REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 40 (2011) no. 1, pp. 14–23
ictp.acad.ro/jnaat

A VORONOVSKAYA TYPE THEOREM FOR q-SZASZ-MIRAKYAN-KANTOROVICH OPERATORS

GÜLEN BAŞCANBAZ-TUNCA[1] and AYŞEGÜL ERENÇIN[2]

Abstract. In this work, we consider a Kantorovich type generalization of q-Szasz-Mirakyan operators via Riemann type q-integral and prove a Voronovskaya type theorem by using suitable machinery of q-calculus.

MSC 2000. 41A25, 41A36.
Keywords. q-Szasz-Mirakyan-Kantorovich operator, Riemann type q-integral, Voronovskaya type theorem.

1. INTRODUCTION

In [5], Butzer introduced the Kantorovich variant of the Szasz-Mirakyan operators as

$$K_n(f;x) = n\mathrm{e}^{-nx}\sum_{k=0}^{\infty}\tfrac{(nx)^k}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\mathrm{d}t; \quad x \geq 0$$

for each $n \in \mathbb{N}$ and $f \in L^1(0,\infty)$, the space of integrable functions on unbounded interval $[0,\infty)$. The $L^p(1 < p < \infty)$ saturation and inverse theorems for the Szasz-Mirakyan-Kantorovich operators were studied by Totik in [19]. Bezier variant of the Szasz-Mirakyan-Kantorovich operators were introduced by Gupta, Vasishtha and Gupta in [8], where the rate of convergence of these operators for functions of bounded variation was measured. Moreover, approximation properties of these operators for locally bounded functions have been investigated recently by Gupta and Xiao-Ming Zeng in [9]. In [7], Duman, Özarslan and Della Vecchia constructed a modified Szasz-Mirakyan-Kantorovich operators and introduced a better error estimation and a Voronovskaya type theorem. Another interesting modification of Szasz-Mirakyan-Kantorovich operators has been carried out by Nowak and Sikorska-Nowak in [15], where the operators are defined for Denjoy-Perron integrable functions f. The authors have obtained an estimate for the rate of pointwise convergence for such operators at the Lebesgue Denjoy points of f.

[1]Ankara University, Faculty of Science, Department of Mathematics, 06100, Tandoğan, Ankara, Turkey, e-mail: `tunca@science.ankara.edu.tr`.
[2]Abant İzzet Baysal University, Faculty of Arts and Sciences, Department of Mathematics, 14280, Bolu, Turkey, e-mail: `erencina@hotmail.com`.

In 1996, Phillips' generalization of Bernstein polynomials by using q-integers [16] has brought up a new point of view to the theory of approximation by linear positive operators. There are numerous works dealing with this approach. Some are in [2], [3], [4], [6], [11], [12], [13], [20], [21] and references therein.

In this work, we consider a Kantorovich type generalization of q-Szasz-Mirakyan operators, introduced by Aral in [4] with the help of the Riemann type q-integral defined in [14] and prove a Voronovskaya type theorem. Observe that a different form of q-Szasz-Mirakyan-Kantorovich operators has been studied in [13] by Mahmudov and Gupta.

2. NOTATION AND CONSTRUCTION OF THE OPERATORS

In this section, we first mention some well-known definitions of q-calculus. Next introduce the q-Szasz-Mirakyan-Kantorovich operators via Riemann type q-integral.

Let $q > 0$ be a fixed real number and n be a nonnegative integer. The respective definitions of the q-integer $[n]_q$ and the q-factorial $[n]_q!$ are (see [1])

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q} & \text{if } q \neq 1 \\ n & \text{if } q = 1 \end{cases},$$

$$[n]_q! = \begin{cases} [1]_q[2]_q \cdots [n]_q & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases}.$$

The two q-analogues of the exponential function e^x are given by

$$e_q(x) = \sum_{n=0}^{\infty} \tfrac{1}{[n]_q!} x^n = \tfrac{1}{((1-q)x;q)_\infty}; \quad |x| < \tfrac{1}{1-q}, \quad |q| < 1$$

and

$$E_q(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{[n]_q!} x^n = (-(1-q)x;q)_\infty; \quad x \in \mathbb{R}, \quad |q| < 1,$$

where $(x;q)_\infty = \prod_{k=1}^{\infty}(1 - xq^{k-1})$ (see [10]). We know that

$$e_q(x)E_q(x) = 1$$

and

$$\lim_{q\to 1^-} e_q(x) = \lim_{q\to 1^-} E_q(x) = \mathrm{e}^x.$$

In [4], Aral defined the q-Szasz-Mirakyan operators as follows:

$$S_n^q(f;x) = E_q\left(-[n]_q \tfrac{x}{b_n}\right) \sum_{k=0}^{\infty} f\left(\tfrac{[k]_q b_n}{[n]_q}\right) \tfrac{([n]_q x)^k}{[k]_q!(b_n)^k}, \quad x \in \left[0, \tfrac{b_n}{(1-q)[n]_q}\right) \tag{2.1}$$

where $n \in \mathbb{N}$, $0 < q < 1$, $f \in C[0,\infty)$ and (b_n) is an increasing sequence of positive numbers such that $\lim_{n\to\infty} b_n = \infty$. He proved a Voronovskaya type theorem using q-derivatives and also obtained convergence properties of these operators and their derivatives. Note that the operators defined by (2.1) are the q-extension of the Szasz-Chlodovsky operators introduced in [18] by Stypinski. From [4] we have

(2.2)
$$S_n^q(e_0;x)=1$$
$$S_n^q(e_1;x)=x$$
$$S_n^q(e_2;x)=qx^2+\frac{b_n}{[n]_q}x$$
$$S_n^q(e_3;x)=q^3x^3+(q^2+2q)\frac{b_n}{[n]_q}x^2+\left(\frac{b_n}{[n]_q}\right)^2x$$
$$S_n^q(e_4;x)=q^6x^4+(q^5+2q^4+3q^3)\frac{b_n}{[n]_q}x^3+(q^3+3q^2+3q)\left(\frac{b_n}{[n]_q}\right)^2x^2+\left(\frac{b_n}{[n]_q}\right)^3x,$$

where $e_m(t):=t^m$, $m=0,1,2,3,4$.

Now assume that $0<a<b$, $0<q<1$ and f is a real-valued function. The q-Jackson integral of f over the interval $[0,b]$ and over a general interval $[a,b]$ are defined by (see[10])

$$\int_0^b f(x)\mathrm{d}_qx=(1-q)b\sum_{j=0}^{\infty}f\left(bq^j\right)q^j$$

and

$$\int_a^b f(x)\mathrm{d}_qx=\int_0^b f(x)\mathrm{d}_qx-\int_0^a f(x)\mathrm{d}_qx,$$

respectively, provided the series converge. It is easily seen that the inequality $f(x)\geq 0$ for all $x\in[a,b]$ does not ensure that $\int_a^b f(x)\mathrm{d}_qx\geq 0$. This fact may not be convenient in order to guarantee the positivity of our operator. On the other hand, since the q-Jackson integral contains two infinite sums, in obtaining the q-analogues of some well-known integral inequalities which are used to estimate the order of approximation of linear positive operators involving q-Jackson integral, some problems are encountered. Because of these circumstances, we consider the Riemann type q-integral introduced in [14]

$$R_q(f;a,b)=\int_a^b f(x)\mathrm{d}_q^Rx=(1-q)(b-a)\sum_{j=0}^{\infty}f\left(a+(b-a)q^j\right)q^j \tag{2.3}$$

to construct a Kantorovich variant of the operator defined by (2.1).

With this motivation, we now introduce the Kantorovich type generalization of the q-Szasz-Mirakyan operators as follows:

$$K_{n,q}(f;x)=E_q\left(-[n]_q\frac{x}{b_n}\right)\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{[k]_q!(b_n)^k}\frac{[n]_q}{q^kb_n}\int_{\frac{[k]_qb_n}{[n]_q}}^{\frac{[k+1]_qb_n}{[n]_q}}f(t)\mathrm{d}_q^Rt, \tag{2.4}$$

where $x \in \big[0, \frac{b_n}{(1-q)[n]_q}\big)$, $n \in \mathbb{N}$, $0 < q < 1$, f is Riemann type q-integrable on $\big[\frac{[k]_q b_n}{[n]_q}, \frac{[k+1]_q b_n}{[n]_q}\big]$ and (b_n) is an increasing sequence of positive numbers having the property $\lim\limits_{n\to\infty} b_n = \infty$.

3. AUXILIARY RESULTS

In this section, we give some lemmas which will be used to prove our main result.

LEMMA 1. *Let m be a nonnegative integer. Then we have*

$$I_{n,k}(e_m) := \int_{\frac{[k]_q b_n}{[n]_q}}^{\frac{[k+1]_q b_n}{[n]_q}} t^m \mathrm{d}_q^R t = \frac{q^k b_n}{[n]_q} \sum_{l=0}^{m} \binom{m}{l} \left(\frac{[k]_q b_n}{[n]_q}\right)^{m-l} C_{m,l}(q, b_n)$$

where $e_m(t) := t^m$ and

$$C_{m,l}(q, b_n) = \left(\frac{b_n}{[n]_q}\right)^{l} \sum_{s=0}^{m-l} \binom{m-l}{s} \frac{(q-1)^s}{[l+s+1]_q}. \tag{3.1}$$

Proof. From (2.3) and binomial formula it follows that

$$\begin{aligned}
I_{n,k}(e_m) &= (1-q)\frac{q^k b_n}{[n]_q} \sum_{j=0}^{\infty} \left(\frac{[k]_q b_n}{[n]_q} + \left(\frac{b_n}{[n]_q} + (q-1)\frac{[k]_q b_n}{[n]_q}\right) q^j\right)^m q^j \\
&= (1-q)\frac{q^k b_n}{[n]_q} \sum_{i=0}^{m} \sum_{j=0}^{\infty} (q^j)^{i+1} \binom{m}{i} \left(\frac{b_n}{[n]_q} + (q-1)\frac{[k]_q b_n}{[n]_q}\right)^i \left(\frac{[k]_q b_n}{[n]_q}\right)^{m-i} \\
&= \frac{q^k b_n}{[n]_q} \sum_{i=0}^{m} \binom{m}{i} \frac{1}{[i+1]_q} \left(\frac{b_n}{[n]_q} + (q-1)\frac{[k]_q b_n}{[n]_q}\right)^i \left(\frac{[k]_q b_n}{[n]_q}\right)^{m-i} \\
&= \frac{q^k b_n}{[n]_q} \sum_{i=0}^{m} \binom{m}{i} \frac{1}{[i+1]_q} \left(\frac{[k]_q b_n}{[n]_q}\right)^{m-i} \sum_{l=0}^{i} \binom{i}{l} \left(\frac{b_n}{[n]_q}\right)^l \left((q-1)\frac{[k]_q b_n}{[n]_q}\right)^{i-l} \\
&= \frac{q^k b_n}{[n]_q} \sum_{i=0}^{m} \binom{m}{i} \frac{1}{[i+1]_q} \sum_{l=0}^{i} \binom{i}{l} \left(\frac{[k]_q b_n}{[n]_q}\right)^{m-l} \left(\frac{b_n}{[n]_q}\right)^l (q-1)^{i-l}.
\end{aligned}$$

Interchanging the order of summations over i and l and then taking $s = i - l$ in the obtained result, one may write

$$\begin{aligned}
I_{n,k}(e_m) &= \frac{q^k b_n}{[n]_q} \sum_{l=0}^{m} \sum_{i=l}^{m} \binom{m}{i}\binom{i}{l} \left(\frac{[k]_q b_n}{[n]_q}\right)^{m-l} \left(\frac{b_n}{[n]_q}\right)^l \frac{(q-1)^{i-l}}{[i+1]_q} \\
&= \frac{q^k b_n}{[n]_q} \sum_{l=0}^{m} \sum_{i=l}^{m} \frac{m!}{(m-i)!} \frac{1}{l!(i-l)!} \left(\frac{[k]_q b_n}{[n]_q}\right)^{m-l} \left(\frac{b_n}{[n]_q}\right)^l \frac{(q-1)^{i-l}}{[i+1]_q} \\
&= \frac{q^k b_n}{[n]_q} \sum_{l=0}^{m} \sum_{s=0}^{m-l} \frac{m!}{l!(m-l)!} \frac{(m-l)!}{s!(m-l-s)!} \left(\frac{[k]_q b_n}{[n]_q}\right)^{m-l} \left(\frac{b_n}{[n]_q}\right)^l \frac{(q-1)^s}{[l+s+1]_q} =
\end{aligned}$$

$$= \frac{q^k b_n}{[n]_q} \sum_{l=0}^{m} \binom{m}{l} \left(\frac{[k]_q b_n}{[n]_q}\right)^{m-l} \left(\frac{b_n}{[n]_q}\right)^{l} \sum_{s=0}^{m-l} \binom{m-l}{s} \frac{(q-1)^s}{[l+s+1]_q}$$
$$= \frac{q^k b_n}{[n]_q} \sum_{l=0}^{m} \binom{m}{l} \left(\frac{[k]_q b_n}{[n]_q}\right)^{m-l} C_{m,l}(q, b_n),$$

where each $C_{m,l}(q, b_n)$ is given by (3.1). Thus the proof is completed. □

Now, by means of the Lemma 1 we can evaluate $K_{n,q}(e_m; x)$.

LEMMA 2. *Let m be a nonnegative integer. Then for the operator $K_{n,q}(f; x)$ defined by* (2.4)*, we have*

$$K_{n,q}(e_m; x) = \sum_{l=0}^{m} \binom{m}{l} C_{m,l}(q, b_n) S_n^q(e_{m-l}; x),$$

where $C_{m,l}(q, b_n)$ and S_n^q are given by (3.1) *and* (2.1)*, respectively.*

Proof. Taking into account the definitions of $K_{n,q}$ and S_n^q we immediately obtain that

$$K_{n,q}(e_m; x) = E_q\left(-[n]_q \frac{x}{b_n}\right) \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{[k]_q!(b_n)^k} \frac{[n]_q}{q^k b_n} I_{n,k}(e_m)$$
$$= \sum_{l=0}^{m} \binom{m}{l} C_{m,l}(q, b_n) \left\{ E_q\left(-[n]_q \frac{x}{b_n}\right) \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{[k]_q!(b_n)^k} \left(\frac{[k]_q b_n}{[n]_q}\right)^{m-l} \right\}$$
$$= \sum_{l=0}^{m} \binom{m}{l} C_{m,l}(q, b_n) S_n^q(e_{m-l}; x).$$

□

COROLLARY 1. *For the operator $K_{n,q}(f; x)$ defined by* (2.4) *we have*

(3.2) $$K_{n,q}(e_0; x) = C_{0,0}(q, b_n)$$

(3.3) $$K_{n,q}(e_1; x) = C_{1,0}(q, b_n)x + C_{1,1}(q, b_n),$$

(3.4)
$$K_{n,q}(e_2; x) = C_{2,0}(q, b_n)qx^2 + \left\{\frac{b_n}{[n]_q} C_{2,0}(q, b_n) + 2C_{2,1}(q, b_n)\right\} x + C_{2,2}(q, b_n)$$

(3.5)
$$K_{n,q}(e_3; x) = C_{3,0}(q, b_n)q^3x^3 + \left\{(q^2 + 2q)\frac{b_n}{[n]_q} C_{3,0}(q, b_n) + 3qC_{3,1}(q, b_n)\right\} x^2$$
$$+ \left\{\left(\frac{b_n}{[n]_q}\right)^2 C_{3,0}(q, b_n) + 3\frac{b_n}{[n]_q} C_{3,1}(q, b_n) + 3C_{3,2}(q, b_n)\right\} x$$
$$+ C_{3,3}(q, b_n)$$

and

(3.6)

$$\begin{aligned}K_{n,q}(e_4;x) =&C_{4,0}(q,b_n)q^6x^4 + \Big\{(q^5+2q^4+3q^3)\tfrac{b_n}{[n]_q}C_{4,0}(q,b_n)\\ &+ 4q^3C_{4,1}(q,b_n)\Big\}x^3 + \Big\{(q^3+3q^2+3q)\left(\tfrac{b_n}{[n]_q}\right)^2C_{4,0}(q,b_n)\\ &+4(q^2+2q)\tfrac{b_n}{[n]_q}C_{4,1}(q,b_n)+6C_{4,2}(q,b_n)\Big\}x^2\\ &+\Big\{\left(\tfrac{b_n}{[n]_q}\right)^3C_{4,0}(q,b_n)+4\left(\tfrac{b_n}{[n]_q}\right)^2C_{4,1}(q,b_n)\\ &+6\tfrac{b_n}{[n]_q}C_{4,2}(q,b_n)+4C_{4,3}(q,b_n)\Big\}x+C_{4,4}(q,b_n).\end{aligned}$$

In the equalities $(3.2)-(3.6)$*, the coefficients* $C_{m,l}(q,b_n)$ *for* $m=0,1,2,3,4$*,* $l=0,1,2,3,4$ *are listed below:*

$$\begin{aligned}
C_{0,0}(q,b_n) &= 1,\\
C_{1,0}(q,b_n) &= 1+\tfrac{q-1}{[2]_q},\\
C_{1,1}(q,b_n) &= \tfrac{1}{[2]_q}\tfrac{b_n}{[n]_q},\\
C_{2,0}(q,b_n) &= 1+\tfrac{2(q-1)}{[2]_q}+\tfrac{(q-1)^2}{[3]_q},\\
C_{2,1}(q,b_n) &= \left(\tfrac{1}{[2]_q}+\tfrac{q-1}{[3]_q}\right)\tfrac{b_n}{[n]_q},\\
C_{2,2}(q,b_n) &= \tfrac{1}{[3]_q}\left(\tfrac{b_n}{[n]_q}\right)^2,\\
C_{3,0}(q,b_n) &= 1+\tfrac{3(q-1)}{[2]_q}+\tfrac{3(q-1)^2}{[3]_q}+\tfrac{(q-1)^3}{[4]_q},\\
C_{3,1}(q,b_n) &= \left(\tfrac{1}{[2]_q}+\tfrac{2(q-1)}{[3]_q}+\tfrac{(q-1)^2}{[4]_q}\right)\tfrac{b_n}{[n]_q},\\
C_{3,2}(q,b_n) &= \left(\tfrac{1}{[3]_q}+\tfrac{q-1}{[4]_q}\right)\left(\tfrac{b_n}{[n]_q}\right)^2,\\
C_{3,3}(q,b_n) &= \tfrac{1}{[4]_q}\left(\tfrac{b_n}{[n]_q}\right)^3,\\
C_{4,0}(q,b_n) &= 1+\tfrac{4(q-1)}{[2]_q}+\tfrac{6(q-1)^2}{[3]_q}+\tfrac{4(q-1)^3}{[4]_q}+\tfrac{(q-1)^4}{[5]_q},\\
C_{4,1}(q,b_n) &= \left(\tfrac{1}{[2]_q}+\tfrac{3(q-1)}{[3]_q}+\tfrac{3(q-1)^2}{[4]_q}+\tfrac{(q-1)^3}{[5]_q}\right)\tfrac{b_n}{[n]_q},\\
C_{4,2}(q,b_n) &= \left(\tfrac{1}{[3]_q}+\tfrac{2(q-1)}{[4]_q}+\tfrac{(q-1)^2}{[5]_q}\right)\left(\tfrac{b_n}{[n]_q}\right)^2,\\
C_{4,3}(q,b_n) &= \left(\tfrac{1}{[4]_q}+\tfrac{q-1}{[5]_q}\right)\left(\tfrac{b_n}{[n]_q}\right)^3,\\
C_{4,4}(q,b_n) &= \tfrac{1}{[5]_q}\left(\tfrac{b_n}{[n]_q}\right)^4.
\end{aligned}$$

Using Lemma 2 and (2.2), it can be proved by direct calculation. So we omit it.

In the light of Corollary 1 and the linearity of $K_{n,q}$, we now introduce the following Lemma without proof.

Lemma 3. *For the operator $K_{n,q}(f;x)$ defined by (2.4), we have*

$$K_{n,q}(\varphi_x;x) = \{C_{1,0}(q,b_n)-1\}\,x + C_{1,1}(q,b_n) \tag{3.7}$$

(3.8)

$$K_{n,q}(\varphi_x^2;x) = \{qC_{2,0}(q,b_n)-2C_{1,0}(q,b_n)+1\}\,x^2 + \left\{\tfrac{b_n}{[n]_q}C_{2,0}(q,b_n)+2C_{2,1}(q,b_n)-2C_{1,1}(q,b_n)\right\}x + C_{2,2}(q,b_n)$$

and

(3.9)

$$\begin{aligned}
K_{n,q}(\varphi_x^4;x) &= \\
&= \Big\{q^6C_{4,0}(q,b_n)-4q^3C_{3,0}(q,b_n)+6qC_{2,0}(q,b_n)-4C_{1,0}(q,b_n)+1\Big\}x^4 \\
&+ \Big\{(q^5+2q^4+3q^3)\tfrac{b_n}{[n]_q}C_{4,0}(q,b_n)+4q^3C_{4,1}(q,b_n)-4(q^2+2q)\tfrac{b_n}{[n]_q}C_{3,0}(q,b_n) \\
&\quad -12qC_{3,1}(q,b_n)+6\tfrac{b_n}{[n]_q}C_{2,0}(q,b_n)+12C_{2,1}(q,b_n)-4C_{1,1}(q,b_n)\Big\}x^3 \\
&+ \Big\{(q^3+3q^2+3q)\left(\tfrac{b_n}{[n]_q}\right)^2C_{4,0}(q,b_n)+4(q^2+2q)\tfrac{b_n}{[n]_q}C_{4,1}(q,b_n) \\
&\quad +6qC_{4,2}(q,b_n)-4\left(\tfrac{b_n}{[n]_q}\right)^2C_{3,0}(q,b_n) \\
&\quad -12\tfrac{b_n}{[n]_q}C_{3,1}(q,b_n)-12C_{3,2}(q,b_n)+6C_{2,2}(q,b_n)\Big\}x^2 \\
&+ \Big\{\left(\tfrac{b_n}{[n]_q}\right)^3C_{4,0}(q,b_n)+4\left(\tfrac{b_n}{[n]_q}\right)^2C_{4,1}(q,b_n)+6\tfrac{b_n}{[n]_q}C_{4,2}(q,b_n) \\
&\quad +4C_{4,3}(q,b_n)-4C_{3,3}(q,b_n)\Big\}x+C_{4,4}(q,b_n),
\end{aligned}$$

where $\varphi_x := e_1 - x$ and the coefficients $C_{m,l}(q,b_n)$ for $m = 0,1,2,3,4$, $l = 0,1,2,3,4$ are given as in Corollary 1.

Now, with the help of the Lemma 3, we can evaluate the following limits which will be essential in the proof of the main theorem.

Lemma 4. *Let (q_n) be a sequence in $(0,1)$ such that $\lim_{n\to\infty} q_n = 1$ and $\lim_{n\to\infty} \frac{b_n}{[n]_{q_n}} = 0$. Assume also that there is some $a > 0$ and some $n_0 \in \mathbb{N}$ such that*

$$a \le \tfrac{b_n}{(1-q_n)[n]_{q_n}} \quad \text{for all} \quad n \ge n_0.$$

Then for every $x \in [0, a)$ we have

$$\lim_{n\to\infty} \tfrac{[n]_{q_n}}{b_n} K_{n,q_n}(\varphi_x; x) = \tfrac{1}{2} \tag{3.10}$$

$$\lim_{n\to\infty} \tfrac{[n]_{q_n}}{b_n} K_{n,q_n}(\varphi_x^2; x) = x \tag{3.11}$$

and

$$\lim_{n\to\infty} \tfrac{[n]_{q_n}}{b_n} K_{n,q_n}(\varphi_x^4; x) = 0. \tag{3.12}$$

Proof. Using the fact $\lim\limits_{n\to\infty} b_n = \infty$ from (3.7) it readily follows that

$$\begin{aligned}
\lim_{n\to\infty} \tfrac{[n]_{q_n}}{b_n} K_{n,q_n}(\varphi_x; x) &= \lim_{n\to\infty} \tfrac{[n]_{q_n}}{b_n} \left\{ \left(C_{1,0}(q_n, b_n) - 1\right) x + C_{1,1}(q_n, b_n) \right\} \\
&= \lim_{n\to\infty} \left(\tfrac{(q_n-1)[n]_{q_n}}{[2]_{q_n} b_n} x + \tfrac{1}{[2]_{q_n}} \right) \\
&= \lim_{n\to\infty} \left(\tfrac{q_n^n-1}{[2]_{q_n} b_n} x + \tfrac{1}{[2]_{q_n}} \right) \\
&= \tfrac{1}{2}.
\end{aligned}$$

Now from (3.8), we can write

$$\begin{aligned}
K_{n,q_n}(\varphi_x^2; x) &= \left\{ qC_{2,0}(q_n, b_n) - 2C_{1,0}(q_n, b_n) + 1 \right\} x^2 \\
&\quad + \left\{ \tfrac{b_n}{[n]_q} C_{2,0}(q_n, b_n) + 2C_{2,1}(q_n, b_n) - 2C_{1,1}(q_n, b_n) \right\} x + C_{2,2}(q_n, b_n) \\
&= \left\{ (q_n - 1) + \tfrac{2(q_n-1)^2}{[2]_{q_n}} + \tfrac{q_n(q_n-1)^2}{[3]_{q_n}} \right\} x^2 + \left(1 + \tfrac{2(q_n-1)}{[2]_{q_n}} + \tfrac{q_n^2-1}{[3]_{q_n}} \right) \tfrac{b_n}{[n]_{q_n}} x \\
&\quad + \tfrac{1}{[3]_{q_n}} \left(\tfrac{b_n}{[n]_{q_n}} \right)^2
\end{aligned}$$

and

$$\begin{aligned}
\tfrac{[n]_{q_n}}{b_n} K_{n,q_n}(\varphi_x^2; x) &= \left\{ (q_n - 1) + \tfrac{2(q_n-1)^2}{[2]_{q_n}} + \tfrac{q_n(q_n-1)^2}{[3]_{q_n}} \right\} \tfrac{[n]_{q_n}}{b_n} x^2 \\
&\quad + \left(1 + \tfrac{2(q_n-1)}{[2]_{q_n}} + \tfrac{q_n^2-1}{[3]_{q_n}} \right) x + \tfrac{1}{[3]_{q_n}} \tfrac{b_n}{[n]_{q_n}} \\
&= \left\{ (1 + \tfrac{2(q_n-1)}{[2]_{q_n}} + \tfrac{q_n(q_n-1)}{[3]_{q_n}} \right\} \tfrac{q_n^n-1}{b_n} x^2 + \left(1 + \tfrac{2(q_n-1)}{[2]_{q_n}} + \tfrac{q_n^2-1}{[3]_{q_n}} \right) x \\
&\quad + \tfrac{1}{[3]_{q_n}} \tfrac{b_n}{[n]_{q_n}}.
\end{aligned}$$

Since $\lim\limits_{n\to\infty} q_n = 1$, $\lim\limits_{n\to\infty} b_n = \infty$ and $\lim\limits_{n\to\infty} \frac{b_n}{[n]_{q_n}} = 0$ this gives

$$\lim_{n\to\infty} \tfrac{[n]_{q_n}}{b_n} K_{n,q_n}(\varphi_x^2; x) = x.$$

Similarly it can be shown that (3.12) holds. □

4. MAIN RESULT

Now, we can establish a Voronovskaya type result for the operator $K_{n,q}(f;x)$ defined by (2.4).

THEOREM 1. *Let* (q_n) *be a sequence in* $(0,1)$ *such that* $\lim_{n\to\infty} q_n = 1$ *and* $\lim_{n\to\infty} \frac{b_n}{[n]_{q_n}} = 0$. *Assume also that there is some* $a > 0$ *and some* $n_0 \in \mathbb{N}$ *such that*

$$a \leq \frac{b_n}{(1-q_n)[n]_{q_n}} \quad \text{for all} \quad n \geq n_0.$$

Then for every $x \in [0,a)$ *and every function* $f \in C_B[0,\infty)$, *the space of continuous and bounded functions on* $[0,\infty)$, *which is twice differentiable at* x *one has*

$$\lim_{n\to\infty} \frac{[n]_{q_n}}{b_n} \left(K_{n,q_n}(f;x) - f(x)\right) = \frac{f'(x)+xf''(x)}{2}.$$

Proof. Follows by Lemma 4 and Theorem 1 in [17]. □

REFERENCES

[1] ANDREWS, G.E., ASKEY, R. and ROY, R., *Special Functions*, Cambridge University Press, 1999.

[2] ARAL, A. and GUPTA, V., *The q-derivative and application to q-Szász-Mirakyan operators*, Calcolo, **43 (3)**, pp. 151–170, 2006.

[3] ARAL, A. and DOĞRU, O., *Bleimann, Butzer and Hahn operators based on the q-integers*, J. Inequal. Appl., Art. ID 79410, 12pp, 2007.

[4] ARAL, A., *A generalization of Szász-Mirakyan operators based on q-integers*, Math. Comput. Modelling., **47**, pp. 1052–1062, 2008.

[5] BUTZER, P.L., *On the extensions of Bernstein polynomials to the infinite interval*, Proc. Amer. Soc., **5**, pp. 547–553, 1954.

[6] DALMANOĞLU, Ö. and DOĞRU, O., *Statistical approximation properties of Kantorovich type q-MKZ operators*, Creat. Math. Inform., **19 (1)**, pp. 15–24, 2010.

[7] DUMAN, O., ÖZARSLAN, M.A. and DELLA VECCHIA, B., *Modified q-Szász-Mirakjan-Kantorovich operators preserving linear functions*, Turkish J. Math., **33 (2)**, pp. 151–158, 2009.

[8] GUPTA, V., VASISHTHA, V. and GUPTA, M.K., *Rate of convergence of the Szasz-Kantorovich-Bezier operators for bounded variation functions*, Publ. Inst. Math. (Beograd) (N.S), **72 (86)**, pp. 137–143, 2002.

[9] GUPTA, V. and ZENG, XIAO-MING, *Approximation by Bézier variant of the Szász-Kantorovich operators in case* $\alpha < 1$, Georgian Math. J., **17(2)**, pp. 253–260, 2010.

[10] KAC, V.G. and CHEUNG, P., *Quantum Calculus*, Universitext, Springer-Verlag, New York, 2002.

[11] KARSLI, H. and GUPTA, V., *Some approximation properties of q-Chlodowsky operators*, Appl. Math. Comput., **195**, pp. 220–229, 2008.

[12] MAHMUDOV, N.I., *On q-parametric Szász-Mirakjan operators*, Mediterr. J. Math., **7(3)**, pp. 297–311, 2010.

[13] MAHMUDOV, N.I. and GUPTA, V., *On certain q-analogue of Szász Kantorovich operators*, Journal of Applied Mathematics and Computing, DOI: 10.1007/s12190-010-0441-4.

[14] MARINKOVIĆ, S., RAJKOVIĆ, P. and STANKOVIĆ, M., *The inequalities for some types of q-integrals*, Comput. Math. Appl., **56**, pp. 2490–2498, 2008.

[15] Nowak, G. and Sikorska-Nowak, A., *Some approximation properties for modified Szasz-Mirakyan-Kantorovich operators*, Rev. Anal. Numér. Théor. Approx., **38(1)**, pp. 73–82, 2009.

[16] Phillips, G.M., *Bernstein polynomials based on the q-integers*, Annals of Num. Math., **4**, pp. 511–518, 1997.

[17] Sikkema, P.C., *On some linear positive operators*, Indag. Math., **32**, pp. 327–337, 1970.

[18] Stypinski, Z., *Theorem of Voronovskaya for Szász-Chlodovsky operators*, Funct. Approximatio Comment. Math., **1**, pp 133–137, 1974.

[19] Totik, V., *Approximation by Szasz-Mirakjan-Kantorovich operators in* $L^p(p > 1)$, Annal. Math., **9(2)**, pp. 147–167, 1983.

[20] Trif, T., *Meyer-König and Zeller operators based on the q-integers*, Rev. Anal. Numér. Théory Approx., **29(2)**, pp. 221–229, 2000.

[21] Wang, H., *Properties of convergence for the q-Meyer-König and Zeller operators*, J. Math. Anal. Appl., **335**, pp. 1360–1373, 2007.

Received by the editors: January 10, 2011.