}{2},\ c_1 = \tfrac{c+c_0}{2},$$

Note also that

(51) $$K \leq \tfrac{1}{2} \Rightarrow K_1 \leq \frac{1}{2} \Rightarrow K_2 \leq \tfrac{1}{2}.$$

but not necessarily vice versa unless if $c_0 = c$.

We complete the study with three numerical examples:

EXAMPLE 8. Let $X = Y = \mathbb{R}$, $x_0 = 0$, and for given parameters d_i, $i = 0,1,2,3$, define function F by

(52) $$F(x) = d_0 + d_1 x + d_2 \sin \mathrm{e}^{d_3 x}.$$

It can easily be seen by (51) that for d_3 large, and d_2 sufficiently small $\frac{c}{c_0}$ can be arbitrarily large.

EXAMPLE 9. Let $X = Y = \mathbb{R}$, $x_0 = 1$, $b \in \left[0, \tfrac{1}{2}\right)$, and define function F by

(53) $$F(x) = x^3 - b.$$

Using (52) we have: $a = \frac{1}{3}(1-b)$, $c_0 = 3 - b$, and $c = 2(2-b)$. Note that

$$c_0 < c \text{ for all } b \in \left[0, \tfrac{1}{2}\right).$$

Condition (44) does not hold, since

$$K = \tfrac{2}{3}(1-b)(2-b) > \tfrac{1}{2} \text{ for all } b \in \left[0, \tfrac{1}{2}\right).$$

That is there is no guarantee that Newton's method (2) starting from $x_0 = 1$ converges to the solution $x^* = \sqrt[3]{b}$.

However, condition (49) holds for all $b \in \left[\frac{5-\sqrt{13}}{3}, \frac{1}{2}\right)$, since

$$K_1 = \tfrac{1}{6}(1-b)[3-b+2(2-b)] \leq \tfrac{1}{2}.$$

Note that $\frac{5-\sqrt{13}}{3} = .46481624....$ Finally condition (49) for holds for $b \in \left[.450339002, \tfrac{1}{2}\right)$.

EXAMPLE 10. Let $X = Y = C[0,1]$, be the space of real-valued continuous functions defined on the interval $[0,1]$ with norm

$$\|x\| = \max_{0 \leq s \leq 1} |x(s)|.$$

Let $d \in [0,1]$ be a given parameter. Consider the cubic integral equation

(54) $$u(s) = u^3(s) + \lambda u(s) \int_0^1 q(s,t) u(t) \mathrm{d}t + y(s) - d.$$

Here the kernel $q(s,t)$ is a continuous function of two variables defined on $[0,1]\times[0,1]$; the parameter λ is a real number called the "albedo" for scattering; $y(s)$ is a given continuous function defined on $[0,1]$ and $x(s)$ is the unknown function sought in $C[0,1]$. Equations of the form (53) arise in the theory of radiative transfer, neutron transport, and the kinetic theory of gasses [2], [6].

For simplicity, we choose $u_0(s) = y(s) = 1$ and $q(s,t) = s/(s+t)$, for all $s \in [0,1]$ and $t \in [0,t]$ with $s + t \neq 0$. If we let $D = U(u_0, 1-d)$, and define the operator f on D by

$$f(x)(s) = x^3(s) + \lambda x(s) \int_0^1 q(s,t)x(t)\mathrm{d}t + y(s) - d \tag{55}$$

for all $s \in [0,1]$, then every zero of f satisfies equation (53). We have the estimate

$$\max_{0 \leq s \leq 1} \left| \int_0^1 s/(s+t)\mathrm{d}t \right| = \ln 2. \tag{56}$$

Therefore if we set $b_0 = \left\| f'(u_0)^{-1} \right\|$, then it follows from (54) and (55) that conditions $a = b_0(|\lambda| \ln\ 2 + 1 - d)$, $c = 2b_0[|\lambda| \ln\ 2 + 3(2-d)]$ and $c_0 = b_0[2\,|\lambda| \ln\ 2 + 3(3-d)]$. Moreover, since $c_0 < c$ we get a wider choice of values λ for which our conditions (45)–(49) or (50) hold than the ones provided by (44) (used in [2], [6], [9].

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