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h-STRONGLY E-CONVEX FUNCTIONS

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Abstract. Starting from strongly E-convex functions introduced by E. A. Youness, and T. Emam, from h-convex functions introduced by S. Varošanec and from the more general concept of h-convex functions introduced by A. Házy we define and study h-strongly E-convex functions. We study some properties of them.

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1. PRELIMINARY NOTIONS AND RESULTS

The concepts of E-convex sets and E-convex functions were introduced by Youness in [8]. Subsequently, Chen introduced a new concept of semi-E-convex functions in [2]. Based upon these approaches, in [9] Youness and Emam introduced the concepts of strongly E-convex sets and strongly E-convex functions. We firstly recall the definitions of convex sets, convex functions, E-convex sets and E-convex functions then of strongly E-convex sets and strongly E-convex functions and finally the definitions of h-convex functions, in the sense of Varošanec [7] and Házy [4].

Definition 1. *A set $A \subset \mathbb{R}^n$ is called convex if $\lambda x + (1-\lambda)\, y \in A$, for every pair of points $x, y \in A$ and every $\lambda \in [0,1]$.*

Definition 2. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called convex on a convex set $A \subset \mathbb{R}^n$ if for every pair of points $x, y \in A$ and every $\lambda \in [0,1]$, the following inequality is satisfied:*

$$f\left(\lambda x + (1-\lambda)\, y\right) \leq \lambda f(x) + (1-\lambda)\, f(y).$$

We consider a function $E : \mathbb{R}^n \to \mathbb{R}^n$.

Definition 3. [8]*A set $A \subset \mathbb{R}^n$ is called E-convex if $\lambda E(x) + (1-\lambda)\, E(y) \in A$, for every pair of points $x, y \in A$ and every $\lambda \in [0,1]$.*

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DEFINITION 4. [8]*A function $f:\mathbb{R}^n\to\mathbb{R}$ is called E-convex on an E-convex set $A\subset\mathbb{R}^n$ if for every pair of points $x,y\in A$ and every $\lambda\in[0,1]$, the following inequality is satisfied:*

$$f\left(\lambda E(x)+(1-\lambda)\,E(y)\right)\leq\lambda f(E(x))+(1-\lambda)\,f(E(y)).$$

DEFINITION 5. [9]*A set $A\subset\mathbb{R}^n$ is called strongly E-convex if*

$$\lambda\left(\alpha x+E\left(x\right)\right)+\left(1-\lambda\right)\left(\alpha y+E\left(y\right)\right)\in A,$$

for every pair of points $x,y\in A$, $\alpha\in[0,1]$ and $\lambda\in[0,1]$.

DEFINITION 6. [9]*A function $f:\mathbb{R}^n\to\mathbb{R}$ is called strongly E-convex on a strongly E-convex set $A\subset\mathbb{R}^n$ if for every pair of points $x,y\in A$, $\alpha\in[0,1]$ and $\lambda\in[0,1]$, the following inequality is satisfied:*

$$f(\lambda\left(\alpha x+E\left(x\right)\right)+\left(1-\lambda\right)\left(\alpha y+E\left(y\right)\right))\leq\lambda f(E(x))+\left(1-\lambda\right)f(E(y)).$$

In the following lines we recall the definition of h-convex functions introduced in [7] by S. Varošanec.

We consider I and J intervals in $\mathbb{R}$, $(0,1)\subseteq J$ and the real non-negative functions $h:J\to\mathbb{R}$, $f:I\to\mathbb{R}, h\neq 0$.

DEFINITION 7. [7]*The function $f:I\to\mathbb{R}$ is called h-convex on I or is said to belong to the class $SX\left(h,I\right)$ if for every pair of points $x,y\in I$ and every $\lambda\in(0,1)$, the following inequality is satisfied:*

$$f\left(\lambda x+\left(1-\lambda\right)y\right)\leq h(\lambda)f(x)+h\left(1-\lambda\right)f(y).$$

In [1] Bombardelli and Varošanec omitted the assumption that f and h are non-negative. We recall now the definitions of h-convex functions introduced in [4] by A. Házy.

Let X be a real (complex) linear space and $A\subset X$ nonempty, convex, open. Let $h:[0,1]\to\mathbb{R}$, $f:A\to\mathbb{R}$.

DEFINITION 8. [4]*The function $f:A\to\mathbb{R}$ is called h-convex on A if for every pair of points $x,y\in A$ and every $\lambda\in[0,1]$, the following inequality is satisfied:*

$$f\left(\lambda x+\left(1-\lambda\right)y\right)\leq h(\lambda)f(x)+h\left(1-\lambda\right)f(y).$$

2. PROPERTIES OF h-STRONGLY E-CONVEX FUNCTIONS

Starting from strongly E-convex functions and from h-convex functions in the sense of Házy we define and study h-strongly E-convex functions.

In the following lines we consider a map $E:\mathbb{R}^n\to\mathbb{R}^n$ and a strongly E-convex set $A\subset\mathbb{R}^n$. We also consider the functions $h:[0,1]\to\mathbb{R}$, $f:A\to\mathbb{R}$.

DEFINITION 9. *A function $f:A\to\mathbb{R}$ is called h-strongly E-convex on A if for every pair of points $x,y\in A$, $\alpha\in[0,1]$ and $\lambda\in[0,1]$, the following inequality is satisfied:*

(1)
$$f(\lambda\left(\alpha x+E\left(x\right)\right)+\left(1-\lambda\right)\left(\alpha y+E\left(y\right)\right))\leq h\left(\lambda\right)f\left(E\left(x\right)\right)+h\left(1-\lambda\right)f\left(E\left(y\right)\right).$$

THEOREM 10. *If $f : A \to \mathbb{R}$ is h-strongly E-convex on A and $h(0) = 0$ then*

$$f\left(\alpha x+E\left(x\right)\right)\leq h\left(1\right)f\left(E\left(x\right)\right). \tag{2}$$

Proof. We put $\lambda = 1$ in (1) and we obtain (2). □

THEOREM 11. *If the functions $f_i : A \to \mathbb{R}$, $i = 1, 2, \ldots, k$ are h-strongly E-convex on A, then, for $a_i \geq 0, i = 1, 2, \ldots, k$ the function $F : A \to \mathbb{R}$, $F(x) = \sum_{i=1}^{k} a_i f_i(x)$ is h-strongly E-convex on A.*

Proof. Since the functions $f_i : A \to \mathbb{R}$, $i = 1, 2, \ldots, k$ are h-strongly E-convex on A, then, for each $x, y \in A$, every $\alpha \in [0, 1]$ and $\lambda \in [0, 1]$, we have

$$\begin{aligned}
&F(\lambda\left(\alpha x+E\left(x\right)\right)+\left(1-\lambda\right)\left(\alpha y+E\left(y\right)\right))\\
&=\sum_{i=1}^{k}a_if_i\left(\lambda\left(\alpha x+E\left(x\right)\right)+\left(1-\lambda\right)\left(\alpha y+E\left(y\right)\right)\right)\\
&\leq h\left(\lambda\right)\sum_{i=1}^{k}a_if_i\left(E\left(x\right)\right)+h\left(1-\lambda\right)\sum_{i=1}^{k}a_if_i\left(E\left(y\right)\right)\\
&=h\left(\lambda\right)F\left(E\left(x\right)\right)+h\left(1-\lambda\right)F\left(E\left(y\right)\right).
\end{aligned}$$

Hence the function F is h-strongly E-convex on A. □

We consider a strongly E-convex set $A \subset \mathbb{R}^n$, a function $f : \mathbb{R}^n \to \mathbb{R}$, and a function $\varphi : \mathbb{R} \to \mathbb{R}$ linear and nondecreasing.

THEOREM 12. *If the function $f : \mathbb{R}^n \to \mathbb{R}$ is h-strongly E-convex on A then the composite function $\varphi \circ f$ is h-strongly E-convex on A.*

Proof. Since f is h-strongly E-convex on A, for each $x, y \in A$, $\alpha \in [0, 1]$ and $\lambda \in [0, 1]$, we have $f(\lambda(\alpha x + E(x)) + (1 - \lambda)(\alpha y + E(y))) \leq h(\lambda) f(E(x)) + h(1 - \lambda) f(E(y))$ and hence

$$\begin{aligned}
&\left(\varphi\circ f\right)\left(\lambda\left(\alpha x+E\left(x\right)\right)+\left(1-\lambda\right)\left(\alpha y+E\left(y\right)\right)\right)\\
&\leq\varphi\left[h\left(\lambda\right)f\left(E\left(x\right)\right)+h\left(1-\lambda\right)f\left(E\left(y\right)\right)\right)]\\
&=h\left(\lambda\right)\left(\varphi\circ f\right)\left(E\left(x\right)\right)+h\left(1-\lambda\right)\left(\varphi\circ f\right)\left(E\left(y\right)\right),
\end{aligned}$$

which implies that $\varphi \circ f$ is h-strongly E-convex on A. □

We denote $E(x)$ by Ex for simplicity.

THEOREM 13. *If the function $f : \mathbb{R}^n \to \mathbb{R}$ is non-negative and differentiable h-strongly E-convex on a strongly E-convex set A and h is a non-negative function with the property $h(\lambda) \leq \lambda$ for every $\lambda \in [0, 1]$ then*

$$\left(Ex-Ey\right)\nabla\left(f\circ E\right)\left(y\right)\leq\left(f\circ E\right)\left(x\right)-\left(f\circ E\right)\left(y\right) \tag{3}$$

for every $x, y \in A$.

Proof. Since f is h-strongly E-convex on A,

$$f(\lambda(\alpha x+Ex)+(1-\lambda)(\alpha y+Ey))\leq h(\lambda)(f\circ E)(x)+h(1-\lambda)(f\circ E)(y)$$

for each $x,y\in A$, $\lambda\in[0,1]$ and $\alpha\in[0,1]$. Since $h(x)\leq x$ for every $x\in[0,1]$ we have

$$\begin{aligned}&f((\alpha y+Ey)+\lambda[(\alpha x+Ex)-(\alpha y+Ey)])\\&\leq\lambda(f\circ E)(x)+(1-\lambda)(f\circ E)(y)\\&=(f\circ E)(y)+\lambda[(f\circ E)(x)-(f\circ E)(y)]\end{aligned}$$

and hence

$$\begin{aligned}&f(\alpha y+Ey)+\lambda[(\alpha x+Ex)-(\alpha y+Ey)])\nabla f(\alpha y+Ey)+O\left(\lambda^2\right)\\&\leq(f\circ E)(y)+\lambda[(f\circ E)(x)-(f\circ E)(y)]\end{aligned}$$

By taking $\alpha\to 0$, we get

$$\begin{aligned}&f(Ey)+\lambda(Ex-Ey)\nabla f(Ey))+O\left(\lambda^2\right)\\&\leq(f\circ E)(y)+\lambda[(f\circ E)(x)-(f\circ E)(y)].\end{aligned}$$

Dividing by $\lambda>0$ and taking $\lambda\to 0$, we obtain

$$(Ex-Ey)\nabla(f\circ E)(y)\leq(f\circ E)(x)-(f\circ E)(y),$$

for each $x,y\in A$. □

The following theorem provides a characterization of h-strongly E-convex functions with respect to the E-monotonicity of the gradient of map, similar with that obtain from E-convex functions, by Soleimani-Damaneh in [3].

Definition 14. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be differentiable. The map $\nabla f:\mathbb{R}^n\to\mathbb{R}^n$ is called E-monotone if*

$$(\nabla f(E(x))-\nabla f(E(y)))(E(x)-E(y))\geq 0,$$

for every $x,y\in R^n$.

Theorem 15. *If the function $f:\mathbb{R}^n\to\mathbb{R}$ is non-negative and differentiable h-strongly E-convex on a strongly E-convex set A and h is a non-negative function with the property $h(\lambda)\leq\lambda$ for every $\lambda\in[0,1]$ then*

$$(\nabla f(E(x))-\nabla f(E(y)))(E(x)-E(y))\geq 0 \tag{4}$$

for every $x,y\in A$.

Proof. Since f is h-strongly E-convex on A, from theorem (13) we have

$$(Ex-Ey)\nabla(f\circ E)(y)\leq(f\circ E)(x)-(f\circ E)(y)$$

and

$$(Ey-Ex)\nabla(f\circ E)(x)\leq(f\circ E)(y)-(f\circ E)(x),$$

for every $x,y\in A$. Adding these two inequalities we obtain

$$(\nabla f(E(x))-\nabla f(E(y)))(E(x)-E(y))\geq 0$$

for every $x, y \in A$. □

THEOREM 16. *Let the functions* $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$ *be* h*-strongly* E*-convex on* $\mathbb{R}^n$*. We consider the set*

$$M = \{x \in R^n \mid g_i(x) \leq 0, i = 1, 2, \ldots m\}.$$

If $E(M) \subseteq M$ *and the function* h *is positively then the set* M *is strongly* E*-convex.*

Proof. Since the functions $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$ are h-strongly E-convex on $\mathbb{R}^n$ then, for every $x, y \in M$, $\alpha \in [0, 1]$ and $\lambda \in [0, 1]$ we have

$$\begin{aligned} & g_i(\lambda(\alpha x + Ex) + (1 - \lambda)(\alpha y + Ey)) \\ & \leq h(\lambda)(g_i \circ E)(x) + h(1 - \lambda)(g_i \circ E)(y) \leq 0, \end{aligned}$$

and hence $\lambda(\alpha x + Ex) + (1 - \lambda)(\alpha y + Ey) \in M$. □

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