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TRAPEZOIDAL OPERATOR PRESERVING THE EXPECTED INTERVAL AND THE SUPPORT OF FUZZY NUMBERS

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Abstract. The problem to find the trapezoidal fuzzy number which preserves the expected interval and the support of a given fuzzy number is discussed. Properties of this new trapezoidal approximation operator are studied.

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1. INTRODUCTION

In [12] the trapezoidal approximation of a fuzzy number is treated as a reasonable compromise between two opposite tendencies: to lose too much information and to introduce too sophisticated form of approximation from the point of view of computation. Many approximation methods of fuzzy numbers with trapezoidal fuzzy numbers were proposed in last years (see [2], [3], [4], [12], [13], [14], [18], [20]). In each case the authors attached to a fuzzy number a trapezoidal fuzzy number by preserving some parameters and/or minimizing the distance between them.

In this paper we propose a trapezoidal approximation operator which preserves the expected interval and the support of a given fuzzy number (in Section 3). We conclude that the approximation is computationally inexpensive and it is not possible for any fuzzy number. Following the list of criteria in [12], in Section 4 we examine important properties of this new trapezoidal approximation operator: translation invariance, linearity, identity, expected value invariance, order invariance with respect different preference relations, uncertainty invariance, correlation invariance, monotonicity and continuity.

2. PRELIMINARIES

A fuzzy number A is a fuzzy subset of the real line $\mathbb{R}$ with the membership function μ_A which is (see [9]) normal, fuzzy convex, upper semicontinuous, supp A is bounded, where the support of A, denoted by supp A, is the closure of the set $\{x \in X : \mu_A(x) > 0\}$.

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The α-cut, $\alpha \in (0,1]$ of a fuzzy number A is a crisp set defined as:

$$A_\alpha = \{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}.$$

Every α-cut, $\alpha \in (0,1]$ of a fuzzy number A is a closed interval $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, where

$$A_L(\alpha) = \inf\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\},$$
$$A_U(\alpha) = \sup\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}.$$

We denote

$$A_0 = [A_L(0), A_U(0)] = supp(A),$$

where the support of A is defined by $supp(A) = cl\{x \in \mathbb{R} : \mu_A(x) > 0\}$ and cl is the closure operator. We denote by $F(\mathbb{R})$ the set of fuzzy numbers.

The expected interval $EI(A)$ of $A \in F(\mathbb{R})$, $A_\alpha = [A_L(\alpha), A_U(\alpha)]$ is defined by (see [10], [15])

$$EI(A) = [E_*(A), E^*(A)] = \left[\int_0^1 A_L(\alpha)\,\mathrm{d}\alpha, \int_0^1 A_U(\alpha)\,\mathrm{d}\alpha\right]$$

and the expected value by (see [15])

$$EV(A) = \tfrac{E_*(A)+E^*(A)}{2}.$$

The core of a fuzzy number A is introduced by (see [1]):

$$core(A) = A_1 = [A_L(1), A_U(1)].$$

A trapezoidal fuzzy number T is characterized by four real numbers $t_1 \leqslant t_2 \leqslant t_3 \leqslant t_4$. It is denoted by $T = (t_1, t_2, t_3, t_4)$ and has the parametric representation $[T_L(\alpha), T_U(\alpha)]$, where

$$T_L(\alpha) = t_1 + (t_2 - t_1)\alpha,$$

$$T_U(\alpha) = t_4 + (t_3 - t_4)\alpha, \alpha \in [0,1]$$

and the expected interval

$$EI(T) = \left[\tfrac{t_1+t_2}{2}, \tfrac{t_3+t_4}{2}\right].$$

We denote by $F^T(\mathbb{R})$ the set of trapezoidal fuzzy numbers.

Let $A, B \in F(\mathbb{R})$,

$$A_\alpha = [A_L(\alpha), A_U(\alpha)],$$
$$B_\alpha = [B_L(\alpha), B_U(\alpha)], \alpha \in [0,1].$$

The quantity

$$D(A,B) = \sqrt{\textstyle\int_0^1 (A_L(\alpha) - B_L(\alpha))^2\,\mathrm{d}\alpha + \int_0^1 (A_U(\alpha) - B_U(\alpha))^2\,\mathrm{d}\alpha} \tag{2.1}$$

gives a distance between A and B (see, e.g., [11]). We consider the sum $A+B$ and the scalar multiplication $\lambda \cdot A$ by (see [8])

$$(A+B)_\alpha = A_\alpha + B_\alpha = [A_L(\alpha) + B_L(\alpha), A_U(\alpha) + B_U(\alpha)], \tag{2.2}$$

and

$$(\lambda \cdot A)_\alpha = \lambda A_\alpha = \begin{cases} [\lambda A_L(\alpha), \lambda A_U(\alpha)], & \text{if } \lambda \geq 0, \\ [\lambda A_U(\alpha), \lambda A_L(\alpha)], & \text{if } \lambda < 0, \end{cases} \tag{2.3}$$

respectively, for every $\alpha \in [0,1]$. In the case of the trapezoidal fuzzy numbers $T = (t_1, t_2, t_3, t_4)$ and $S = (s_1, s_2, s_3, s_4)$ we obtain

$$T + S = (t_1 + s_1, t_2 + s_2, t_3 + s_3, t_4 + s_4), \tag{2.4}$$

respectively

$$\lambda \cdot T = \begin{cases} (\lambda t_1, \lambda t_2, \lambda t_3, \lambda t_4), & \text{if } \lambda \geq 0, \\ (\lambda t_4, \lambda t_3, \lambda t_2, \lambda t_1), & \text{if } \lambda < 0. \end{cases} \tag{2.5}$$

Another kind of fuzzy number (see [5]) is defined by

$$\mu_A(x) = \begin{cases} \left(\frac{x-a}{b-a}\right)^n, & \text{if } x \in [a,b] \\ 1, & \text{if } x \in [b,c] \\ \left(\frac{d-x}{d-c}\right)^n, & \text{if } x \in [c,d] \\ 0, & \text{else,} \end{cases}$$

where $n > 0$, and denoted by $A = (a,b,c,d)_n$. We have

$$A_L(\alpha) = a + (b-a)\sqrt[n]{\alpha}$$
$$A_U(\alpha) = d - (d-c)\sqrt[n]{\alpha}, \alpha \in [0,1].$$

3. MAIN RESULT AND EXAMPLES

For a fuzzy number A, $A_\alpha = [A_L(\alpha), A_U(\alpha)], \alpha \in [0,1]$, the problem is to find the trapezoidal fuzzy number, $T(A) = (t_1(A), t_2(A), t_3(A), t_4(A)) = (t_1, t_2, t_3, t_4)$, which preserves the expected interval and the support of A, that is

$$\left[\int_0^1 A_L(\alpha)\,\mathrm{d}\alpha, \int_0^1 A_U(\alpha)\,\mathrm{d}\alpha\right] = \left[\tfrac{t_1+t_2}{2}, \tfrac{t_3+t_4}{2}\right] \tag{3.1}$$

and

$$\operatorname{supp}(A) = \operatorname{supp}(T(A)). \tag{3.2}$$

Let us denote

$$F_{ES}(\mathbb{R}) = \left\{ A \in F(\mathbb{R}) : 2\int_0^1 [A_U(\alpha) - A_L(\alpha)]\,\mathrm{d}\alpha \geq A_U(0) - A_L(0) \right\}.$$

THEOREM 1. *If $A, A_\alpha = [A_L(\alpha), A_U(\alpha)]$, $A \in F_{ES}(\mathbb{R})$ then*

$$T(A) = (t_1(A), t_2(A), t_3(A), t_4(A)) = (t_1, t_2, t_3, t_4)$$

the trapezoidal fuzzy number which preserves the expected interval and the support of the fuzzy number A, is given by

$$T(A) = \left(A_L(0), 2\int_0^1 A_L(\alpha)\,\mathrm{d}\alpha - A_L(0), 2\int_0^1 A_U(\alpha)\,\mathrm{d}\alpha - A_U(0), A_U(0) \right).$$

Proof. Conditions (3.1) and (3.2) imply that

$$\int_0^1 A_L(\alpha)\,\mathrm{d}\alpha = \tfrac{t_1+t_2}{2},$$
$$\int_0^1 A_U(\alpha)\,\mathrm{d}\alpha = \tfrac{t_3+t_4}{2},$$
$$A_L(0) = t_1$$

and

$$A_U(0) = t_4,$$

under restriction as $T(A)$ is a trapezoidal fuzzy number, that is

$$t_1 \leq t_2 \leq t_3 \leq t_4. \tag{3.3}$$

We obtain

$$t_2 = 2\int_0^1 A_L(\alpha)\,\mathrm{d}\alpha - A_L(0)$$

and

$$t_3 = 2\int_0^1 A_U(\alpha)\,\mathrm{d}\alpha - A_U(0).$$

Because $A_L(0) \leq A_L(\alpha)$ and $A_U(0) \geq A_U(\alpha)$, for every $\alpha \in [0,1]$, we have

$$t_2 = 2\int_0^1 A_L(\alpha)\,\mathrm{d}\alpha - A_L(0) \geq A_L(0) = t_1,$$
$$t_3 = 2\int_0^1 A_U(\alpha)\,\mathrm{d}\alpha - A_U(0) \leq A_U(0) = t_4.$$

The condition

$$t_2 \leq t_3$$

becomes

$$2\int_0^1 A_L(\alpha)\,\mathrm{d}\alpha - A_L(0) \leq 2\int_0^1 A_U(\alpha)\,\mathrm{d}\alpha - A_U(0),$$

so $T(A)$ is a trapezoidal fuzzy number if $A \in F(\mathbb{R})$ satisfies the condition

$$2\int_0^1 [A_U(\alpha) - A_L(\alpha)]\,\mathrm{d}\alpha \geq A_U(0) - A_L(0),$$

that is

$$A \in F_{ES}(\mathbb{R}).$$

□

Example 2. Let us consider a fuzzy number A given by

$$A_L(\alpha) = 1 + \alpha^2,$$
$$A_U(\alpha) = 3 - \alpha^2, \quad \alpha \in [0,1].$$

Because $A \in F_{ES}(\mathbb{R})$, the trapezoidal fuzzy number which preserves the expected interval and the support of A is

$$T(A) = \left(1, \tfrac{5}{3}, \tfrac{7}{3}, 3\right). \quad \square$$

If $A \notin F_{ES}(\mathbb{R})$ then it doesn't exist a trapezoidal fuzzy number which preserves the expected interval and the support of the fuzzy number A, as the following example proves.

Example 3. Let us consider a fuzzy number A given by

$$A_L(\alpha) = 1 + \sqrt{\alpha},$$
$$A_U(\alpha) = 45 - 35\sqrt{\alpha}, \quad \alpha \in [0,1].$$

Because

$$2\int_0^1 [A_U(\alpha) - A_L(\alpha)]\,\mathrm{d}\alpha = 40 < 44 = A_U(0) - A_L(0)$$

we get $A \notin F_{ES}(\mathbb{R})$ and not exists a trapezoidal fuzzy number which preserves the expected interval and the support of A. $\square$

Corollary 4. *If* $A = (a,b,c,d)_n$, *is a fuzzy number,* $n \in \mathbb{R}_+^*$ *and*

$$2n(b-c) \le (n-1)(a-d)$$

then

$$T(A) = \left(a, \tfrac{2bn-an+a}{n+1}, \tfrac{2cn-dn+d}{n+1}, d\right)$$

is the trapezoidal fuzzy number which preserves the expected interval and the support of A.

Proof. For a fuzzy number $A = (a,b,c,d)_n$ we have

$$A_L(\alpha) = (b-a)\sqrt[n]{\alpha} + a,$$
$$A_U(\alpha) = d - (d-c)\sqrt[n]{\alpha}, \quad \alpha \in [0,1],$$

therefore

$$\int_0^1 A_L(\alpha)\,\mathrm{d}\alpha = \tfrac{a+nb}{n+1}$$
$$\int_0^1 A_U(\alpha)\,\mathrm{d}\alpha = \tfrac{d+nc}{n+1}.$$

Because $A \in F_{ES}(\mathbb{R})$ if and only if

$$d\left(\tfrac{d+cn}{n+1} - \tfrac{a+nb}{n+1}\right) \ge d - a$$

according to Theorem 1 the conclusion is immediate. $\square$

EXAMPLE 5. If $B = (5, 8, 12, 14)_{\frac{1}{3}}$ then

$$T(B) = \left(5, \tfrac{13}{2}, 13, 14\right)$$

is the trapezoidal fuzzy number which preserves the expected interval and the support of B. □

4. PROPERTIES

In [12] Grzegorzewski and Mrowka proposed a number of criteria which the trapezoidal approximations should or just possess: α-cut invariance, translation invariance, identity, nearest criterion, expected value invariance, expected interval invariance, continuity, compatibility with the extension principle, order invariance.

THEOREM 6. *The trapezoidal operator preserving the expected interval and the support* $T : F_{ES}(\mathbb{R}) \longrightarrow F^T(\mathbb{R})$ *given in Theorem* 1 *has the following properties:*

(i) *is invariant to translations, that is*

$$T(A + z) = T(A) + z$$

for every $A \in F_{ES}(\mathbb{R})$ *and* $z \in \mathbb{R}$;

(ii) *is linear, that is*

$$T(\lambda \cdot A) = \lambda \cdot T(A)$$
$$T(A + B) = T(A) + T(B)$$

for every $A, B \in F_{ES}(\mathbb{R})$ *and* $\lambda \in \mathbb{R}^*$;

(iii) *fulfills the identity criterion, that is* $F^T(\mathbb{R}) \subset F_{ES}(\mathbb{R})$ *and*

$$T(A) = A,$$

for every $A \in F^T(\mathbb{R})$;

(iv) *is order invariant with respect to the preference relation* $\succ$ *defined by (see* [19]*)*

$$A \succ B \Leftrightarrow EV(A) \geqslant EV(B)$$

that is

$$A \succ B \Leftrightarrow T(A) \succ T(B)$$

for every A, $B \in F_{ES}(\mathbb{R})$;

(v) *is order invariant with respect to the preference relation* M *defined by (see* [17]*)*

$$M(A, B) = \begin{cases} 0, & \text{if } E^*(A) - E_*(B) < 0 \\ \frac{E^*(A) - E_*(B)}{E^*(A) - E_*(B) - (E_*(A) - E^*(B))}, & \text{if } 0 \in [E_*(A) - E^*(B), E^*(A) - E_*(B)] \\ 1, & \text{if } E_*(A) - E^*(B) > 0 \end{cases}$$

that is

$$M(T(A), T(B)) = M(A, B)$$

for every $A, B \in F_{ES}(\mathbb{R})$*;*

(vi) *is uncertainty invariant with respect to the nonspecificity measure defined by (see* [6]*)*

$$w(A) = \int_{-\infty}^{\infty} \mu_A(x) \mathrm{d}x$$

that is

$$w(A) = w(T(A))$$

for every $A \in F_{ES}(\mathbb{R})$*;*

(vii) *is correlation invariant, that is*

$$\rho(T(A), T(B)) = \rho(A, B),$$

for every $A, B \in F_{ES}(\mathbb{R})$*, where* $\rho(A, B)$ *denotes the correlation coefficient between* A *and* B*, defined as (see* [16]*)*

$$\rho(A, B) = \frac{E_*(A)E_*(B) + E^*(A)E^*(B)}{\sqrt{(E_*(A))^2 + (E^*(A))^2}\sqrt{(E_*(B))^2 + (E^*(B))^2}}.$$

Proof. (i) Let A be a fuzzy number and $z \in \mathbb{R}$. Then

$$(A+z)_L(\alpha) = (A)_L(\alpha) + z$$

and

$$(A+z)_U(\alpha) = (A)_U(\alpha) + z,$$

for every $\alpha \in [0,1]$, that is

$$\int_0^1 (A+z)_L(\alpha)\,\mathrm{d}\alpha = \int_0^1 A_L(\alpha)\,\mathrm{d}\alpha + z$$
$$\int_0^1 (A+z)_U(\alpha)\,\mathrm{d}\alpha = \int_0^1 A_U(\alpha)\,\mathrm{d}\alpha + z.$$

Because $A \in F_{ES}(\mathbb{R})$ we have

$$2\int_0^1 [A_U(\alpha) - A_L(\alpha)]\,\mathrm{d}\alpha - [A_U(0) - A_L(0)] \geq 0$$

and

$$2\int_0^1 [(A+z)_U(\alpha) - (A+z)_L(\alpha)]\,\mathrm{d}\alpha - [(A+z)_U(0) - (A+z)_L(0)] \geq 0$$

so

$$A + z \in F_{ES}(\mathbb{R}).$$

According to Theorem 1 we obtain

$$\begin{aligned}
t_1(A+z) &= (A+z)_L(0) = A_L(0) + z = t_1(A) + z,\\
t_2(A+z) &= 2\int_0^1 (A+z)_L(\alpha)\,\mathrm{d}\alpha - (A+z)_L(0)\\
&= 2\int_0^1 A_L(\alpha)\,\mathrm{d}\alpha + 2z - (A)_L(0) - z\\
&= t_2(A) + z\\
t_3(A+z) &= 2\int_0^1 (A+z)_U(\alpha)\,\mathrm{d}\alpha - (A+z)_U(0)\\
&= 2\int_0^1 A_U(\alpha)\,\mathrm{d}\alpha + 2z - (A)_U(0) - z\\
&= t_3(A) + z
\end{aligned}$$

and

$$t_4(A+z) = (A+z)_U(0) = A_U(0) + z = t_4(A) + z$$

therefore

$$T(A+z) = T(A) + z.$$

(ii) In the case $\lambda > 0$ and $A \in F_{ES}(\mathbb{R})$ we have

$$2\int_0^1 [A_U(\alpha) - A_L(\alpha)]\,\mathrm{d}\alpha - [A_U(0) - A_L(0)] \geq 0$$

and

$$2\int_0^1 [(\lambda\cdot A)_U(\alpha) - (\lambda\cdot A)_L(\alpha)]\,\mathrm{d}\alpha - [(\lambda\cdot A)_U(0) - (\lambda\cdot A)_L(0)] \geq 0$$

so

$$\lambda\cdot A \in F_{ES}(\mathbb{R}).$$

Then ((2.5) is used here),

$$\begin{aligned}
T(\lambda\cdot A) = \Big(&(\lambda\cdot A)_L(0), 2\int_0^1 (\lambda\cdot A)_L(\alpha)\,\mathrm{d}\alpha - (\lambda\cdot A)_L(0),\\
&2\int_0^1 (\lambda\cdot A)_U(\alpha)\,\mathrm{d}\alpha - (\lambda\cdot A)_U(0), (\lambda\cdot A)_U(0)\Big)\\
&= \lambda\cdot\Big(A_L(0), 2\int_0^1 A_L(\alpha)\,\mathrm{d}\alpha - A_L(0),\\
&2\int_0^1 A_U(\alpha)\,\mathrm{d}\alpha - A_U(0), A_U(0)\Big)\\
&= \lambda\cdot T(A).
\end{aligned}$$

In the case $\lambda < 0$ and $A \in F_{ES}(\mathbb{R})$,

$$2\int_0^1 \left[A_U(\alpha) - A_L(\alpha)\right] \mathrm{d}\alpha - \left[A_U(0) - A_L(0)\right] \geq 0$$

and

$$2\lambda\int_0^1 \left[A_U(\alpha) - A_L(\alpha)\right] \mathrm{d}\alpha - \lambda\left[A_U(0) - A_L(0)\right] \leq 0$$

therefore

$$2\int_0^1 \left[(\lambda \cdot A)_U(\alpha) - (\lambda \cdot A)_L(\alpha)\right] \mathrm{d}\alpha - \left[(\lambda \cdot A)_U(0) - (\lambda \cdot A)_L(0)\right] \geq 0$$

so

$$\lambda \cdot A \in F_{ES}(\mathbb{R}).$$

Then ((2.5) is used here),

$$\begin{aligned} T(\lambda \cdot A) &= \bigg((\lambda \cdot A)_L(0), 2\int_0^1 (\lambda \cdot A)_L(\alpha)\,\mathrm{d}\alpha - (\lambda \cdot A)_L(0), \\ &\qquad 2\int_0^1 (\lambda \cdot A)_U(\alpha)\,\mathrm{d}\alpha - (\lambda \cdot A)_U(0), (\lambda \cdot A)_U(0)\bigg) \\ &= \lambda \cdot \bigg(A_U(0), 2\int_0^1 A_U(\alpha)\,\mathrm{d}\alpha - A_U(0), \\ &\qquad 2\int_0^1 A_L(\alpha)\,\mathrm{d}\alpha - A_L(0), A_L(0)\bigg) \\ &= \lambda \cdot T(A). \end{aligned}$$

If A, $B \in F_{ES}(\mathbb{R})$ then

$$2\int_0^1 \left[A_U(\alpha) - A_L(\alpha)\right] \mathrm{d}\alpha - \left[A_U(0) - A_L(0)\right] \geq 0$$

and

$$2\int_0^1 \left[B_U(\alpha) - B_L(\alpha)\right] \mathrm{d}\alpha - \left[B_U(0) - B_L(0)\right] \geq 0,$$

imply that

$$\begin{aligned} &2\int_0^1 \left[(A_U(\alpha) + B_U(\alpha)) - (A_L(\alpha) + B_L(\alpha))\right] \mathrm{d}\alpha \\ &\quad - \left[(A_U(0) + B_U(0)) - (A_L(0) + B_L(0))\right] \geq 0 \end{aligned}$$

therefore ((2.2) is used here)

$$2\int_0^1 \left[(A+B)_U(\alpha) - (A+B)_L(\alpha)\right] \mathrm{d}\alpha - \left[(A+B)_U(0) - (A+B)_L(0)\right] \geq 0,$$

so

$$A + B \in F_{ES}(\mathbb{R}).$$

Applying Theorem 1 and (2.4) we get,

$$T(A+B) = \Big(A_L(0)+B_L(0), 2\int_0^1 (A_L(\alpha)+B_L(\alpha))\,\mathrm{d}\alpha-(A_L(0)+B_L(0)),$$
$$2\int_0^1 (A_U(\alpha)+B_U(\alpha))\,\mathrm{d}\alpha-(A_U(0)+B_U(0)), A_U(0)+B_U(0)\Big)$$
$$= T(A)+T(B).$$

(iii) For $A \in F^T(\mathbb{R}), A=(t_1,t_2,t_3,t_4)$, we have

$$A_L(\alpha) = t_1+(t_2-t_1)\alpha,$$
$$A_U(\alpha) = t_4+(t_3-t_4)\alpha, \alpha \in [0,1],$$

and $A \in F_{ES}(\mathbb{R})$ because

$$2\int_0^1 (A_U(\alpha)-A_L(\alpha))\,\mathrm{d}\alpha \geq A_U(0)-A_L(0)$$

is equivalent to

$$t_3-t_2-t_1+t_4 \geq t_4-t_1$$

that is

$$t_3 \geq t_2.$$

According to Theorem 1 the trapezoidal fuzzy number which preserves the expected interval and the support of A is

$$T(A) = (t_1(A), t_2(A), t_3(A), t_4(A)),$$

where

$$t_1(A) = A_L(0) = t_1$$
$$t_2(A) = 2\int_0^1 A_L(\alpha)\,\mathrm{d}\alpha - A_L(0) = 2\int_0^1 (t_1+(t_2-t_1)\alpha)\,\mathrm{d}\alpha - t_1 = t_2$$
$$t_3(A) = 2\int_0^1 A_U(\alpha)\,\mathrm{d}\alpha - A_U(0) = 2\int_0^1 (t_4+(t_3-t_4)\alpha)\,\mathrm{d}\alpha - t_4 = t_3$$

and

$$t_4(A) = A_U(0) = t_4$$

therefore

$$T(A) = A.$$

(iv), (v), (vi), (vii) For $A, B \in F_{ES}(\mathbb{R})$ we have

$$E_*(A) = E_*(T(A)),$$
$$E^*(A) = E^*(T(A)),$$
$$E_*(B) = E_*(T(B))$$

and

$$E^*(B) = E^*(T(B)),$$

therefore

$$A \succ B \Leftrightarrow T(A) \succ T(B),$$
$$M(T(A), T(B)) = M(A, B)$$

$$\begin{aligned} w(A) &= w(EI(A)) \\ &= w(EI(T(A) \\ &= w(T(A)) \end{aligned}$$

and

$$\rho(T(A), T(B)) = \rho(A, B).$$

□

Two important parameters, ambiguity and value, were introduced to capture the relevant information, to simplify the task of representing and handling fuzzy numbers. The ambiguity of a fuzzy number A (see [7]), denoted by $Amb(A)$, is defined by

$$Amb(A) = \int_0^1 \alpha (A_U(\alpha) - A_L(\alpha))\, \mathrm{d}\alpha$$

and the value of a fuzzy number A (see [7]), denoted by $Val(A)$, is defined by

$$Val(A) = \int_0^1 \alpha (A_U(\alpha) + A_L(\alpha))\, \mathrm{d}\alpha.$$

For a trapezoidal fuzzy number

$$T = (t_1, t_2, t_3, t_4)$$

we have

$$Amb(T) = \tfrac{t_3 - t_2}{2} + \tfrac{(t_4 - t_3) + (t_2 - t_1)}{6} \tag{4.1}$$

and

$$Val(T) = \tfrac{t_3 + t_2}{2} + \tfrac{(t_4 - t_3) - (t_2 - t_1)}{6}. \tag{4.2}$$

The trapezoidal operator T in Theorem 1 does not preserve the value and the ambiguity, that is there exists $A \in F_{ES}(\mathbb{R})$ such that

$$Val(A) \neq Val(T(A))$$

and

$$Amb(A) \neq Amb(T(A)),$$

as the following example proves.

Example 7. The trapezoidal fuzzy number which preserves the expected interval and the support of $A \in F(\mathbb{R})$ given by

$$A_\alpha = \left[\alpha^2 + 1, 30 - 27\alpha^2\right], \alpha \in [0, 1]$$

is (see Theorem 1)

$$T(A) = \left(1, \tfrac{5}{3}, 12, 30\right).$$

After elementary calculus and taking into account (4.1), (4.2) we obtain

$$Val(A) = 9 \neq \tfrac{175}{18} = Val(T(A)),$$
$$Amb(A) = \tfrac{15}{2} \neq \tfrac{149}{18} = Amb(T(A)). \quad \square$$

The continuity and monotony are between the criteria which a trapezoidal approximation operator should or just can possess ([12]). The continuity constraint means if two fuzzy numbers are close (in some sense) then their approximations should also be close. The criterion of monotony is verified for an trapezoidal approximation operator T if for any $A, B \in F(\mathbb{R}), A \subseteq B$ implies $T(A) \subseteq T(B)$. Here, for two fuzzy numbers A and $B, A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$, for every $x \in \mathbb{R}$. In fact, $A \subseteq B$ if and only if $A_L(\alpha) \geq B_L(\alpha)$ and $A_U(\alpha) \leq B_U(\alpha)$, for every $\alpha \in [0,1]$ and it is immediate that $(t_1, t_2, t_3, t_4) \subseteq (t_1', t_2', t_3', t_4')$ if and only if $t_1 \geq t_1', t_2 \geq t_2', t_3 \leq t_3'$ and $t_4 \leq t_4'$. Unfortunately, the operator in Theorem 1 is not continuous with respect to metric D (see (2.1)) and not monotonic, as the following examples prove.

Example 8. If $A \in F(\mathbb{R}), A_n \in F(\mathbb{R}), n \geq 2$ is given by

$$(A_n)_L(\alpha) = \begin{cases} (n+1)\alpha, & \text{if } \alpha \in \left[0, \frac{1}{n}\right] \\ \alpha + 1, & \text{if } \alpha \in \left[\frac{1}{n}, 1\right] \end{cases}$$
$$(A_n)_U(\alpha) = 4 - \alpha,$$

and

$$A_L(\alpha) = 1 + \alpha,$$
$$A_U(\alpha) = 4 - \alpha, \alpha \in [0,1]$$

then the trapezoidal fuzzy numbers which preserve the expected interval and the support of A and A_n are (Theorem 1)

$$T(A) = (1,2,3,4),$$
$$T(A_n) = \left(0, \tfrac{3n-1}{n}, 3, 4\right), n \geq 2.$$

We get

$$\begin{aligned} \lim_{n\to\infty} D^2(T(A_n), T(A)) &= \lim_{n\to\infty} \left(\int_0^1 \left[(T(A_n))_L(\alpha) - (T(A))_L(\alpha) \right]^2 \mathrm{d}\alpha \right. \\ &\quad \left. + \int_0^1 \left[(T(A_n))_U(\alpha) - (T(A))_U(\alpha) \right]^2 \mathrm{d}\alpha \right) \\ &= \lim_{n\to\infty} \int_0^1 \left(\tfrac{2n-1}{n}\alpha - 1 \right)^2 \mathrm{d}\alpha \\ &= \tfrac{1}{3} \end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty} D^2(A_n, A) \\
&= \lim_{n\to\infty} \left(\int_0^1 [(A_n)_L(\alpha) - (A)_L(\alpha)]^2 \, \mathrm{d}\alpha + \int_0^1 [(A_n)_U(\alpha) - (A)_U(\alpha)]^2 \, \mathrm{d}\alpha \right) \\
&= \lim_{n\to\infty} \int_0^{\frac{1}{n}} [(n+1)\alpha - 1 - \alpha]^2 \, \mathrm{d}\alpha \\
&= 0,
\end{aligned}$$

therefore

$$\lim_{n\to\infty} D(T(A_n), T(A)) = \tfrac{\sqrt{3}}{3} \neq 0 = \lim_{n\to\infty} D(A_n, A). \quad \square$$

EXAMPLE 9. Let us consider $A, B \in F(\mathbb{R})$ given by

$$\begin{aligned}
A_L(\alpha) &= e^{2\alpha} + e^{-2\alpha}, \\
A_U(\alpha) &= 10 - \alpha, \alpha \in [0,1]
\end{aligned}$$

and

$$\begin{aligned}
B_L(\alpha) &= e^{2\alpha}, \\
B_U(\alpha) &= 10 - \alpha, \alpha \in [0,1].
\end{aligned}$$

Because

$$\begin{aligned}
2\int_0^1 (A_U(\alpha) - A_L(\alpha))\,\mathrm{d}\alpha &= 19 - e^2 + \tfrac{1}{e^2} > 8 = A_U(0) - A_L(0), \\
2\int_0^1 (B_U(\alpha) - B_L(\alpha))\,\mathrm{d}\alpha &= 20 - e^2 > 9 = B_U(0) - B_L(0)
\end{aligned}$$

we have $A, B \in F_{ES}(\mathbb{R})$ and, by applying Theorem 1,

$$\begin{aligned}
T(A) &= \left(2, e^2 - \tfrac{1}{e^2} - 2, 9, 10\right), \\
T(B) &= \left(1, e^2 - 2, 9, 10\right),
\end{aligned}$$

therefore $T(A) \subsetneq T(B)$ even if $A \subseteq B$. $\square$

5. CONCLUSION

In this paper we have introduced the trapezoidal fuzzy number preserving the expected interval and the support of a fuzzy number. We have proved that this trapezoidal approximation fulfills properties like: translation invariance, linearity and identity, it does not preserve the value and the ambiguity of the fuzzy number and the criteria of continuity and monotony are not satisfied. The expected value invariance, order invariance, correlation invariance and uncertainty invariance are true because their definitions are based on the expected value.

REFERENCES

[1] Allahviranloo, T. and Adabitabar Firozja, M., *Note on trapezoidal approximation of fuzzy numbers*, Fuzzy Sets and Systems, **158**, pp. 755–756, 2007.

[2] Ban, A., *Approximation of fuzzy numbers by trapezoidal fuzzy numbers preserving the expected value*, Fuzzy Sets and Systems, **159**, pp. 1327–1344, 2008.

[3] Ban, A., *Triangular and parametric approximations of fuzzy numbers—inadvertences and corrections*, Fuzzy Sets and Systems, **160**, pp. 3048–3058, 2009.

[4] Ban, A., *On the nearest parametric approximation of a fuzzy number – Revisited*, Fuzzy Sets and Systems, **160**, pp. 3027–3047, 2009.

[5] Bodjanova, S., *Median value interval of a fuzzy number*, Information Sciences, **172**, pp. 73–89, 2005.

[6] Chanas, S., *On the interval approximation of a fuzzy number*, Fuzzy Sets and Systems, **122**, pp. 353—356, 2001.

[7] Delgado, M., Vila, M. A. and Voxman, W., *On a canonical representation of fuzzy numbers*, Fuzzy Sets and Systems, **93**, pp. 125–135, 1998.

[8] Diamond, P. and Kloeden, P., *Metric Spaces of Fuzzy Sets. Theory and Applications*, World Scientific, Singapore 1994.

[9] Dubois, D. and Prade, H., *Operations on fuzzy numbers*, Internat. J. Systems Sci, **9**, pp. 613-626, 1978.

[10] Dubois, D. and Prade, H., *The mean value of a fuzzy number*, Fuzzy Sets and Systems, **24**, pp. 279-300, 1987.

[11] Grzegorzewski, P., *Metrics and orders of fuzzy numbers*, Fuzzy Sets and Systems, **97**, pp. 83-94, 1998.

[12] Grzegorzewski, P. and Mrowka, E., *Trapezoidal approximations of fuzzy numbers*, Fuzzy Sets and Systems, **153**, pp. 115–135, 2005.

[13] Grzegorzewski, P. and Mrowka, E., *Trapezoidal approximations of fuzzy numbers - revisited*, Fuzzy Sets and Systems, **158**, pp. 757-768, 2007.

[14] Guerra, M. L. and Stefanini, L., *Approximate fuzzy arithmetic operations using monotonic interpolations*, Fuzzy Sets and Systems, **150**, pp. 5–33, 2005.

[15] Heilpern, S., *The expected value of a fuzzy number*, Fuzzy Sets and Systems, **47**, pp. 81–86, 1992.

[16] Hung, W. and Hu, J., *A note on the correlation of fuzzy numbers by expected interval*, Internat. J. Uncertainty Fuzziness and Knowledge-based System, **9**, pp. 517-523, 2001.

[17] Jimenez, M., *Ranking fuzzy numbers through of its expected interval*, Internat. J. Uncertainty Fuzziness and Knowledge-based System, **4**, pp. 379-388, 1996.

[18] Nasibov, E. N. and Peker, S., *On the nearest parametric approximation of a fuzzy number*, Fuzzy Sets and Systems, **159**, pp. 1365-1375, 2008.

[19] Yager, R. R., *A procedure for ordering fuzzy subset of the unit interval*, IInformation Sciences, **24**, pp. 143–161, 1981.

[20] Yeh, C.-T., *A note on trapezoidal approximation of fuzzy numbers*, Fuzzy Sets and Systems, **158**, pp. 747–754, 2007.

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