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THE GENERALIZATION OF SOME RESULTS FOR SCHURER AND SCHURER-STANCU OPERATORS

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Abstract. In the present paper we generalize some results for Schurer and Schurer-Stancu operators. Firstly, we establish a general formula concerning calculation of test functions by Schurer operators. Secondly, using this relationship and some known results we prove in every case a Voronovskaja type theorem, the uniform convergence and the order of approximation for Schurer and Schurer-Stancu operators.

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1. INTRODUCTION

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The operators $B_n : C([0,1]) \to C([0,1])$ given by

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\tfrac{k}{n}\right), \tag{1.1}$$

where $p_{n,k}(x)$ are the fundamental Bernstein's polynomials defined by

$$p_{n,k}(x) = \tbinom{n}{k} x^k (1-x)^{n-k}, \tag{1.2}$$

for any $x \in [0,1]$, $k \in \{0,1,\dots,n\}$ and $n \in \mathbb{N}$, are called Bernstein operators and were first introduced in [8]. In what follows, let $p \in \mathbb{N}_0$ be a fixed natural number and let the real parameters α, β be given such that $0 \le \alpha \le \beta$. The operators $\tilde{B}_{n,p} : C([0,1+p]) \to C([0,1])$ given by

$$\tilde{B}_{n,p}(f;x) = \sum_{k=0}^{n+p} \tilde{p}_{n,k}(x) f\left(\tfrac{k}{n}\right), \tag{1.3}$$

where $\tilde{p}_{n,k}(x)$ are the fundamental Schurer's polynomials defined by

$$\tilde{p}_{n,k}(x) = \tbinom{n+p}{k} x^k (1-x)^{n+p-k}, \tag{1.4}$$

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for any $x \in [0,1]$, $k \in \{0,1,\dots,n+p\}$ and $n \in \mathbb{N}$, are called Schurer operators [20]. The operators $\tilde{S}_{n,p}^{(\alpha,\beta)} : C([0,1+p]) \to C([0,1])$ defined by

$$\tilde{S}_{n,p}^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n+p} \tilde{p}_{n,k}(x) f\left(\tfrac{k+\alpha}{n+\beta}\right), \tag{1.5}$$

for any $x \in [0,1]$, $k \in \{0,1,\dots,n+p\}$ and $n \in \mathbb{N}$, where $\tilde{p}_{n,k}(x)$ are the fundamental Schurer's polynomials given at (1.4), are called Schurer-Stancu operators and were first introduced in [9], then studied intensively in [6].

Remark 1.1. More results and properties concerning (1.3) and (1.5) can be found also in monographs [2], [3], [7]. □

The aim of this paper is to generalize some results for the presented operators. Firstly, we establish a general formula concerning calculation of the test functions by Schurer operators and next, taking this into account, we will prove a Voronovskaja type theorem in every case for Schurer and Schurer-Stancu operators. Using some known results, which will be cited at the adequate moment we shall prove the uniform convergence, general Voronovskaja type formulas and the order of approximation up to twice continuously differentiable function for the Bernstein type operators.

2. PRELIMINARIES

Of the greatest utility in the calculus of finite differences, in number theory, in the summation of series, in the calculation of the Bernstein polynomials are the numbers introduced in 1730 by J. Stirling in his *Methodus differentialis* [21], subsequently called "Stirling numbers" of the first and second kind. For any $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$, let

$$(x)_n := \prod_{i=0}^{n-1} (x-i),$$

where $(x)_0 := 1$ is the falling factorial denoted by Pochhammer symbol. It is well known that

$$x^j = \sum_{i=0}^{j} S(j,i)(x)_i \tag{2.1}$$

holds for any $x \in \mathbb{R}$ and $j \in \mathbb{N}_0$, where $S(j,i)$ are the Stirling numbers of second kind. Now, let $i,j \in \mathbb{N}_0$ be natural numbers, then the Stirling numbers of second kind have the following properties:

(2.2)

$$S(j,i) := \begin{cases} 1, & \text{if } j=i=0;\ j=i \text{ or } j>1, i=1 \\ 0, & \text{if } j>0, i=0 \\ 0, & \text{if } j<i \\ i \cdot S(j-1,i) + S(j-1,i-1), & \text{if } j,i>1. \end{cases}$$

Let $e_j(x) = x^j$, with $j \in \mathbb{N}_0$ be the test functions. The main result established in [16], by O.T. Pop and M. Farcaş concerning calculation of the test functions in general case by Bernstein operators is given by the following:

Proposition 2.1. [16] *If* $n, j \in \mathbb{N}$, *then*

$$B_n(e_j; x) = \tfrac{1}{n^j} \sum_{i=1}^{j} S(j,i)(n)_i x^i. \tag{2.3}$$

During the preparation of the present paper, making some researches we discovered that, the relation (2.3) had been proved earlier by S. Karlin and Z. Ziegler [13]. As a special case, we can find the same relation in the article [1], where the asymptotic expansion of multivariate Bernstein polynomials on a simplex are considered. Later, in [19] the authors O.T. Pop, D. Bărbosu and P.I. Braica proved another result concerning calculation of the test functions by Bernstein operators. In [14] we established that the result proved in [19] does not differ by the result given at (2.3).

In this section we recall some results from [17] and [18], which we shall use in the present paper. Let I, J be real intervals and $I \cap J \neq \emptyset$. For any $n, k \in \mathbb{N}_0$, $n \neq 0$ consider the functions $\varphi_{n,k} : J \to \mathbb{R}$, with the property that $\varphi_{n,k}(x) \geq 0$, for any $x \in J$ and also consider the linear positive functionals $A_{n,k} : E(I) \to \mathbb{R}$. For any $n \in \mathbb{N}$ define the operator $L_n : E(I) \to F(J)$, by

$$L_n(f; x) = \sum_{k=0}^{n} \varphi_{n,k}(x) A_{n,k}(f), \tag{2.4}$$

where $E(I)$ is a linear space of real-valued functions defined on I and $F(J)$ is a subset of the set of real-valued functions defined on J.

Remark 2.2. [17] The operators $(L_n)_{n\in\mathbb{N}}$ are linear and positive on $E(I \cap J)$. □

For $n \in \mathbb{N}$ and $i \in \mathbb{N}_0$ define $T^*_{n,i}$ by

$$T^*_{n,i}(L_n; x) = n^i L_n\left(\psi_x^i; x\right) = n^i \sum_{k=0}^{n} \varphi_{n,k}(x) A_{n,k}\left(\psi_x^i\right), \quad x \in I \cap J, \tag{2.5}$$

where $\psi_x^i = (t-x)^i$, $t \in I \cap J$.

In what follows $s \in \mathbb{N}_0$ is even and we assume that the next two conditions hold:

- there exists the smallest $\alpha_s,\ \alpha_{s+2} \in [0, +\infty)$, so that

$$\lim_{n\to\infty} \frac{T^*_{n,j}(L_n; x)}{n^{\alpha_j}} = B_j(x) \in \mathbb{R}, \tag{2.6}$$

  for any $x \in I \cap J$ and $j \in \{s, s+2\}$,

$$\alpha_{s+2} < \alpha_s + 2 \tag{2.7}$$

- $I \cap J$ is an interval.

THEOREM 2.1. [17, 18] *If $f \in E(I)$ is a function s times differentiable in a neighborhood of $x \in I \cap J$, then*

$$\lim_{n\to\infty} n^{s-\alpha_s}\left(L_n(f;x) - \sum_{i=0}^{s} \tfrac{f^{(i)}(x)}{i!\cdot n^i} T^*_{n,i}(L_n;x)\right) = 0. \tag{2.8}$$

Assume that f is s times differentiable function on I and there exists an interval $K \subseteq I \cap J$, such that, there exist $n(s) \in \mathbb{N}$ and the constants $k_j \in \mathbb{R}$ depending on K, so that for $n \geq n(s)$ and $x \in K$, the following

$$\frac{T^*_{n,j}(L_n;x)}{n^{\alpha_j}} \leq k_j, \tag{2.9}$$

holds, for $j \in \{s, s+2\}$.

Then, the convergence expressed by (2.8) is uniform on K and moreover

$$n^{s-\alpha_s}\left|L_n(f;x) - \sum_{i=0}^{s} \tfrac{f^{(i)}(x)}{i!\cdot n^i} T^*_{n,i}(L_n;x)\right| \leq \tag{2.10}$$

$$\leq \tfrac{1}{s!}(k_s + k_{s+2})\omega_1\left(f^{(s)}; \tfrac{1}{\sqrt{n^{2+\alpha_s-\alpha_{s+2}}}}\right),$$

for any $x \in K$ and $n \geq n(s)$, where $\omega_1(f;\delta)$ denotes the modulus of continuity of the function f.

3. MAIN RESULTS

In the case of Schurer operators, we get:

PROPOSITION 3.1. *For any $j, n \in \mathbb{N}$ and $x \in [0,1]$, the following holds*

$$\tilde{B}_{n,p}(e_j;x) = \tfrac{1}{n^j}\sum_{i=0}^{j-1} S(j,j-i)(n+p)_{j-i}x^{j-i}. \tag{3.1}$$

Proof. For the proof of this proposition we take into account the same idea used in [16]. The relation (2.1) can be written also in the following form

$$x^j = \sum_{i=0}^{j-1} S(j,j-i)(x)_{j-i}, \tag{3.2}$$

because $S(j,0) = 0$, see (2.2). Using (3.2), we get

$$\begin{aligned}
\tilde{B}_{n,p}(e_j;x) &= \sum_{k=0}^{n+p} \tbinom{n+p}{k} x^k (1-x)^{n+p-k} \left(\tfrac{k}{n}\right)^j \\
&= \tfrac{1}{n^j}\sum_{k=0}^{n+p} \tbinom{n+p}{k} x^k (1-x)^{n+p-k} \sum_{i=0}^{j-1} S(j,j-i)(k)_{j-i} \\
&= \tfrac{1}{n^j}\sum_{i=0}^{j-1} S(j,j-i) \sum_{k=0}^{n+p} (k)_{j-i} \tbinom{n+p}{k} x^k (1-x)^{n+p-k} =
\end{aligned}$$

$$= \frac{1}{n^j}\sum_{i=0}^{j-1} S(j,j-i)(n+p)_{j-i}x^{j-i}.$$

□

In the following, we assume that the first three cases concerning calculation of the test functions by Schurer, respectively Schurer-Stancu operators are well known and for more details we recommend the reader our paper [15].

3.1. Schurer operators. Using the construction form preliminaries, we assume that $I=[0,1+p]$, $J=[0,1]$, $E(I)=C([0,1+p])$, $F(J)=C([0,1])$ and the role of n is played by $n+p$. Then let the functions $\varphi_{n+p,k}:[0,1]\to\mathbb{R}$ be defined by $\varphi_{n+p,k}(x):=\tilde{p}_{n,k}(x)$, for any $x\in[0,1]$, $n,k\in\mathbb{N}_0$, $n\neq 0$ and the functionals $A_{n+p,k}:C([0,1+p])\to\mathbb{R}$ let be defined by $A_{n+p,k}(f):=f\left(\frac{k}{n}\right)$, for any $n,k\in\mathbb{N}_0$, $n\neq 0$. In this case one obtains the Schurer operators, with

$$\begin{aligned}(3.3)\qquad T^*_{n,i}\left(\tilde{B}_{n,p};x\right) &= n^i\sum_{k=0}^{n+p}\tilde{p}_{n,k}(x)A_{n+p,k}\left(\psi_x^i\right)\\ &= n^i\sum_{k=0}^{n+p}\tilde{p}_{n,k}(x)\left(\tfrac{k}{n}-x\right)^i\\ &= \sum_{k=0}^{n+p}\tilde{p}_{n,k}(x)(k-(n+p)x+px)^i\\ &= \sum_{k=0}^{n+p}\tilde{p}_{n,k}(x)\sum_{l=0}^{i}\tbinom{i}{l}(k-(n+p)x)^l(px)^{i-l}\\ &= \sum_{l=0}^{i}\tbinom{i}{l}(px)^{i-l}T_{n+p,l}(x),\end{aligned}$$

where

$$(3.4)\qquad T_{n+p,l}(x)=\sum_{k=0}^{n+p}\tilde{p}_{n,k}(x)(k-(n+p)x)^l.$$

Application 3.2. *For $j\in\{3,4\}$ we present the calculation of the test functions by Schurer operators, taking into account* (3.1).

Case 1. $j=3$

$$\begin{aligned}\tilde{B}_{n,p}(e_3;x) &= \frac{1}{n^3}\sum_{i=0}^{2}S(3,3-i)(n+p)_{3-i}x^{3-i}\\ &= \tfrac{1}{n^3}\left(S(3,3)(n+p)_3x^3+S(3,2)(n+p)_2x^2+S(3,1)(n+p)_1x\right)\\ &= \tfrac{1}{n^3}\left((n+p)_3x^3+3(n+p)_2x^2+(n+p)_1x\right),\end{aligned}$$

where $S(3,2)=2\cdot S(2,2)+S(2,1)=3.$

Case 2. $j = 4$

$$\begin{aligned}\tilde{B}_{n,p}(e_4;x) &= \tfrac{1}{n^4}\sum_{i=0}^{3} S(4,4-i)(n+p)_{4-i}x^{4-i}\\ &= \tfrac{1}{n^4}\big(S(4,4)(n+p)_4x^4\\ &\quad +S(4,3)(n+p)_3x^3+S(4,2)(n+p)_2x^2+S(4,1)(n+p)_1x\big)\\ &= \tfrac{1}{n^4}\left((n+p)_4x^4+6(n+p)_3x^3+7(n+p)_2x^2+(n+p)_1x\right),\end{aligned}$$

where $S(4,2) = 2\cdot S(3,2)+S(3,1) = 7$ *and* $S(4,3) = 3\cdot S(3,3)+S(3,2) = 6$.

Remark 3.3. Regarding the polynomials $T_{n+p,l}(x)$, which were first introduced in [4], we shall give a proof relied on Application 3.2. □

Lemma 3.4. *The polynomials* $T_{n+p,l}(x)$ *satisfy the following*

$$\begin{aligned}&T_{n+p,0}(x) = 1,\\ &T_{n+p,1}(x) = 0,\\ &T_{n+p,2}(x) = (n+p)x(1-x),\\ &T_{n+p,3}(x) = (n+p)x(1-x)(1-2x),\\ &T_{n+p,4}(x) = 3(n+p)^2x^2(1-x)^2+(n+p)\left(x(1-x)-6x^2(1-x)^2\right).\end{aligned}$$

Proof. Using (3.4) and Application 3.2, it follows

$$\begin{aligned}T_{n+p,0}(x) &= \tilde{B}_{n,p}(e_0;x) = 1;\\ T_{n+p,1}(x) &= n\tilde{B}_{n,p}(e_1;x)-(n+p)x\tilde{B}_{n,p}(e_0;x) = 0;\\ T_{n+p,2}(x) &= n^2\tilde{B}_{n,p}(e_2;x)-2n(n+p)x\tilde{B}_{n,p}(e_1;x)+((n+p)x)^2\tilde{B}_{n,p}(e_0;x)\\ &= (n+p)x(1-x);\\ T_{n+p,3}(x) &= n^3\tilde{B}_{n,p}(e_3;x)-3n^2(n+p)x\tilde{B}_{n,p}(e_2;x)+3n((n+p)x)^2\tilde{B}_{n,p}(e_1;x)\\ &\quad -((n+p)x)^3\tilde{B}_{n,p}(e_0;x) = (n+p)x(1-x)(1-2x);\end{aligned}$$

$$\begin{aligned}T_{n+p,4}(x) &= n^4\tilde{B}_{n,p}(e_4;x)-4n^3(n+p)x\tilde{B}_{n,p}(e_3;x)+6(nx(n+p))^2\tilde{B}_{n,p}(e_2;x)\\ &\quad -4n((n+p)x)^3\tilde{B}_{n,p}(e_1;x)+((n+p)x)^4\tilde{B}_{n,p}(e_0;x)\\ &= 3(n+p)^2x^2(1-x)^2+(n+p)\left(x(1-x)-6x^2(1-x)^2\right).\end{aligned}$$

□

Lemma 3.5. *For any $x \in [0,1]$ and $n \in \mathbb{N}$, the following hold:*

$$T^*_{n,0}\left(\tilde{B}_{n,p};x\right) = 1,$$
$$T^*_{n,1}\left(\tilde{B}_{n,p};x\right) = px,$$
$$T^*_{n,2}\left(\tilde{B}_{n,p};x\right) = (px)^2 + (n+p)x(1-x),$$
$$T^*_{n,3}\left(\tilde{B}_{n,p};x\right) = (px)^3 + 3p(n+p)x^2(1-x) + (n+p)x(1-x)(1-2x),$$
$$T^*_{n,4}\left(\tilde{B}_{n,p};x\right) = (px)^4 + 6p^2(n+p)x^3(1-x) + 4p(n+p)x^2(1-x)(1-2x) + 3(n+p)^2x^2(1-x)^2 + (n+p)\left(x(1-x) - 6x^2(1-x)^2\right).$$

Proof. Using (3.3), (3.4) and Lemma 3.4, the identities follow. □

Lemma 3.6. *For any $x \in [0,1]$, the following relations hold*

$$\lim_{n\to\infty} T^*_{n,0}\left(\tilde{B}_{n,p};x\right) = 1, \tag{3.5}$$

$$\lim_{n\to\infty} \frac{T^*_{n,2}\left(\tilde{B}_{n,p};x\right)}{n} = x(1-x), \tag{3.6}$$

$$\lim_{n\to\infty} \frac{T^*_{n,4}\left(\tilde{B}_{n,p};x\right)}{n^2} = 3(x(1-x))^2 \tag{3.7}$$

and there exist

$$T^*_{n,0}\left(\tilde{B}_{n,p};x\right) = 1 = k_0, \tag{3.8}$$

$$\frac{T^*_{n,2}\left(\tilde{B}_{n,p};x\right)}{n} \le \frac{1}{4} = k_2, \tag{3.9}$$

$$\frac{T^*_{n,4}\left(\tilde{B}_{n,p};x\right)}{n^2} \le \frac{3}{16} = k_4, \tag{3.10}$$

for any $x \in [0,1]$ and $n \in \mathbb{N}$.

Proof. The identities (3.5)–(3.7) follow immediately from Lemma 3.5, while (3.8)–(3.10) yield from (3.5)–(3.7). □

Theorem 3.1. *Let $f \in C([0,1+p])$ be a function. If $x \in [0,1]$ and f is s times differentiable in a neighborhood of x, then*

$$\lim_{n\to\infty} \tilde{B}_{n,p}(f;x) = f(x), \tag{3.11}$$

for $s = 0$;

$$\lim_{n\to\infty} n\left(\tilde{B}_{n,p}(f;x) - f(x)\right) = pxf^{(1)}(x) + \frac{x(1-x)}{2}f^{(2)}(x), \tag{3.12}$$

for $s=2$;

$$\lim_{n\to\infty} n^2\left(\tilde{B}_{n,p}(f;x)-f(x)-\frac{px}{n}f^{(1)}(x)-\frac{(px)^2+(n+p)x(1-x)}{2n^2}f^{(2)}(x)\right)= \tag{3.13}$$
$$=\frac{3px^2(1-x)+x(1-x)(1-2x)}{6}f^{(3)}(x)+\frac{(x(1-x))^2}{8}f^{(4)}(x),$$

for $s=4$ *and*

$$\lim_{n\to\infty} n^{s-\alpha_s}\left(\tilde{B}_{n,p}(f;x)-\sum_{i=0}^{s}\frac{f^{(i)}(x)}{i!\cdot n^i}T^*_{n,i}\left(\tilde{B}_{n,p};x\right)\right)=0, \tag{3.14}$$

for $s\geq 4$.

Assume that f *is* s *times differentiable on* $[0,1+p]$, *then the convergence in* (3.11)–(3.14) *is uniform on* $[0,1]\subset[0,1+p]$. *Moreover, we get*

$$\left|\tilde{B}_{n,p}(f;x)-f(x)\right|\leq\frac{5}{4}\cdot\omega_1\left(f;\frac{1}{\sqrt{n}}\right), \tag{3.15}$$

for $s=0$ *and*

$$n\left|\tilde{B}_{n,p}(f;x)-f(x)-\frac{px}{n}f^{(1)}(x)-\frac{(px)^2+(n+p)x(1-x)}{2n^2}f^{(2)}(x)\right|\leq\frac{7}{32}\cdot\omega_1\left(f^{(2)};\frac{1}{\sqrt{n}}\right), \tag{3.16}$$

for $s=2$.

Proof. It follows from Theorem 2.1, with $\alpha_0=0$, $\alpha_2=1$ and $\alpha_4=2$, taking into account Lemma 3.5 and Lemma 3.6. □

REMARK 3.7. The above theorem, by the relation (3.14) generalizes the asymptotic behavior of the Schurer operators and in the particular case $s=2$ we recover formula (3.12), for twice continuously differentiable function, proved first by D. Bărbosu [5]. We get also the asymptotic behavior in the particular case $s=4$ and quantitative forms in terms of the modulus of continuity, for the same operators. Various quantitative forms of Voronovskaja's 1932 result [22] dealing with the asymptotic behavior of the Bernstein type operators are discussed also in several recent papers [10], [11] and [12], where better estimate close to the endpoints 0 and 1 then the global one was established. □

3.2. Schurer-Stancu operators. Using the same construction form preliminaries, we assume that $I=[0,1+p]$, $J=[0,1]$, $E(I)=C([0,1+p])$, $F(J)=C([0,1])$, the role of n is played by $n+p$. Then the functions $\varphi_{n+p,k}:[0,1]\to\mathbb{R}$ are defined by $\varphi_{n+p,k}(x):=\tilde{p}_{n,k}(x)$, for any $x\in[0,1]$, $n,k\in\mathbb{N}_0$, $n\neq 0$ and the functionals $A^{(\alpha,\beta)}_{n+p,k}:C([0,1+p])\to\mathbb{R}$ are defined by

$$A^{(\alpha,\beta)}_{n+p,k}(f):=f\left(\frac{k+\alpha}{n+\beta}\right),\ \text{for any } n,k\in\mathbb{N}_0, n\neq 0.$$

In this case one obtains the Schurer-Stancu operators, with

(3.17)
$$T^*_{n,i}\left(\tilde{S}^{(\alpha,\beta)}_{n,p};x\right)=n^i\sum_{k=0}^{n+p}\tilde{p}_{n,k}(x)A^{(\alpha,\beta)}_{n+p,k}\left(\psi^i_x\right)=n^i\sum_{k=0}^{n+p}\tilde{p}_{n,k}(x)\left(\tfrac{k+\alpha}{n+\beta}-x\right)^i$$
$$=\left(\tfrac{n}{n+\beta}\right)^i\sum_{k=0}^{n+p}\tilde{p}_{n,k}(x)\left(k-(n+p)x+\alpha+px-\beta x\right)^i$$
$$=\left(\tfrac{n}{n+\beta}\right)^i\sum_{k=0}^{n+p}\tilde{p}_{n,k}(x)\sum_{l=0}^{i}\tbinom{i}{l}(k-(n+p))^l(\alpha+px-\beta x)^{i-l}$$
$$=\left(\tfrac{n}{n+\beta}\right)^i\sum_{l=0}^{i}\tbinom{i}{l}(\alpha+px-\beta x)^{i-l}T_{n+p,l}(x),$$

where $T_{n+p,l}(x)$ were given in (3.4).

Lemma 3.8. *For any $x\in[0,1]$ and $n\in\mathbb{N}$, the following hold:*

$$T^*_{n,0}\left(\tilde{S}^{(\alpha,\beta)}_{n,p};x\right)=1,$$
$$T^*_{n,1}\left(\tilde{S}^{(\alpha,\beta)}_{n,p};x\right)=\tfrac{n}{n+\beta}(\alpha+px-\beta x),$$
$$T^*_{n,2}\left(\tilde{S}^{(\alpha,\beta)}_{n,p};x\right)=\left(\tfrac{n}{n+\beta}\right)^2\left((\alpha+px-\beta x)^2+(n+p)x(1-x)\right),$$
$$T^*_{n,3}\left(\tilde{S}^{(\alpha,\beta)}_{n,p};x\right)=\left(\tfrac{n}{n+\beta}\right)^3\left((\alpha+px-\beta x)^3\right.$$
$$\left.+3(\alpha+px-\beta x)(n+p)x(1-x)+(n+p)x(1-x)(1-2x)\right),$$
$$T^*_{n,4}\left(\tilde{S}^{(\alpha,\beta)}_{n,p};x\right)=\left(\tfrac{n}{n+\beta}\right)^4\left((\alpha+px-\beta x)^4\right.$$
$$+6(\alpha+px-\beta x)^2(n+p)x(1-x)$$
$$+4(\alpha+px-\beta x)(n+p)x(1-x)(1-2x)$$
$$\left.+3(n+p)^2(x(1-x))^2+(n+p)\left(x(1-x)-6(x(1-x))^2\right)\right).$$

Proof. Using (3.17) and Lemma 3.4, it follows the identities. □

Lemma 3.9. *For any $x\in[0,1]$, the following relations hold*

(3.18)
$$\lim_{n\to\infty}T^*_{n,0}\left(\tilde{S}^{(\alpha,\beta)}_{n,p};x\right)=1,$$

(3.19)
$$\lim_{n\to\infty}\frac{T^*_{n,2}\left(\tilde{S}^{(\alpha,\beta)}_{n,p};x\right)}{n}=x(1-x),$$

(3.20)
$$\lim_{n\to\infty}\frac{T^*_{n,4}\left(\tilde{S}^{(\alpha,\beta)}_{n,p};x\right)}{n^2}=3(x(1-x))^2,$$

and there exist

$$T^*_{n,0}\left(\tilde{S}^{(\alpha,\beta)}_{n,p};x\right)=1=k_0, \tag{3.21}$$

$$\frac{T^*_{n,2}\left(\tilde{S}^{(\alpha,\beta)}_{n,p};x\right)}{n}\le\frac{1}{4}=k_2, \tag{3.22}$$

$$\frac{T^*_{n,4}\left(\tilde{S}^{(\alpha,\beta)}_{n,p};x\right)}{n^2}\le\frac{3}{16}=k_4, \tag{3.23}$$

for any $x\in[0,1]$ *and* $n\in\mathbb{N}$.

Proof. The identities (3.18)–(3.20) follow immediately from Lemma 3.8, while (3.21)–(3.23) yield from (3.18)–(3.20). □

THEOREM 3.2. *Let* $f\in C([0,1+p])$ *be a function. If* $x\in[0,1]$ *and* f *is* s *times differentiable in a neighborhood of* x, *then*

$$\lim_{n\to\infty}\tilde{S}^{(\alpha,\beta)}_{n,p}(f;x)=f(x), \tag{3.24}$$

for $s=0$;

$$\lim_{n\to\infty}n\left(\tilde{S}^{(\alpha,\beta)}_{n,p}(f;x)-f(x)\right)=(\alpha+px-\beta x)f^{(1)}(x)+\frac{x(1-x)}{2}f^{(2)}(x), \tag{3.25}$$

for $s=2$;

$$\begin{aligned}\lim_{n\to\infty}n^2\Bigg(\tilde{S}^{(\alpha,\beta)}_{n,p}(f;x)-f(x)-\frac{\alpha+px-\beta x}{n+\beta}f^{(1)}(x)\\ -\frac{(\alpha+px-\beta x)^2+(n+p)x(1-x)}{2(n+\beta)^2}f^{(2)}(x)\Bigg)=\\ =\frac{3(\alpha+px-\beta x)x(1-x)+x(1-x)(1-2x)}{6}f^{(3)}(x)+\frac{(x(1-x))^2}{8}f^{(4)}(x),\end{aligned} \tag{3.26}$$

for $s=4$ *and*

$$\lim_{n\to\infty}n^{s-\alpha_s}\left(\tilde{S}^{(\alpha,\beta)}_{n,p}(f;x)-\sum_{i=0}^{s}\frac{f^{(i)}(x)}{i!\cdot n^i}T^*_{n,i}\left(\tilde{S}^{(\alpha,\beta)}_{n,p};x\right)\right)=0, \tag{3.27}$$

for $s\ge 4$. *Assume that* f *is* s *times differentiable on* $[0,1+p]$, *then the convergence from* (3.24)–(3.27) *is uniform on* $[0,1]\subset[0,1+p]$. *Moreover, we get*

$$\left|\tilde{S}^{(\alpha,\beta)}_{n,p}(f;x)-f(x)\right|\le\frac{5}{4}\cdot\omega_1\left(f;\frac{1}{\sqrt{n}}\right), \tag{3.28}$$

for $s=0$ *and*

$$\begin{aligned}n\left|\tilde{S}^{(\alpha,\beta)}_{n,p}(f;x)-f(x)-\frac{\alpha+px-\beta x}{n+\beta}f^{(1)}(x)-\frac{(\alpha+px-\beta x)^2+(n+p)^2x(1-x)}{2(n+\beta)^2}f^{(2)}(x)\right|\le\\ \le\frac{7}{32}\cdot\omega_1\left(f^{(2)};\frac{1}{\sqrt{n}}\right),\end{aligned} \tag{3.29}$$

for $s=2$.

Proof. It follows from Theorem 2.1, with $\alpha_0 = 0$, $\alpha_2 = 1$ and $\alpha_4 = 2$, taking into account Lemma 3.8 and Lemma 3.9. □

REMARK 3.10. The above theorem, by the relation (3.27) and by some particular cases given at (3.25), respectively (3.26) generalizes the asymptotic behavior of the Schurer-Stancu operators. Concerning quantitative forms in terms of the modulus of continuity, it is easily to remark that, for the Schurer-Stancu operators we get estimates as good as in the case of the Schurer operators. □

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