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STRONG ASYMPTOTICS OF EXTREMAL POLYNOMIALS ON THE SEGMENT IN THE PRESENCE OF DENUMERABLE SET OF MASS POINTS

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Abstract. The strong asymptotics of the monic extremal polynomials with respect to a $L_p(\sigma)$ norm are studied. The measure σ is concentrated on the segment $[-1,1]$ plus a denumerable set of mass points which accumulate at the boundary points of $[-1,1]$ only. Under the assumptions that the mass points satisfy Blaschke's condition and that the absolutely continuous part of σ satisfies Szegő's condition.

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1. INTRODUCTION

Let $0 < p < \infty$ and σ be a positive Borel measure supported on an infinite compact set E of the complex plane. We can then define for $n = 1, 2, 3, ...$

$$m_{n,p}(\sigma) := \min_{Q \in \mathcal{P}_{n-1}} \|z^n - Q(z)\|_{L_p(\sigma)},$$

where $\mathcal{P}_{n-1}$ denotes the class of complex polynomials of degree at most $n-1$. It is easily seen that there is at least one monic polynomial $T_{n,p}(\sigma, z) = z^n + ... \in \mathcal{P}_n$ such that

$$\|T_{n,p}(\sigma, z)\|_{L_p(\sigma)} = m_{n,p}(\sigma). \tag{1}$$

We call $T_{n,p}(\sigma, z)$ an L_p extremal polynomial with respect to the measure σ. We define also the nomalized extremal polynomials

$$P_{n,p}(\sigma, z) := T_{n,p}(\sigma, z)/m_{n,p}(\sigma),$$

$n = 1, 2, 3, ...$, satisfying

$$\|P_{n,p}(\sigma, z)\|_{L_p(\sigma)} = 1.$$

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When $p = 2$, $P_{n,2}(\sigma, z) = \kappa_n z^n + ... \in \mathcal{P}_n$ $(\kappa_n = 1/m_{n,2}(\sigma) > 0)$ is just the orthonormal polynomial of degree n with respect to the measure σ i.e.

$$(P_{n,2}, z^k)_{L_p(\sigma)} := \int_E P_{n,2}(\xi)\overline{\xi}^k \mathrm{d}\sigma(\xi) = \kappa_n^{-1}\delta_{nk}, k = 0, 1, ..., n.$$

A special area of research in this subject has been the study of the asymptotic behavior of $T_{n,p}(z)$ when n tends to infinity. There exists an extensive literature on orthogonal polynomials, but not enough on extremal polynomials. Beginning by Geronimus results in 1952 [1], who considered the case where the support E of the measure is a rectifiable Jordan curve, in particular, Widom [11] investigated the case $p = \infty$. Then, in 1987, Lubinsky and Saff [7] proved the asymptotic of $m_{n,p}(\sigma)$ and $T_{n,p}$ outside the segment $[-1, 1]$ under a general condition on the weight function. Another result on the zero distributions of the extremal polynomials on the unit circle, was presented by X. Li and K. Pan in [6]. In 1992, Kaliaguine [2], obtained the power asymptotic for extremal polynomials when E is a rectifiable Jordan curve plus a finite set of mass points and in 2004, Khaldi presented in [4] an extension of Kaliaguine's results, where he studied the case of a measure supported on a rectifiable Jordan curve plus an infinite set of mass points. Recently, Khaldi [5], solved this problem for a measure supported on the segment $[-1, 1]$ plus a finite set of mass points.

We mentioned that in the special case $p = 2$ of orthogonal polynomials, Peherstorfer and Yudiskii in [8] established the asymptotic for such polynomial on a segment $[-2, +2]$ plus a infinite set of mass points.

In this paper, we generalize the work of Peherstorfer and Yudiskii in [8] in the case where $p \geq 2$, more precisely we establish the strong asymptotic of the L_p extremal polynomials $\{T_{n,p}(\sigma, z)\}$ associated with the measure σ which has a decomposition of the form $\sigma = \alpha + \gamma$, where α is a measure with $\mathrm{supp}(\alpha) = [-1, 1]$, absolutely continuous with respect to the Lebesgue measure on the segment $[-1, 1]$ i.e.

$$\mathrm{d}\alpha(x) = \rho(x)\mathrm{d}x, \quad \rho \geq 0, \quad \int_{-1}^{+1} \rho(x)\mathrm{d}x < +\infty, \tag{2}$$

satisfying Szegő's condition and γ is a discrete measure supported on the infinite set of points $\{z_k\}_{k=1}^{\infty} \subset \mathbb{C}\backslash[-1, +1]$ i.e.

$$\gamma = \sum_{k=1}^{\infty} A_k \delta(z - z_k); \quad A_k > 0, \quad \sum_{k=1}^{\infty} A_k < \infty. \tag{3}$$

2. PRELIMINARY MATERIALS

2.1. Hardy space and Szegő function. Let $E = [-1, 1]$, $\Omega = \{\mathbb{C}\backslash E\} \cup \{\infty\}$, $G = \{w \in \mathbb{C} : |w| > 1\} \cup \{\infty\}$. The conformal mapping $\Phi : \Omega \to G$ is defined by $\Phi(z) = z + \sqrt{z^2 - 1}$, its inverse $\Psi(w) = \frac{1}{2}\left(w + \frac{1}{w}\right)$, and the capacity $C(E) = \lim_{z\to\infty}\left(\frac{z}{\Phi(z)}\right) = \frac{1}{2}$.

Let ρ be an integrable non negative weight function on E satisfying the Szegő's condition

$$\int_{-1}^{1} \frac{\mathrm{Log}\rho(x)}{\sqrt{1-x^2}}\mathrm{d}x > -\infty. \tag{4}$$

Then we can easily see that the weight function λ defined on the unit circle by

$$\lambda(e^{i\theta}) = \begin{cases} \rho(\xi)/\left|\Phi'_{-}(\xi)\right|, & \xi = \Psi(e^{i\theta}), \pi < \theta < 2\pi \\ \rho(\xi)/\left|\Phi'_{+}(\xi)\right|, & \xi = \Psi(e^{i\theta}), 0 < \theta < \pi \end{cases}$$

satisfies the following usual Szegő's condition

$$\int_{-\pi}^{\pi} \mathrm{Log}(\lambda(e^{\mathrm{i}\theta}))\mathrm{d}\theta > -\infty.$$

Thus the Szegő function associated with the unit circle $T = \{t : |t| = 1\}$ and the weight function λ is defined by

$$D(w) = \exp\left\{-\frac{1}{2p\pi}\int_{0}^{2\pi} \mathrm{Log}(\rho(\cos\theta)\left|\sin\theta\right|)\frac{1+we^{-\mathrm{i}\theta}}{1-we^{-\mathrm{i}\theta}}\mathrm{d}\theta\right\}, |w| < 1, \tag{5}$$

satisfying the following properties:

1) D is analytic on the open unit disk $U = \{w : |w| < 1\}$, $D(w) \neq 0$, $\forall w \in U$, and $D(0) > 0$.

2) D has boundary values, almost everywhere on the unit circle T such that

$$\lambda(e^{\mathrm{i}\theta}) = \rho(\cos\theta)\left|\sin\theta\right| = |D(t)|^{-p}$$

a.e. for $t = e^{\mathrm{i}\theta} \in T$.

Definition 2.1. *An analytic function f on Ω, belongs to $H^p(\Omega,\rho)$ if and only if $f(\Psi(w))/D(1/w) \in H^p(G)$,where $H^p(G)$ is the usual Hardy space associated with G, the exterior of the unit circle.*

Any function $f \in H^p(\Omega,\rho)$ has boundary values f_+ and f_- on both sides of E, and f_+, $f_- \in L_p(\alpha)$.

In the Hardy space $H^p(\Omega,\rho)$ we will define

$$\|f\|^p_{H^p(\Omega,\rho)} = \oint_E |f(x)|^p\,\rho(x)\mathrm{d}x = \lim_{R\to 1^+}\frac{1}{\pi R}\int_{E_R}\frac{|f(z)|^p}{|D(z)|^p}\left|\Phi'(z)\right||\mathrm{d}z|,$$

where $E_R = \{z \in \Omega : |\Phi(z)| = R\}$.

2.2. Notations and lemmas. Let $1 \le p < \infty$. We denote by $\mu(\rho)$ and $\mu(\sigma)$ respectively the extremal values of the following problems:

$$\mu(\rho) = \inf\left\{\|\varphi\|^p_{H^p(\Omega,\rho)} : \varphi \in H^p(\Omega,\rho), \varphi(\infty) = 1\right\}, \tag{6}$$

(7)

$$\mu(\sigma) = \inf\left\{\|\varphi\|^p_{H^p(\Omega,\rho)} : \varphi \in H^p(\Omega,\rho), \varphi(\infty) = 1, \varphi(z_k) = 0, k = 1, 2, ...\right\}$$

We denote by φ^* and ψ^* the extremal functions of the problems (6) and (7) respectively.

Notice that $\varphi^*(z) = D\left(1/\Phi(z)\right)/D\left(0\right)$ is an extremal function of the problem (6) and $\mu(\rho) = 2/\left[D\left(0\right)\right]^p$ (see [3]).

Lemma 2.2. *The extremal functions* φ^* *and* ψ^* *are related by* $\psi^* = \frac{1}{B(\infty)} B\varphi^*$ *and* $\mu(\sigma) = \left[B(\infty)\right]^{-p} \mu(\rho)$, *where*

$$B(z) = \prod_{k=1}^{\infty} \tfrac{\Phi(z)-\Phi(z_k)}{\Phi(z)\overline{\Phi(z_k)}-1} \tfrac{|\Phi(z_k)|}{\Phi(z_k)} \tag{8}$$

is a Blaschke product.

Proof. This lemma is proved for a curve in [4, p. 374]. This proof is valid in this case, too. □

Definition 2.3. *A measure* $\sigma = \alpha + \sum_{k=1}^{\infty} A_k \delta(z - z_k)$ *is said to belong to a class A, if the absolutely continuous part* α *satisfies the Szegö's condition* (4) *and the discrete part satisfies the the Blaschke's condition*

$$\left(\sum_{k=1}^{\infty} |\Phi(z_k)| - 1 \right) < \infty. \tag{9}$$

Lemma 2.4. *Let* $\sigma = \alpha + \sum_{k=1}^{\infty} A_k \delta(z - z_k)$ *be a measure which belongs to a class A, then we have*

$$\limsup_{n\to\infty} 2^n m_{n,p}(\sigma) \leq \left[\mu\left(\sigma\right)\right]^{1/p}.$$

Proof. This lemma is proved for $p = 2$ by Peherstorfer and Yudiskii in [8]. We will prove this lemma for $p > 2$ following the same ideas as in [8].

Without loss of generality, we assume the boundness from below of the weight function $\frac{1}{|D|}$, so $\frac{1}{|D|} \geq 2$. Let $\frac{1}{|D_\varepsilon|}$ be a smooth function such that $\frac{1}{|D_\varepsilon|} \geq 1$ and

$$\int_T \left| \tfrac{1}{|D_\varepsilon|^p} - \tfrac{1}{|D_\varepsilon|^p} \right| \mathrm{d}m < \epsilon \tag{10}$$

for $\varepsilon > 0$. Let us choose $\eta > 0$ such that $\max \frac{1}{|D_\varepsilon|} \leq \frac{1}{\eta}$ and denote by $E_\pm$ and $\widetilde{E}_\pm$ the vicinities of ± 1 of the form

$$E_\pm = \left\{ t \in T, |t \pm 1| \leq \tfrac{\eta}{2} \right\}, \quad \widetilde{E}_\pm = \left\{ t \in T, |t \pm 1| \leq \eta \right\}.$$

Introduce a smooth function as follows

$$F_{\varepsilon,\eta}\left(t\right) = \begin{cases} \left|D_\varepsilon\left(t\right)\right|, & t \in T \backslash \widetilde{E}_+ \cup \widetilde{E}_- \\ \left|t \pm 1\right|^2, & t \in E_\pm \end{cases}$$

and for $t \in \widetilde{E}_\pm \backslash E_\pm$ is such that

$$\left|t \pm 1\right|^2 \leq \left|F_{\varepsilon,\quad \eta}\left(t\right)\right| \leq \left|D_\varepsilon\left(t\right)\right|.$$

By the above settings it yields

$$\begin{aligned} 0 &\leq \log\frac{1}{F_{\varepsilon,\eta}(0)} - \log\frac{1}{D_{\varepsilon}(0)} \leq \int_{\widetilde{E}_+\cup\widetilde{E}_-} \log\left|\frac{D_{\varepsilon}(t)}{F_{\varepsilon,\eta}(t)}\right| \mathrm{d}m \\ &\leq \int_{\widetilde{E}_+} \log\frac{1}{|t+1|^2}\mathrm{d}m + \int_{\widetilde{E}_-} \log\frac{1}{|t-1|^2}\mathrm{d}m = o(1), \quad \text{as } \eta\to 0 \end{aligned}$$

In view of (10), we get

$$F_{\varepsilon,\eta}(0) = D(0) + o(1); \quad \varepsilon\to 0, \eta\to 0. \tag{11}$$

Let $b(t)$ be the Blaschke product

$$b(t) = \prod_{k=1}^{\infty} \frac{t-t_k}{t\overline{t_k}-1}\frac{\overline{t_k}}{|t_k|}$$

with $t_k = \frac{1}{\Phi(z_k)}, k = 0,1,2,....$ We see that $b(t)$ oscillates only in vicinities of the points ± 1, moreover, we have

$$\sup\left\{\left|b'(t)\left|t^2-1\right|^2, t\in T\right|\right\} < \infty,$$

Consequently, $(bF_{\varepsilon,\eta})' = b'F_{\varepsilon,\eta} + bF'_{\varepsilon,\eta} \in L_\infty$, and the Fourier series of $bF_{\varepsilon,\eta}$ converges to this function uniformly on T. Let

$$(bF_{\varepsilon,\eta})(t) = Q_{n,\varepsilon,\eta}(t) + t^{n+1}g_{n,\varepsilon,\eta}(t), g_{n,\varepsilon,\eta} \in H^\infty.$$

Putting

$$2^nP_{n,\varepsilon,\eta}(z) = 2^nP_{n,\varepsilon,\eta}(\Psi(\xi)) = \frac{\zeta^{-n}Q_{n,\varepsilon,\eta}(\zeta)+\zeta^nQ_{n,\varepsilon,\eta}\left(\frac{1}{\zeta}\right)}{2^{1/p}}.$$

We have the following estimate of the norm of the polynomial $2^nP_{n,\varepsilon,\eta}(z)$ for the absolutely continuous part of the measure,

$$\left\|\frac{2^nP_{n,\varepsilon,\eta}(z(t))}{D_\varepsilon(t)}\right\|_{L_p} \leq \left\|\frac{t^{-n}F_{\varepsilon,\eta}(t)b(t)+t^nF_{\varepsilon,\eta}(\bar{t})b(\bar{t})}{D_\varepsilon(t)2^{1/p}}\right\|_{L_p} + \left\|\frac{tg_{n,\varepsilon,\eta}(t)+\bar{t}g_{n,\varepsilon,\eta}(\bar{t})}{D_\varepsilon(t)2^{1/p}}\right\|_{L_p}$$

From the fact that $\|g_{n,\varepsilon,\eta}\|_{L_\infty} \to 0$ as $n\to\infty$ and $\left|\frac{F_{\varepsilon,\eta}(z(t))}{D_\varepsilon(t)}\right| \leq 1$, we conclude

$$\left\|\frac{2^nP_{n,\varepsilon,\eta}(z(t))}{D_\varepsilon(t)}\right\|_{L_p} \leq 1 + o(1).$$

Since $P_{n,\varepsilon,\eta}$ is uniformly bounded, using (10), we get

$$\left\|\frac{2^nP_{n,\varepsilon,\eta}(z(t))}{D(t)}\right\|_{L_p} \leq 1 + C\varepsilon + o(1). \tag{12}$$

Finally by using the extremal property of the polynomials $T_{n,p}$ and the fact that $P_{n,\varepsilon,\eta}(z) = \frac{(bF_{\varepsilon,\eta})(0)}{2^{1/p}}z^n + ...$ we get with the help of (12)

$$2^nm_{n,p}(\sigma) \leq \frac{\|2^nP_{n,\varepsilon,\eta}\|_{L_p(\sigma)}}{(bF_{\varepsilon,\eta})(0)/2^{1/p}} \leq \frac{1+C\varepsilon+o(1)}{(bF_{\varepsilon,\eta})(0)/2^{1/p}},$$

From here, lemma 1 and (11), it yields

$$\limsup_{n\to\infty} 2^n m_{n,p}(\sigma) \leq \tfrac{2^{1/p}}{b(0)D(0)} = \tfrac{[\mu(\rho)]^{1/p}}{B(\infty)} = [\mu(\sigma)]^{1/p}.$$

The proof is complete. □

Now we give the main result of this paper:

THEOREM 2.5. *Let a measure $\sigma = \alpha + \sum_{k=1}^{\infty} A_k \delta(z - z_k)$ be a measure which belongs to a class A. Associate with the measure σ the functions D, B and the extremal values $m_{n,p}(\sigma)$ and $\mu(\sigma)$ given respectively by* (5), (8), (1) *and* (7). *Then the monic extremal polynomials $T_{n,p}(\sigma, z)$ have the following asymptotic behavior as $n \to \infty$*

(1) $\lim 2^n m_{n,p}(\sigma) = [\mu(\sigma)]^{1/p}$.

(2) $T_{n,p}(\sigma, z) = \{\Phi(z)/2\}^n B(z) \frac{D(1/\Phi(z))}{D(0)} [1 + \chi_n(z)]$, *where* $\chi_n(z) \to 0$ *uniformly on compact subsets of* Ω.

Proof. We recall that by putting $t_k = \frac{1}{\Phi(z_k)}$, $\xi = \Psi(t)$ where $t = e^{i\theta}$, we get $B(\xi) = b(\bar{t})$ and $B(\infty) = b(0) = \prod_{k=1}^{\infty} \left|\frac{1}{\Phi(z_k)}\right|$, with $b(t) = \prod_{k=1}^{\infty} \frac{t - t_k}{t\overline{t_k} - 1} \frac{\overline{t_k}}{|t_k|}$.

Now we consider the following integral

$$I_n = \int_0^{2\pi} \left| \frac{2^{1/p} t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma) D(t)} - \left(b(t) + \frac{t^{2n} b(\bar{t}) D(\bar{t})}{D(t)} \right) \right|^2 \frac{\mathrm{d}\theta}{2\pi},$$

and transform it in a standard way as the following sum

$$\begin{aligned} I_n \;=\; & \int_0^{2\pi} \left| \frac{2^{1/p} t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma) D(t)} \right|^2 \frac{\mathrm{d}\theta}{2\pi} + \int_0^{2\pi} \left| b(t) + \frac{t^{2n} b(\bar{t}) D(\bar{t})}{D(t)} \right|^2 \frac{\mathrm{d}\theta}{2\pi} \\ & -2\mathcal{R}e \int_0^{2\pi} \frac{2^{1/p} t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma) D(t)} \overline{\left(b(t) + \frac{t^{2n} b(\bar{t}) D(\bar{t})}{D(t)} \right)} \frac{\mathrm{d}\theta}{2\pi}. \end{aligned} \tag{13}$$

Then, applying the Hölder inequality to the first term of (13) for $p \geq 2$ we get

$$\begin{aligned} \int_0^{2\pi} \left| \frac{2^{1/p} t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma) D(t)} \right|^2 \frac{\mathrm{d}\theta}{2\pi} &\leq \left(\int_0^{2\pi} \left| \frac{2^{1/p} t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma) D(t)} \right|^p \frac{\mathrm{d}\theta}{2\pi} \right)^{2/p} \left(\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \right)^{1-2/p} \\ &= \left[\left(\int_0^{2\pi} \left| \frac{T_{n,p}(\Psi(t))}{m_{n,p}(\sigma)} \right|^p |D(t)|^{-p} \frac{\mathrm{d}\theta}{\pi} \right)^{1/p} \right]^2 \\ &\leq \left[\frac{1}{m_{n,p}(\sigma)} \left(\frac{2}{\pi} \int_{-1}^{+1} |T_{n,p}(x)|^p \rho(x) \mathrm{d}x \right)^{1/p} \right]^2 \leq 2 \end{aligned} \tag{14}$$

For the second term of (13) we transform it as the following sum

$$\begin{aligned}
&\int_0^{2\pi} \left|b(t) + \tfrac{t^{2n} b(\bar t) D(\bar t)}{D(t)}\right|^2 \tfrac{d\theta}{2\pi} = \\
&= \int_0^{2\pi} |b(t)|^2 \tfrac{d\theta}{2\pi} + \int_0^{2\pi} \left|b(\bar t)\right|^2 \tfrac{d\theta}{2\pi} + 2\mathcal{R}e \int_0^{2\pi} t^{-2n} b(t) \overline{\left(\tfrac{b(\bar t) D(\bar t)}{D(t)}\right)} \tfrac{d\theta}{2\pi} \\
&= 2 + 2\mathcal{R}e \int_0^{2\pi} t^{-2n} b(t) \overline{\left(\tfrac{b(\bar t) D(\bar t)}{D(t)}\right)} \tfrac{d\theta}{2\pi}.
\end{aligned}$$

Since the last term approaches 0 when n tends to ∞ then we have

$$\int_0^{2\pi} \left|\tfrac{b(t)}{D(0)} + \tfrac{t^{2n} b(\bar t) D(\bar t)}{D(0) D(t)}\right|^2 \tfrac{d\theta}{2\pi} = 2 + \alpha_n \tag{15}$$

where $\alpha_n \to 0$, as $n \to \infty$.

In order to estimate the last integral of (13), we transform it as follows

$$\begin{aligned}
J_n &= \int_0^{2\pi} \tfrac{2^{1/p} t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma) D(t)} \overline{\left(b(t) + \tfrac{t^{2n} b(\bar t) D(\bar t)}{D(t)}\right)} \tfrac{d\theta}{2\pi} = \\
&= \int_0^{2\pi} \tfrac{2^{1/p} t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma) D(t)} \overline{b(t)} \tfrac{d\theta}{2\pi} + \int_0^{2\pi} \tfrac{2^{1/p} t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma) D(t)} \overline{\left(\tfrac{t^n b(\bar t) D(\bar t)}{D(t)}\right)} \tfrac{d\theta}{2\pi} \\
&= 2 \int_0^{2\pi} \tfrac{2^{1/p} t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma) D(t)} \overline{b(t)} \tfrac{dt}{2\pi i t} \\
&= 2 \int_0^{2\pi} \tfrac{2^{1/p} t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma) D(t)} \left(\overline{b(t)} - \overline{b_l(t)} + \overline{b_l(t)}\right) \tfrac{dt}{2\pi i t} \\
&= 2 \int_0^{2\pi} \tfrac{2^{1/p} t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma) D(t)} \left(\overline{b(t)} - \overline{b_l(t)}\right) \tfrac{dt}{2\pi i t} + 2 \int_0^{2\pi} \tfrac{2^{1/p} t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma) D(t)} \overline{b_l(t)} \tfrac{dt}{2\pi i t} \qquad (16)
\end{aligned}$$

where $b_l(t) = \prod_{k=1}^{l} \frac{t - t_k}{t\overline{t_k} - 1} \frac{\overline{t_k}}{|t_k|}$ be the finite Blaschke product with zeros $t_k = \frac{1}{\Phi(z_k)}$, $k = 1, 2, ...l$.

By applying the Hölder inequality to the first term of (16) we get

$$\begin{aligned}
&\left|\int_0^{2\pi} \tfrac{2^{1/p} t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma) D(t)} \left(\overline{b(t)} - \overline{b_l(t)}\right) \tfrac{dt}{2\pi i t}\right| \le \\
&\le \left(\int_0^{2\pi} \left|\tfrac{2^{1/p} t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma) D(t)}\right|^p \tfrac{d\theta}{2\pi}\right)^{\frac{1}{p}} \left(\int_0^{2\pi} |b(t) - b_l(t)|^q \tfrac{d\theta}{2\pi}\right)^{\frac{1}{q}} \\
&= \tfrac{2^{1/p}}{m_{n,p}(\sigma)} \left(\int_0^{2\pi} |T_{n,p}(\Psi(t))|^p |D(t)|^{-p} \tfrac{d\theta}{2\pi}\right)^{\frac{1}{p}} \left(\int_0^{2\pi} |b(t) - b_l(t)|^q \tfrac{d\theta}{2\pi}\right)^{\frac{1}{q}} \\
&= \tfrac{2^{1/p+1}}{m_{n,p}(\sigma)} \left(\int_{-1}^{+1} |T_{n,p}(x)|^p \rho(x) dx\right)^{\frac{1}{p}} \left(\int_0^{2\pi} |b(t) - b_l(t)|^q \tfrac{d\theta}{2\pi}\right)^{\frac{1}{q}} \le
\end{aligned}$$

$$\leq \quad 2^{1/p+1}\left(\int_0^{2\pi}|b(t)-b_l(t)|^q\,\frac{\mathrm{d}\theta}{2\pi}\right)^{\frac{1}{q}}. \tag{17}$$

For the last term of (16) by using the residue Theorem we get

$$\int_T \frac{t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma)D(t)}\overline{b_l(t)}\frac{\mathrm{d}t}{2\pi\mathrm{i}t} = \int_T \frac{t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma)D(t)b_l(t)}\,|b_l(t)|^2\,\frac{\mathrm{d}t}{2\pi\mathrm{i}t}$$

$$= \quad \frac{2^{1/p}}{2^n m_{n,p}(\sigma)D(0)b_l(0)} + \frac{2^{1/p}}{m_{n,p}(\sigma)}\sum_{k=1}^{l}\frac{t_k^{n-1}T_{n,p}(z_k)}{D(t_k)b_l'(t_k)}, \tag{18}$$

the last term of (18) can be estimated as

$$\frac{1}{m_{n,p}(\sigma)}\left|\sum_{k=1}^{l}\frac{t_k^{n-1}T_{n,p}(z_k)}{D(t_k)b_l'(t_k)}\right| \leq$$

$$\leq \frac{1}{m_{n,p}(\sigma)}\left[\sum_{k=1}^{l}|T_{n,p}(z_k)|^p A_k\right]^{1/p}\left[\sum_{k=1}^{l}\left(\left|\frac{1}{D(t_k)b_l'(t_k)}\right|\frac{|t_k^{n-1}|}{A_k^{1/p}}\right)^q\right]^{1/q}$$

$$\leq \left[\sum_{k=1}^{l}\left(\left|\frac{1}{D(t_k)b_l'(t_k)}\right|\frac{|t_k^{n-1}|}{A_k^{1/p}}\right)^q\right]^{1/q}, \quad \frac{1}{p}+\frac{1}{q}=1.$$

So, (18) becomes

$$\int_0^{2\pi}\frac{2^{1/p}t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma)D(t)}\overline{b_l(t)}\frac{\mathrm{d}t}{2\pi\mathrm{i}t} = \frac{2^{1/p}}{2^n m_{n,p}(\sigma)D(0)b_l(0)} + \beta_n$$

where $\beta_n \to 0$, as $n \to \infty$

So, first choosing l big enough and then n we conclude that

$$\int_0^{2\pi}\frac{t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma)D(t)}\frac{\mathrm{d}\theta}{2\pi} = \frac{2^{1/p}}{2^n m_{n,p}(\sigma)D(0)b(0)} + o(1) \tag{19}$$

Substituting (14),(15) and (19) we obtain

$$0 \leq I_n \leq 2+2+\alpha_n - \frac{4\left(2^{1/p}\right)}{2^n m_{n,p}(\sigma)D(0)b(0)} + o(1) \tag{20}$$

where $\alpha_n \to 0$ as $n \to \infty$.

Finally using the previous estimate we get

$$\liminf_{n\to\infty} 2^n m_{n,p}(\sigma) \geq \frac{2^{1/p}}{D(0)b(0)} = \frac{[\mu(\rho)]^{1/p}}{B(\infty)} = [\mu(\sigma)]^{1/p}.$$

This with Lemma 2 prove the first statement of Theorem.

Now, to prove (2) of Theorem, first we estimate the following integral

$$\left|\int_T\left[\frac{2^{1/p}t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma)D(t)} - \left(b(t)+\frac{t^{2n}b(\bar{t})D(\bar{t})}{D(t)}\right)\right]\frac{1}{1-w\bar{t}}\frac{\mathrm{d}t}{2\pi\mathrm{i}t}\right|^2 \leq$$

$$\leq \frac{1}{1-|w|}\int_T\left|\frac{2^{1/p}t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma)D(t)} - \left(b(t)+\frac{t^{2n}b(\bar{t})D(\bar{t})}{D(t)}\right)\right|^2\frac{\mathrm{d}\theta}{2\pi} = \frac{1}{1-|w|}I_n \tag{21}$$

As an immediate consequence of (20) and the first statement of Theorem, we get

$$\lim_{n\to\infty} I_n = 0.$$

So, from (21) yields

$$\int_T \left[\frac{2^{1/p}t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma)D(t)} - \left(b(t) + \frac{t^{2n}b(\bar{t})D(\bar{t})}{D(t)}\right)\right] \frac{1}{1-w\bar{t}}\frac{\mathrm{d}t}{2\pi \mathrm{i}t} = o(1). \tag{22}$$

On the other hand we have

$$\int_T \left[\frac{2^{1/p}t^n T_{n,p}(\Psi(t))}{m_{n,p}(\sigma)D(t)} - \left(b(t) + \frac{t^{2n}b(\bar{t})D(\bar{t})}{D(t)}\right)\right] \frac{1}{1-w\bar{t}}\frac{\mathrm{d}t}{2\pi \mathrm{i}t} =$$

$$= \int_T \chi_n(\Psi(t))\frac{1}{1-w\bar{t}}\frac{\mathrm{d}t}{2\pi \mathrm{i}t} - \int_T \frac{t^{2n}b(\bar{t})D(\bar{t})}{D(t)}\frac{1}{1-w\bar{t}}\frac{\mathrm{d}t}{2\pi \mathrm{i}t}. \tag{23}$$

Applying the Cauchy formula to the first term in (23), we can see that

$$\int_T \chi_n(\Psi(t))\frac{1}{1-w\bar{t}}\frac{\mathrm{d}t}{2\pi \mathrm{i}t} = \chi_n(z), z = \Psi(w) \in \Omega. \tag{24}$$

Since the last term in (23) approaches 0 as $n \to \infty$, we conclude from (22), (23) and (24), the second statement of Theorem. □

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