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THE APPROXIMATION OF BIVARIATE FUNCTIONS BY BIVARIATE OPERATORS AND GBS OPERATORS

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Abstract. In this paper we demonstrate a general approximation theorems for the bivariate functions by bivariate operators and GBS (Generalized Boolean Sum) operators.

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1. INTRODUCTION

In the paper [5], [7] we proved a Voronovskaja-type theorem and approximation theorem for a class of bivariate operators defined by finite sum, respectively by infinite sum. In the papers [8], and [9] we studied the approximation of bivariate functions by GBS operators.

The aim of this paper is to demonstrate a general approximation theorem of bivariate functions by special bivariate operators.

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

In this section we recall some notions which we will use in this paper.

We consider $I \subset \mathbb{R}$, I an interval and we shall use the function sets: $B(I) = \{f | f : I \to \mathbb{R},\ f \text{ bounded on } I\}$, $C(I) = \{f | f : I \to \mathbb{R},\ f \text{ continuous on } I\}$ and $C_B(I) = B(I) \cap C(I)$. For any $x \in I$, let the functions $\psi_x : I \to \mathbb{R}$, $\psi_x(t) = t - x$, for any $t \in I$ and $e_0 : I \to \mathbb{R}$, $e_0(x) = 1$ for any $x \in I$.

If $I \subset \mathbb{R}$ is a given interval and $f \in B(I)$, then the first order modulus of smoothness of f is the function $\omega(f; \cdot) : [0, \infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by

$$\omega(f;\delta) = \sup\left\{|f(x') - f(x'')| : x',\ x' \in I,\ |x' - x''| \leq \delta\right\}. \tag{1}$$

Let $I_1, I_2, J_1, J_2 \subset \mathbb{R}$ be intervals, $E(I_1 \times I_2)$, $F(J_1 \times J_2)$ which are subsets of the set of real functions defined on $I_1 \times I_2$, respectively $J_1 \times J_2$ and $L : E(I_1 \times I_2) \to F(J_1 \times J_2)$ be a linear positive operator. The operator $UL : E(I_1 \times I_2) \to F((I_1 \cap J_1) \times (I_2 \cap J_2))$ defined for any function $f \in E(I_1 \times I_2)$, any $(x, y) \in (I_1 \cap J_1) \times (I_2 \cap J_2)$ by

$$(ULf)(x, y) = L\left(f(x, *) + f(\cdot, y) - f(\cdot, *)\right)(x, y) \tag{2}$$

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is called GBS operator ("Generalized Boolean Sum" operator) associated to the operator L, where "$\cdot$" and "$*$" stand for the first and second variable (see [1]).

If $f \in E(I_1 \times I_2)$ and $(x,y) \in I_1 \times I_2$, let the functions $f_x = f(x,*)$, $f^y = f(\cdot,y) : I_1 \times I_2 \to \mathbb{R}$, $f_x(s,t) = f(x,t)$, $f^y(s,t) = f(s,y)$ for any $(s,t) \in I_1 \times I_2$. Then, we can consider that f_x, f^y are functions of real variable, $f_x : I_2 \to \mathbb{R}$, $f_x(t) = f(x,t)$ for any $t \in I_2$ and $f^y : I_1 \to \mathbb{R}$, $f^y(s) = f^y(s,y)$ for any $s \in I_1$.

Let $I_1, I_2 \subset \mathbb{R}$ be given intervals and $f : I_1 \times I_2 \to \mathbb{R}$ be a bounded function. The function $\omega_{total}(f;\cdot,*) : [0,\infty) \times [0,\infty) \to \mathbb{R}$, defined for any $(\delta_1,\delta_2) \in [0,\infty) \times [0,\infty)$ by

$$\omega_{total}(f;\delta_1,\delta_2) = \sup\left\{|f(x,y) - f(x',y')| : (x,y),(x',y') \in I_1 \times I_2, \right. \\ \left. |x-x'| \le \delta_1, |y-y'| \le \delta_2\right\} \tag{3}$$

is called the first order modulus of smoothness of function f or total modulus of continuity of function f (see [11]).

The first order modulus of smoothness for bivariate functions has properties similar to the properties of the first modulus of smoothness for univariate functions.

If $(L_m)_{m\ge1}$ is a sequence of operators, $L_m : E(I) \to F(J)$, $m \in \mathbb{N}$, for $m \in \mathbb{N}$ and $i \in \mathbb{N}_0$ define $T_{m,i}$ by

$$(T_{m,i}L_m)(x) = m^i \left(L_m \psi_x^i\right)(x) \tag{4}$$

for any $x \in I \cap J$, where $E(I)$, $F(J)$ are subsets of the set of real functions defined on I, respectively J.

2. PRELIMINARIES

In this section let $p_m = m$ for any $m \in \mathbb{N}$ or $p_m = \infty$ for any $m \in \mathbb{N}$ and similarly is defined q_n, $n \in \mathbb{N}$.

Let $I_1, I_2, J_1, J_2 \subset \mathbb{R}$ be intervals with $I_1 \cap J_1 \ne \emptyset$ and $I_2 \cap J_2 \ne \emptyset$. For $m,n \in \mathbb{N}$ and $k \in \{0,1,\ldots,p_m\} \cap \mathbb{N}_0$, $j \in \{0,1,\ldots,q_n\} \cap \mathbb{N}_0$, we consider $\varphi_{m,k} : J_1 \to \mathbb{R}$, $\varphi_{m,k}(x) \ge 0$ for any $x \in J_1$, $\psi_{n,j} : J_2 \to \mathbb{R}$, $\psi_{n,j}(y) \ge 0$ for any $y \in J_2$ and the linear positive functionals $A_{m,k} : E_1(I_1) \to \mathbb{R}$, $B_{n,j} : E_2(I_2) \to \mathbb{R}$.

Definition 2.1. For $m,n \in \mathbb{N}$ define the sequences of operators $(L_m)_{m\ge1}$ and $(K_n)_{n\ge1}$ by

$$(L_m f)(x) = \sum_{k=0}^{p_m} \varphi_{m,k}(x) A_{m,k}(f), \tag{5}$$

$$(K_n g)(y) = \sum_{j=0}^{q_n} \psi_{n,j}(y) B_{n,j}(g) \tag{6}$$

for any $f \in E_1(I_1)$, $g \in E_2(I_2)$, $x \in J_1$ and $y \in J_2$, where $E_1(I_1)$, $E_2(I_2)$ are subsets of the set of real functions defined on I_1, respectively I_2.

PROPOSITION 2.2. *The operators* $(L_m)_{m\geq 1}$ *and* $(K_n)_{n\geq 1}$ *are linear positive on* $E_1(I_1 \cap J_1)$ *and* $E_2(I_2 \cap J_2)$ *respectively.*

Proof. The proof follows immediately. □

In the following let $s \in \mathbb{N}_0$, s even. We suppose that the operators $(L_m)_{m\geq 1}$, $(K_n)_{n\geq 1}$ verify the conditions: there exist the smallest $\alpha_j, \beta_j \in [0,\infty)$, $j \in \{0,2,4,\dots,s+2\}$, such that

$$\lim_{m\to\infty} \frac{(T_{m,j}L_m)(x)}{m^{\alpha_j}} = a_j(x) \tag{7}$$

for any $x \in I_1 \cap J_1$,

$$\lim_{n\to\infty} \frac{(T_{n,j}K_n)(y)}{n^{\beta_j}} = b_j(y) \tag{8}$$

for any $y \in I_2 \cap J_2$ and if we note

$$\gamma_s = \max\left\{\alpha_{s-2l+\beta_{2l}} : l \in \left\{0,1,\dots,\tfrac{s}{2}\right\}\right\}, \tag{9}$$

then

$$\begin{cases} \alpha_{s-2l} + \beta_{2l+2} - \gamma_s - 2 < 0 \\ \alpha_{s-2l+2} + \beta_{2l} - \gamma_s - 2 < 0 \\ \alpha_{s-2l+2} + \beta_{2l+2} - \gamma_s - 4 < 0 \end{cases} \tag{10}$$

where $l \in \left\{0,1,2,\dots \frac{s}{2}\right\}$.

In the following we consider the set $E(I_1 \times I_2) = \big\{f | f : I_1 \times I_2 \to \mathbb{R}$, $f_x \in E_2(I_2)$ for any $x \in I_1$ and $f^y \in E_1(I_1)$ for any $y \in I_2\big\}$.

For $m,n \in \mathbb{N}$, let the linear positive functionals $A_{m,n,k,j} : E(I_1 \times I_2) \to \mathbb{R}$ with the property

$$A_{m,n,k,j}\left((\cdot - x)^i(* - y)^l\right) = A_{m,k}\left((\cdot - x)^i\right) B_{n,j}\left((* - y)^l\right) \tag{11}$$

for any $k \in \{0,1,\dots,p_m\} \cap \mathbb{N}_0$, $j \in \{0,1,\dots,q_n\} \cap \mathbb{N}_0$, $i,l \in \{0,1,\dots,s\}$ and $x \in I_1$, $y \in I_2$.

DEFINITION 2.3. Let $m,n \in \mathbb{N}$. The operator $L^*_{m,n}$ defined for any function $f \in E(I_1 \times I_2)$ and any $(x,y) \in J_1 \times J_2$ by

$$\left(L^*_{m,n}f\right)(x,y) = \sum_{k=0}^{p_m}\sum_{j=0}^{q_n} \varphi_{m,k}(x)\psi_{n,j}(y)A_{m,n,k,j}(f) \tag{12}$$

is named the bivariate operator of LK-type.

PROPOSITION 2.4. *The operators* $\left(L^*_{m,n}\right)_{m,n\geq 1}$ *are linear and positive on* $E\left((I_1 \cap J_1) \times (I_2 \cap J_2)\right)$.

Proof. The proof follows immediately. □

In the following we consider that

$$(T_{m,0}L_m)(x) = A_{m,0}(e_0) = 1 \tag{13}$$

for any $x \in I_1 \cap J_1$, $m \in \mathbb{N}$ and

$$(T_{n,0}K_n)(y) = B_{n,0}(e_0) = 1 \tag{14}$$

for any $y \in I_2 \cap J_2$, $n \in \mathbb{N}$.

From (13), (14) it results immediately that

$$\sum_{k=0}^{p_m} \varphi_{m,k}(x) = 1 \tag{15}$$

for any $x \in I_1 \cap J_1$, $m \in \mathbb{N}$ and

$$\sum_{j=0}^{q_n} \psi_{n,j}(y) = 1 \tag{16}$$

for any $y \in I_2 \cap J_2$, $n \in \mathbb{N}$.

Remark 2.5. *From* (13) *and* (14) *it results that* $\alpha_0 = \beta_0 = 0$. □

3. MAIN RESULTS

We recall the following theorem from [5].

Theorem 3.1. *Let* $I_1, I_2 \subset \mathbb{R}$ *be intervals,* $(a,b) \in I_1 \times I_2$, $n \in \mathbb{N}_0$ *and the function* $f : I_1 \times I_2 \to \mathbb{R}$, f *admits partial derivatives of order* n *continuous in a neighborhood* V *of the point* (a,b)*. According to Taylor's expansion theorem for the function* f *around* (a,b)*, for* $(x,y) \in V$ *we have*

$$f(x,y) = \sum_{k=0}^{n} \tfrac{1}{k!} \left(\tfrac{\partial}{\partial x}(x-a) + \tfrac{\partial}{\partial y}(y-b) \right)^k f(a,b) + \\ + \rho^n(x,y)\mu(x-a, y-b) \tag{17}$$

where

$$\left(\tfrac{\partial}{\partial x}(x-a) + \tfrac{\partial}{\partial y}(y-b) \right)^k f(a,b) = \\ = \sum_{i=0}^{k} \tbinom{k}{i} \tfrac{\partial^k f}{\partial x^{k-i} \partial y^i}(a,b)(x-a)^{k-i}(y-b)^i, \tag{18}$$

$k \in \{0,1,\dots,n\}$, μ *is a bounded function with* $\lim\limits_{(x,y)\to(a,b)} \mu(x-a, y-b) = 0$ *and*

$$\rho(x,y) = \sqrt{(x-a)^2 + (y-b)^2}. \tag{19}$$

Then for any $\delta_1, \delta_2 > 0$*, any* $(x, y) \in V$ *we have*

(20) $$|\mu(x-a, y-b)| \leq$$
$$\leq \frac{1}{n!}\left(1+\delta_1^{-2}(x-a)^2\right)\left(1+\delta_2^{-2}(y-b)^2\right)\sum_{i=0}^{n}\binom{n}{i}\omega_{total}\left(\frac{\partial^n f}{\partial x^{n-i}\partial y^i};\delta_1,\delta_2\right).$$

THEOREM 3.2. *Let* $f : I_1 \times I_2 \to \mathbb{R}$ *be a bivariate function.*
If $(x, y) \in (I_1 \cap J_1) \times (I_2 \cap J_2)$ *and* f *admits partial derivatives of order* s *continuous in a neighborhood of the point* (x, y)*, then*

(21)
$$\lim_{m\to\infty} m^{s-\gamma_s}\Bigg[\left(L^*_{m,m}f\right)(x,y)-$$
$$-\sum_{i=0}^{s}\frac{1}{m^i i!}\sum_{l=0}^{i}\frac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)\left(T_{m,i-l}L_m\right)(x)\left(T_{m,l}K_m\right)(y)\Bigg]=0.$$

If f *admits partial derivatives of order* s *continuous on* $(I_1 \cap J_1) \times (I_2 \cap J_2)$ *and there exist the intervals* $K_1 \subset I_1 \cap J_1$*,* $K_2 \subset I_2 \cap J_2$ *such that there exist* $m(s) \in \mathbb{N}$ *and* $a_{2l}, b_{2l} \in \mathbb{R}$ *depending on* K_1*, respectively* K_2*, so that for any* $m \in \mathbb{N}$*,* $m \geq m(s)$ *and for any* $(x, y) \in K_1 \times K_2$ *we have*

(22) $$\frac{(T_{m,2l}L_m)(x)}{m^{\alpha_{2l}}} \leq a_{2l},$$

(23) $$\frac{(T_{m,2l}K_m)(y)}{m^{\beta_{2l}}} \leq b_{2l},$$

where $l \in \left\{0, 1, \ldots, \frac{s}{2}+1\right\}$*, then the convergence given in* (21) *is uniform on* $K_1 \times K_2$ *and*

(24) $$m^{s-\gamma_s}\Bigg|\left(L^*_{m,m}f\right)(x,y)-$$
$$-\sum_{i=0}^{s}\frac{1}{m^i i!}\sum_{l=0}^{i}\frac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)\left(T_{m,i-l}L_m\right)(x)\left(T_{m,l}K_m\right)(y)\Bigg|\leq$$
$$\leq\frac{1}{s!}\sum_{l=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{l}(a_{s-2l}+a_{s-2l+2})(b_{2l}+b_{2l+2})\cdot$$
$$\cdot\sum_{i=0}^{s}\binom{s}{i}\omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial\tau^i};\frac{1}{\sqrt{m^{\delta_s}}},\frac{1}{\sqrt{m^{\delta_s}}}\right)$$

for any $(x, y) \in K_1 \times K_2$*, any* $m \in \mathbb{N}$ *with* $m \geq m(s)$*, where*

(25) $$\delta_s = -\max\Big\{\alpha_{s-2l}+\beta_{2l+2}-\gamma_s-2, \alpha_{s-2l+2}+\beta_{2l}-\gamma_s-2,$$
$$\frac{1}{2}\left(\alpha_{s-2l+2}+\beta_{2l+2}-\gamma_s-4\right) : l \in \left\{0,1,\ldots,\frac{s}{2}\right\}\Big\}.$$

Proof. Let $m, n \in \mathbb{N}$. According to Taylor's formula for the function f around (x, y), we have

$$f(t,\tau) = \sum_{i=0}^{s} \frac{1}{i!} \left(\frac{\partial}{\partial t}(t-x) + \frac{\partial}{\partial \tau}(\tau - y)\right)^i f(x,y) + \rho^s(t,\tau)\mu(t-x,\tau-y),$$

from where

$$(26)\qquad f(t,\tau) = \sum_{i=0}^{s} \frac{1}{i!} \sum_{l=0}^{i} \binom{i}{l} \frac{\partial^i f}{\partial t^{i-l}\partial \tau^l}(x,y)(t-x)^{i-l}(\tau-y)^l + \\ + \rho^s(t,\tau)\mu(t-x,\tau-y),$$

where μ is a bounded function and $\lim\limits_{(t,\tau)\to(x,y)} \mu(t-x,\tau-y) = 0$. Because $A_{m,n,k,j}$ is linear positive functional and verifies (11), from (26) we have

$$A_{m,n,k,j}(f) = \sum_{i=0}^{s} \frac{1}{i!} \sum_{l=0}^{i} \binom{i}{l} \frac{\partial^i f}{\partial t^{i-l}\partial \tau^l}(x,y) A_{m,k}\left((\cdot - x)^{i-l}\right) B_{n,j}\left((* - y)^l\right) + \\ + A_{m,n,k,j}\left(\rho^s(\cdot,*)\mu_{xy}\right),$$

where $\mu_{xy} : (I_1 \cap J_1) \times (I_2 \cap J_2) \to \mathbb{R}$, $\mu_{x,y}(t,\tau) = \mu(t-x,\tau-y)$ for any $(t,\tau) \in (I_1 \cap J_1) \times (I_2 \cap J_2)$. Multiplying by $\varphi_{m,k}(x)\psi_{n,j}(y)$ and summing after k, j where $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$, $j \in \{0, 1, \ldots, q_n\} \cap \mathbb{N}_0$, we obtain

$$(L^*_{m,n} f)(x,y) = \sum_{i=0}^{s} \frac{1}{i!} \sum_{l=0}^{i} \binom{i}{l} \frac{\partial^i f}{\partial^{i-l}\partial \tau^l}(x,y) \frac{1}{m^{i-l}} \frac{1}{n^l} (T_{m,i-l}L_m)(x) \cdot \\ \cdot (T_{n,l}K_n) + \sum_{k=0}^{p_m}\sum_{j=0}^{q_m} \varphi_{m,k}(x)\psi_{n,j}(y) A_{m,n,k,j}\left(\rho^s(\cdot,*)\mu_{xy}\right),$$

from which

$$(27)\qquad m^{s-\gamma_s}\Bigg[(L^*_{m,m}f)(x,y) - \\ - \sum_{i=0}^{s} \frac{1}{m^i i!} \sum_{l=0}^{i} \frac{\partial^i f}{\partial t^{i-l}\partial \tau^l}(x,y)(T_{m,i-l}L_m)(x)(T_{m,l}K_m)(y)\Bigg] = \\ = (R_{m,m}f)(x,y),$$

where

$$(28)\qquad (R_{m,m}f)(x,y) = m^{s-\gamma_s} \sum_{k=0}^{p_m}\sum_{j=0}^{q_m} \varphi_{m,k}(x)\psi_{m,j}(y) A_{m,m,k,j}\left(\rho^s(\cdot,*)\mu_{xy}\right).$$

Then

$$|(R_{m,m}f)(x,y)| \le m^{s-\gamma_s} \sum_{k=0}^{p_m}\sum_{j=0}^{q_m} \varphi_{m,k}(x)\psi_{m,j}(y) \left| A_{m,m,k,j}\left(\rho^s(\cdot,*)\mu_{xy}\right)\right|,$$

from where

$$(29)\qquad |(R_{m,m}f)(x,y)| \leq \\ \leq m^{s-\gamma_s} \sum_{k=0}^{p_m} \sum_{j=0}^{q_m} \varphi_{m,k}(x) \psi_{m,j}(y) A_{m,m,k,j}\left(\rho^s(\cdot,*)|\mu_{xy}|\right).$$

According to the relation (20), for any $\delta_1, \delta_2 > 0$ and for any $(t,\tau) \in (I_1 \cap J_1) \times (I_2 \cap J_2)$, we have that

$$|\mu_{xy}(t,\tau)| = |\mu(t-x,\tau-y)| \leq \frac{1}{s!}\big(1+\delta_1^{-2}(t-x)^2+\delta_2^{-2}(\tau-y)^2+ \\ +\delta_1^{-2}\delta_2^{-2}(t-x)^2(\tau-y)^2\big) \sum_{i=0}^{s} \binom{s}{i} \omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial \tau^i}, \delta_1, \delta_2\right)$$

and taking $\rho^s(t,\tau) = \sum_{l=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{l} (t-x)^{s-2l}(\tau-y)^{2l}$ into account, (30) results

$$(30)\quad A_{m,m,k,j}\left(\rho^s(\cdot,*)|\mu_{xy}|\right) \leq \frac{1}{s!} \sum_{l=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{l} \Big[A_{m,k}\big(\psi_x^{s-2l}\big) B_{m,j}\big(\psi_y^{2l}\big)+ \\ +\delta_1^{-2} A_{m,k}\big(\psi_x^{s-2l+2}\big) B_{m,j}\big(\psi_y^{2l}\big) + \delta_2^{-2} A_{m,k}\big(\psi_x^{s-2l}\big) B_{m,j}\big(\psi_y^{2l+2}\big)+ \\ +\delta_1^{-2}\delta_2^{-2} A_{m,k}\big(\psi_x^{s-2l+2}\big) B_{m,j}\big(\psi_y^{2l+2}\big)\Big] \sum_{i=0}^{s} \binom{s}{i} \omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial \tau^i}; \delta_1, \delta_2\right).$$

From (29) and (30), it results that

$$|(R_{m,m}f)(x,y)| \leq \\ \leq \frac{1}{s!} m^{s-\gamma_s} \sum_{l=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{l} \sum_{k=0}^{p_m} \sum_{j=0}^{q_m} \varphi_{m,k}(x)\psi_{m,j}(y) \Big[A_{m,k}\big(\psi_x^{s-2l}\big) B_{m,j}\big(\psi_y^{2l}\big)+ \\ +\delta_1^{-2} A_{m,k}\big(\psi_x^{s-2l+2}\big) B_{m,j}\big(\psi_y^{2l}\big) + \delta_2^{-2} A_{m,k}\big(\psi_x^{s-2l}\big) B_{m,j}\big(\psi_y^{2l+2}\big)+ \\ +\delta_1^{-2}\delta_2^{-2} A_{m,k}\big(\psi_x^{s-2l+2}\big) B_{m,j}\big(\psi_y^{2l+2}\big)\Big] \sum_{i=0}^{s} \binom{s}{i} \omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial \tau^i}; \delta_1, \delta_2\right),$$

or

$$|(R_{m,m}f)(x,y)| \leq$$
$$\leq \tfrac{1}{s!} m^{s-\gamma_s} \sum_{l=0}^{\frac{s}{2}} \Bigg[\tfrac{(T_{m,s-2l}L_m)(x)}{m^{s-2l}} \tfrac{(T_{m,2l}K_m)(y)}{m^{2l}} +$$
$$+ \delta_1^{-2} \tfrac{(T_{m,s-2l+2}L_m)(x)}{m^{s-2l+2}} \tfrac{(T_{m,2l}K_m)(y)}{m^{2l}} + \delta_2^{-2} \tfrac{(T_{m,s-2l}L_m)(x)}{m^{s-2l}} \tfrac{(T_{m,2l+2}K_m)(y)}{m^{2l+2}} +$$
$$+ \delta_1^{-2}\delta_2^{-2} \tfrac{(T_{m,s-2l+2}L_m(x)}{m^{s-2l+2}} \tfrac{(T_{m,2l+2}K_m)(y)}{m^{2l+2}} \Bigg] \sum_{i=0}^{s} \tbinom{s}{i} \omega_{total}\Big(\tfrac{\partial^s f}{\partial t^{s-i} \partial \tau^i}; \delta_1, \delta_2\Big),$$

so

$$|(R_{m,m}f)(x,y)| \leq$$
$$\leq \tfrac{1}{s!} \sum_{l=0}^{\frac{s}{2}} \tbinom{\frac{s}{2}}{l} \Bigg[\tfrac{(T_{m,s-2l}L_m)(x)}{m^{\alpha_{s-2l}}} \tfrac{(T_{m,2l}K_m)(y)}{m^{\beta_{2l}}} m^{\alpha_{s-2l}+\beta_{2l}-\gamma_s} +$$
$$+ \delta_1^{-2} \tfrac{(T_{m,s-2l+2}L_m)(x)}{m^{\alpha_{s-2l+2}}} \tfrac{(T_{m,2l}K_m)(y)}{m^{\beta_{2l}}} m^{\alpha_{s-2l+2}+\beta_{2l}-\gamma_s-2} +$$
$$+ \delta_2^{-2} \tfrac{(T_{m,s-2l}L_m)(x)}{m^{\alpha_{s-2l}}} \tfrac{(T_{m,2l+2}K_m)(y)}{m^{\beta_{2l+2}}} m^{\alpha_{s-2l}+\beta_{2l+2}-\gamma_s-2} +$$
$$+ \delta_1^{-2}\delta_2^{-2} \tfrac{(T_{m,s-2l+2}L_m)(x)}{m^{\alpha_{s-2l+2}}} \tfrac{(T_{m,2l+2}K_m)(y)}{m^{\beta_{2l+2}}} m^{\alpha_{s-2l+2}+\beta_{2l+2}-\gamma_s-4} \Bigg] \cdot$$
$$\cdot \sum_{i=0}^{s} \tbinom{s}{i} \omega_{total}\left(\tfrac{\partial^s f}{\partial t^{s-i} \partial \tau^i}; \delta_1, \delta_2\right).$$

We have $\delta_s \leq -(\alpha_{s-2l} + \beta_{2l+2} - \gamma_s - 2)$, $\delta_s \leq -(\alpha_{s-2l+2} + \beta_{2l} - \gamma_s - 2)$, $\delta_s \leq -(\alpha_{s-2l+2} + \beta_{2l+2} - \gamma_s - 4)$ for any $l \in \left\{0, 1, \ldots, \frac{s}{2}\right\}$, from where $\delta_s + \alpha_{s-2l} + \beta_{2l+2} - \gamma_s - 2 \leq 0$, $\delta_s + \alpha_{s-2l+2} + \beta_{2l} - \gamma_s - 2 \leq 0$, $2\delta_s + \alpha_{s-2l+2} + \beta_{2l+2} - \gamma_s - 4 \leq 0$, for any $l \in \left\{0, 1, \ldots, \frac{s}{2}\right\}$. From the relation (9) and from the inequalities above, we have

$$\begin{aligned} m^{\alpha_{s-2l}+\beta_{2l}-\gamma_s} &\leq 1, \\ m^{\delta_s+\alpha_{s-2l+2}+\beta_{2l}-\gamma_s-2} &\leq 1 \\ m^{\delta_s+\alpha_{s-2l}+\beta_{2l+2}-\gamma_s-2} &\leq 1, \\ m^{2\delta_s+\alpha_{s-2l+2}+\beta_{2l+2}-\gamma_s-4} &\leq 1, \end{aligned}$$

where $l \in \left\{0, 1, \ldots, \frac{s}{2}\right\}$.

Considering $\delta_1 = \delta_2 = \frac{1}{\sqrt{m^{\delta_s}}}$, we have

$$(31)\qquad |(R_{m,m}f)(x,y)| \le$$

$$(32)\qquad \le \frac{1}{s!} \sum_{l=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{l} \left[\frac{(T_{m,s-2l}L_m)(x)}{m^{\alpha_{s-2l}}} \frac{(T_{m,2l}K_m)(y)}{m^{\beta_{2l}}} + \right.$$

$$+ \frac{(T_{m,s-2l+2}L_m)(x)}{m^{\alpha_{s-2l+2}}} \frac{(T_{m,2l}K_m)(y)}{m^{\beta_{2l}}} + \frac{(T_{m,s-2l}L_m)(x)}{m^{\alpha_{s-2l}}} \frac{(T_{m,2l+2}K_m)(y)}{m^{\beta_{2l+2}}}$$

$$\left. + \frac{(T_{m,s-2l+2}L_m)(x)}{m^{\alpha_{s-2l+2}}} \frac{(T_{m,2l+2}K_m)(y)}{m^{\beta_{2l+2}}} \right] \cdot$$

$$\cdot \sum_{i=0}^{s} \binom{s}{i} \omega_{total} \left(\frac{\partial^s f}{\partial t^{s-i} \partial \tau^i} ; \frac{1}{\sqrt{m^{\delta_s}}} , \frac{1}{\sqrt{m^{\delta_s}}} \right).$$

Taking (7), (8) into account and considering the fact that

$$\lim_{m\to\infty} \omega_{total} \left(\frac{\partial^s f}{\partial t^{s-i} \partial \tau^i} ; \frac{1}{\sqrt{m^{\delta_s}}} , \frac{1}{\sqrt{m^{\delta_s}}} \right) = \omega_{total} \left(\frac{\partial^s f}{\partial t^{s-i} \partial \tau^i} ; 0, 0 \right) = 0,$$

$i \in \{0, 1, \ldots, s\}$, from (31) we have that

$$(33)\qquad \lim_{m\to\infty} (R_{m,m}f)(x,y) = 0.$$

From (27) and (33), (22) follows.

If in addition (23), (24) take place, then (31) becomes

$$(34)\qquad |(R_{m,m}f)(x,y)| \le \frac{1}{s!} \sum_{l=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{l} (a_{s-2l} + a_{s-2l+2})(b_{2l} + b_{2l+2}) \cdot$$

$$\cdot \sum_{i=0}^{s} \binom{s}{i} \omega_{total} \left(\frac{\partial^s f}{\partial t^{s-i} \partial \tau^i} ; \frac{1}{\sqrt{m^{\delta_s}}} , \frac{1}{\sqrt{m^{\delta_s}}} \right)$$

for any $m \in \mathbb{N}$, $m \ge m(s)$ and for any $(x,y) \in K_1 \times K_2$, from which, the convergence from (21) is uniform on $K_1 \times K_2$. From (27) and (34), (24) follows. □

Corollary 3.3. *Let $f : I_1 \times I_2 \to \mathbb{R}$ be a bivariate function.*
If $(x,y) \in (I_1 \cap J_1) \times (I_2 \cap J_2)$ and f is continuous in (x,y), then

$$(35)\qquad \lim_{m\to\infty} \left(L^*_{m,m} f \right)(x,y) = f(x,y).$$

If f is continuous on $(I_1 \cap J_1) \times (I_2 \cap J_2)$ and there exist the intervals $K_1 \subset I_1 \cap J_1$, $K_2 \subset I_2 \cap J_2$ such that there exist $m(0) \in \mathbb{N}$ and $a_2, b_2 \in \mathbb{R}$ depending on K_1, respectively K_2, so that for any $m \in \mathbb{N}$, $m \ge m(0)$ and any $(x,y) \in K_1 \times K_2$ we have

$$(36)\qquad \frac{(T_{m,2}L_m)(x)}{m^{\alpha_2}} \le a_2,$$

$$(37)\qquad \frac{(T_{m,2}K_m)(y)}{m^{\beta_2}} \le b_2,$$

then the convergence given in (35) *is uniform on* $K_1 \times K_2$ *and*

$$\left|\left(L^*_{m,m} f\right)(x,y) - f(x,y)\right| \leq (1+a_2)(1+b_2)\omega_{total}\left(f; \frac{1}{\sqrt{m^{\delta_0}}}, \frac{1}{\sqrt{m^{\delta_0}}}\right) \tag{38}$$

for any $(x,y) \in K_1 \times K_2$*, any* $m \in \mathbb{N}$*,* $m \geq m(0)$*.*

Proof. It result from Theorem 3.2 for $s = 0$ and one verifies immediately that $\alpha_0 = \beta_0 = \gamma_0 = 0$, $a_0 = b_0 = 1$, $\delta_0 = -\max\left\{\beta_2 - 2, \alpha_2 - 2, \frac{1}{2}(\alpha_2 + \beta_2 - 4)\right\}$. □

In the following, in addition we suppose that

$$\alpha_{s+2} < \alpha_s + 2,\ \beta_{s+2} < \beta_s + 2 \tag{39}$$

and for any $f \in E(I_1 \times I_2)$ we have

$$A_{m,n,k,j}(f_x) = B_{n,j}(f_x), \tag{40}$$

$$A_{m,n,k,j}(f^y) = A_{m,k}(f^y), \tag{41}$$

$$A_{m,n,k,j}(f) = A_{m,k}(B_{n,j}(f_x)) = B_{n,j}(A_{m,k}(f^y)) \tag{42}$$

for any $x \in I_1$, $y \in I_2$, $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$, $j \in \{0, 1, \ldots, q_n\} \cap \mathbb{N}_0$, $m, n \in \mathbb{N}$.

In [9] are given the following results, where if $P_m = L_m$, $m \in \mathbb{N}$ then $I = I_1$, $J = J_1$, $\eta_j = \alpha_j$, $k_j = a_j$, $j \in \{s, s+2\}$ and if $P_m = K_m$, $m \in \mathbb{N}$ then $I = I_2$, $J = J_2$, $\eta_j = \beta_j$, $k_j = b_j$, $j \in \{s, s+2\}$.

Theorem 3.4. *Let* $f : I \to \mathbb{R}$ *be a function. If* $x \in I \cap J$ *and* f *is a* s *times differentiable in* x *with* $f^{(s)}$ *continuous in* x*, then*

$$\lim_{m\to\infty} m^{s-\eta_s}\left[(P_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!}(T_{m,i}P_m)(x)\right] = 0. \tag{43}$$

Assume that f *is* s *times differentiable function on* I*, with* $f^{(s)}$ *continuous in* I *and there exists an interval* $M \subset I \cap J$ *such that there exist* $m(s) \in \mathbb{N}$ *and* $k_j \in \mathbb{R}$ *depending on* M *so that for any* $m \geq m(s)$ *and any* $x \in M$ *we have*

$$\frac{(T_{m,j}P_m)(x)}{m^{\eta_j}} \leq k_j \tag{44}$$

where $j \in \{s, s+2\}$*. Then the convergence given in* (43) *is uniform on* M *and*

$$m^{s-\eta_s}\left|(P_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!}(T_{m,i}P_m)(x)\right| \leq \frac{1}{s!}(k_s + k_{s+2})\omega\left(f^{(s)}; \frac{1}{\sqrt{m^{2+\eta_s-\eta_{s+2}}}}\right)$$

for any $x \in M$ *and* $m \geq m(s)$*.*

Now, let $\left(UL^*_{m,m}\right)_{m,n\geq 1}$ be the GBS operators associated to the $\left(L^*_{m,n}\right)_{m,n\geq 1}$ operators.

Lemma 3.5. *If* $m, n \in \mathbb{N}$*, then* $UL^*_{m,n}$ *have the form*

$$\left(UL^*_{m,n} f\right)(x,y) = (K_n f_x)(y) + (L_m f^y)(x) - \left(L^*_{m,n} f\right)(x,y) \tag{45}$$

for any $(x,y) \in (I_1 \cap J_1) \times (I_2 \cap J_2)$*, any* $f \in E(I_1 \times I_2)$*.*

Proof. We have

$$\begin{aligned}\left(UL^*_{m,n}f\right)(x,y) &= \left(L^*_{m,n}(f(x,*)+f(\cdot,y)-f(\cdot,*)\right)(x,y)\\ &= \left(L^*_{m,n}f(x,*)\right)(x,y)+\left(L^*_{m,n}f(\cdot,y)\right)(x,y)-\left(L^*_{m,n}f\right)(x,y)\\ &= \sum_{k=0}^{p_m}\sum_{j=0}^{q_n}\varphi_{m,k}(x)\psi_{n,j}(y)A_{m,n,k,j}(f_x)+\\ &\quad+\sum_{k=0}^{p_m}\sum_{j=0}^{q_n}\varphi_{m,k}(x)\psi_{n,j}(y)A_{m,n,k,j}(f^y)-\left(L^*_{m,n}f\right)(x,y)\end{aligned}$$

and taking (41), (42) into account, we obtain

$$\begin{aligned}\left(UL^*_{m,n}f\right)(x,y) &= \sum_{k=0}^{p_m}\sum_{j=0}^{q_n}\varphi_{m,k}(x)\psi_{n,j}(y)B_{n,j}(f_x)+\\ &\quad+\sum_{k=0}^{p_m}\sum_{j=0}^{q_n}\varphi_{m,k}(x)\psi_{n,j}(y)A_{m,k}(f^y)-\left(L^*_{m,n}f\right)(x,y)\\ &= \left(\sum_{k=0}^{p_m}\varphi_{m,k}(x)\right)\left(\sum_{j=0}^{q_n}\psi_{n,j}(y)B_{n,j}(f_x)\right)+\\ &\quad+\left(\sum_{k=0}^{p_m}\varphi_{m,k}(x)A_{m,k}(f^y)\right)\left(\sum_{j=0}^{q_n}\psi_{n,j}(y)\right)-\left(L^*_{m,n}f\right)(x,y).\end{aligned}$$

From (5), (6), (15) and (16), the relation (45) is obtained. □

THEOREM 3.6. *Let $f: I_1\times I_2\to\mathbb{R}$ be a bivariate function.*

If $(x,y)\in(I_1\cap J_1)\times(I_2\cap J_2)$ and f admits partial derivatives of order s continuous in a neighborhood of the point (x,y), then

$$\begin{aligned}\lim_{m\to\infty} m^{s-\gamma_s}\Bigg\{&\left(UL^*_{m,m}f\right)(x,y)-\\ &-\sum_{i=0}^{s}\frac{1}{m^i i!}\Bigg[\left(\frac{\partial^i f}{\partial\tau^i}(x,y)\left(T_{m,i}K_m\right)(y)+\frac{\partial^i f}{\partial t^i}(x,y)\left(T_{m,i}L_m\right)(x)\right)-\\ &-\sum_{l=0}^{i}\binom{i}{l}\frac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)\left(T_{m,i-l}L_m\right)(x)\left(T_{m,l}K_m\right)(y)\Bigg]\Bigg\}=0. \tag{46}\end{aligned}$$

If f admits partial derivatives of order s continuous on $(I_1\cap J_1)\times(I_2\cap J_2)$ and there exist the intervals $K_1\subset I_1\cap J_1$, $K_2\subset I_2\cap J_2$ such that there exist $m(s)\in\mathbb{N}$ and $a_{2l},b_{2l}\in\mathbb{R}$ depending on K_1, respectively K_2, so that for any $m\in\mathbb{N}$, $m\geq m(s)$ and any $x\in K_1$, $y\in K_2$ we have

$$\frac{\left(T_{m,2l}L_m\right)(x)}{m^{\alpha_{2l}}}\leq a_{2l}, \tag{47}$$

$$\frac{(T_{m,2l}K_m)(y)}{m^{\beta_{2l}}} \leq b_{2l} \tag{48}$$

where $l \in \left\{0, 1, \dots, \frac{s}{2}+1\right\}$, *then the convergence given in* (46) *is uniform on* $K_1 \times K_2$ *and*

(49)

$$m^{s-\gamma_s}\Bigg|(UL^*_{m,m}f)(x,y) - \sum_{i=0}^{s}\frac{1}{m^i i!}\left[\frac{\partial^i f}{\partial \tau^i}(x,y)(T_{m,i}K_m)(y) + \frac{\partial^i f}{\partial t^i}(x,y)(T_{m,i}L_m)(x) - \sum_{l=0}^{i}\binom{i}{l}\frac{\partial^i f}{\partial t^{i-l}\partial \tau^i}(x,y)(T_{m,i-l}L_m)(x)(T_{m,l}K_m)(y)\right]\Bigg| \leq$$

$$\leq \frac{1}{s!}\Bigg[(b_s+b_{s+2})\omega\left(\frac{\partial^s f_x}{\partial \tau^s};\frac{1}{\sqrt{m^{2+\beta_s-\beta_{s+2}}}}\right)+(a_s+a_{s+2})\omega\left(\frac{\partial^s f^y}{\partial t^s};\frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\right)+ \\ + \sum_{l=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{l}(a_{s-2l}+a_{s-2l+2})(b_{2l}+b_{2l+2})\sum_{i=0}^{s}\binom{s}{i}\cdot \\ \cdot\, \omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial \tau^i};\frac{1}{\sqrt{m^{\delta_s}}},\frac{1}{\sqrt{m^{\delta_s}}}\right)\Bigg]$$

for any $(x,y) \in K_1 \times K_2$, *any* $m \in \mathbb{N}$, $m \geq m(s)$, *where* δ_s *is given in* (25).

Proof. From (9), it results that $\gamma_s \geq \alpha_s$, $\gamma_s \geq \beta_s$, and then $s-\gamma_s \leq s-\alpha_s$ and $s-\gamma_s \leq s-\beta_s$. We use the (43) relation from Theorem 3.4 for the functions f_x and K_n, $n \in \mathbb{N}$ operators and for the function f^y and L_m, $m \geq 1$ operators, the (21) relation from Theorem 3.2 for the function f and then we obtain the (46) relation. If we note by S the left member of (49) relation, we can write

$$S = m^{s-\gamma_s}\Bigg|\left[(K_m f_x)(y) - \sum_{i=0}^{s}\frac{1}{m^i i!}\frac{\partial^i f}{\partial \tau^i}(x,y)\,(T_{m,i}K_m)(y)\right]+ \\ + \left[(L_m f^y)(x) - \sum_{i=0}^{s}\frac{1}{m^i i!}\frac{\partial^i f}{\partial t^i}(x,y)\,(T_{m,i}L_m)(x)\right]+ \\ + \left[\sum_{i=0}^{s}\frac{1}{m^i i!}\sum_{l=0}^{i}\binom{i}{l}\frac{\partial^i f}{\partial t^{i-l}\partial \tau^l}(x,y)\,(T_{m,i-l}L_m)(x)\,(T_{m,l}K_m)(y) - \\ - \left(L^*_{m,m}f\right)(x,y)\right]\Bigg| \leq$$

$$\leq m^{s-\alpha_s}\left|(L_m f_x)(y)-\sum_{i=0}^{s}\frac{1}{m^i i!}\frac{\partial^i f}{\partial \tau^i}(x,y)\,(T_{m,i}K_m)(y)\right|+$$
$$+\,m^{s-\gamma_s}\left|(L_m f^y)(x)-\sum_{i=0}^{s}\frac{1}{m^i i!}\frac{\partial^i f}{\partial t^i}(x,y)\,(T_{m,i}L_m)(x)\right|+$$
$$+\,m^{s-\gamma_s}\Bigg|\left(L^*_{m,m}f\right)(x,y)-\sum_{i=0}^{s}\frac{1}{m^i i!}\sum_{l=0}^{i}\binom{i}{l}\frac{\partial^i f}{\partial t^{i-l}\partial \tau^l}(x,y)\cdot$$
$$\cdot\,(T_{m,i-l}L_m)(x)\,(T_{m,l}K_m)(y)\Bigg|$$

and taking Theorem 3.4 and relation (24) into account we obtain the first inequality from (49). From (49) the uniform convergence for (46) results. □

Corollary 3.7. *Let $f : I_1 \times I_2 \to \mathbb{R}$ be a bivariate function.*
If $(x,y) \in (I_1 \cap J_1) \times (I_2 \cap J_2)$ and f is continuous in (x,y), then

$$\lim_{m\to\infty}\left(UL^*_{m,m}f\right)(x,y)=f(x,y). \tag{50}$$

Assume that f is continuous on $(I_1 \cap J_1) \times (I_2 \cap J_2)$ and there exist the intervals $K_1 \subset I_1 \cap J_1$, $K_2 \subset I_2 \cap J_2$ such that there exist $m(0) \in \mathbb{N}$ and $a_2, b_2 \in \mathbb{R}$ depending on K_1, respectively K_2 so that for any $m \in \mathbb{N}$, $m \geq m(0)$ and any $x \in K_1$, $y \in K_2$, we have

$$\frac{(T_{m,2}L_m)(x)}{m^{\alpha_2}} \leq a_2 \tag{51}$$

and

$$\frac{(T_{m,2}K_m)(y)}{m^{\beta_2}} \leq b_2. \tag{52}$$

Then the convergence given in (50) is uniform on $K_1 \times K_2$ and

$$\begin{aligned}&\left|\left(UL^*_{m,m}f\right)(x,y)-f(x,y)\right|\leq\\&\leq(1+b_2)\omega\left(f_x;\tfrac{1}{\sqrt{m^{2-\beta_2}}}\right)+(1+a_2)\omega\left(f^y;\tfrac{1}{\sqrt{m^{2-\alpha_2}}}\right)+\\&\quad+(1+a_2)(1+b_2)\omega_{total}\left(f;\tfrac{1}{\sqrt{m^{\delta_0}}},\tfrac{1}{\sqrt{m^{\delta_0}}}\right)\leq(1+b_2)\omega\left(f_x;\tfrac{1}{\sqrt{m^{\delta_0}}}\right)+\\&\quad+(1+a_2)\omega\left(f^y;\tfrac{1}{\sqrt{m^{\delta_0}}}\right)+(1+a_2)(1+b_2)\omega_{total}\left(f;\tfrac{1}{\sqrt{m^{\delta_0}}},\tfrac{1}{\sqrt{m^{\delta_0}}}\right)\end{aligned} \tag{53}$$

for any $(x,y) \in K_1 \times K_2$ and any $m \in \mathbb{N}$, $m \geq m(0)$, where

$$\delta_0=-\max\left\{\beta_2-2,\alpha_2-2,\tfrac{1}{2}\left(\alpha_2+\beta_2-4\right)\right\}.$$

Proof. It results from Theorem 3.6 for $s = 0$. □

Because every application is a simple substitute in the results of this section, we won't replace anything.

Application 3.8. *If* $I_1 = J_1 = [0,1]$, $I_2 = J_2 = [0,\infty)$, $E_1(I_1) = C([0,1])$, $E_2(I_2) = C_2([0,\infty))$, $p_m = m$, $q_n = \infty$, $\varphi_{m,k}(x) = p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}$, $\psi_{n,j}(y) = \mathrm{e}^{-ny}\frac{(ny)^j}{j!}$, $A_{m,k}(f^y) = f\left(\frac{k}{m}, y\right)$, $B_{n,j}(f_x) = f\left(x, \frac{j}{n}\right)$, $A_{m,n,j,k}(f) = f\left(\frac{k}{m}, \frac{j}{n}\right)$ *for any* $(x,y) \in [0,1] \times [0,\infty)$, $m,n \in \mathbb{N}$, $k \in \{0,1,\ldots,m\}$, $j \in \mathbb{N}_0$ *and* $f \in E([0,1]\times[0,\infty))$. *Then starting from* $(B_m)_{m\geq 1}$ *the Bernstein operators and* $(S_n)_{n\geq 1}$ *the Mirakjan-Favard-Szász operators, we obtain the operators* $\left(L^*_{m,n}\right)_{m,n\geq 1}$ *and* $\left(UL^*_{m,n}\right)_{m,n\geq 1}$ *defined for any function* $f \in E([0,1]\times[0,\infty))$, *any* $(x,y) \in [0,1]\times[0,\infty)$ *and* $m,n \in \mathbb{N}$ *by*

$$\left(L^*_{m,n} f\right)(x,y) = \sum_{k=0}^{m}\sum_{j=0}^{\infty} \binom{m}{k} x^k (1-x)^{m-k} \mathrm{e}^{-ny}\frac{(ny)^j}{j!} f\left(\frac{k}{m}, \frac{j}{n}\right), \tag{54}$$

$$\left(UL^*_{m,n} f\right)(x,y) = (S_n f_x)(y) + (B_m f^y)(x) - \left(L^*_{m,n} f\right)(x,y). \tag{55}$$

In this case $\alpha_2 = \beta_2 = 1$, $\delta_0 = 1$, $K_1 = [0,1]$, $K_2 = [0,b]$, $b > 0$, $a_2 = \frac{5}{4}$, $b_2 = b$ *and* $m(0) = 1$ (see [3]).

Application 3.9. *If* $I_1 = J_1 = [0,\infty)$, $I_2 = J_2 = [0,1]$, $E_1(I_1) = C_2([0,\infty))$, $E_2(I_2) = L_1([0,1])$, $p_m = \infty$, $q_n = n$, $\varphi_{m,k}(x) = (1+x)^{-m}\binom{m+k-1}{k}\left(\frac{x}{1+x}\right)^k$, $\psi_{n,j}(y) = \binom{n}{j} y^j (1-y)^{n-j}$,

$$\begin{aligned} A_{m,k}(f^y) &= f\left(\frac{k}{m}, y\right), \\ B_{n,j}(f_x) &= (n+1)\int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} f(x,t)\,\mathrm{d}t, \\ A_{m,n,k,j}(f) &= (n+1)\int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} f\left(\frac{k}{m}, t\right)\mathrm{d}t, \end{aligned}$$

for any $(x,y) \in [0,\infty)\times[0,1]$, $m,n \in \mathbb{N}$, $k \in \mathbb{N}_0$, $j \in \{0,1,\ldots,n\}$ *and* $f \in E([0,\infty)\times[0,1])$. *Then with the aid of* $(V_m)_{m\geq 1}$ *Baskakov operators and* $(K_n)_{n\geq 1}$ *Kantorovich operators, we obtain the operators* $\left(L^*_{m,n}\right)_{m,n\geq 1}$ *and* $\left(UL^*_{m,n}\right)_{m,n\geq 1}$ *defined for any function* $f \in E([0,\infty)\times[0,1])$, $(x,y) \in [0,\infty)\times[0,1]$ *and* $m,n \in \mathbb{N}$ *by*

(56)

$$\begin{aligned} &\left(L^*_{m,n} f\right)(x,y) = \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{n} (n+1)\binom{m+k-1}{k}\binom{n}{j}(1+x)^{-m}\left(\frac{x}{1+x}\right)^k y^j (1-y)^{n-j} \int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} f\left(\frac{k}{m}, t\right)\mathrm{d}t, \end{aligned}$$

$$\left(UL^*_{m,n} f\right)(x,y) = (K_n f_x)(y) + (V_m f^y)(x) - \left(L^*_{m,n} f\right)(x,y). \tag{57}$$

In this case $\alpha_2 = \beta_2 = 1$, $\delta_0 = 1$, $K_1 = [0, b]$, $b > 0$, $K_2 = [0, 1]$, $a_2 = b(b+1)$, $b_2 = 1$ *and* $m(0) = 3$ (see [2] and [3]).

APPLICATION 3.10. *If* $I_1 = I_2 = J_1 = J_2 = [0, 1]$, $E_1(I_1) = C([0, 1])$, $E_2(I_2) = L_1([0, 1])$, $p_m = \infty$, $q_n = n$, $\varphi_{m,k}(x) = \binom{m+k}{k}(1-x)^{m+1}x^k$, $\psi_{n,j}(y) = p_{n,j}(y)$,

$$\begin{aligned} A_{m,k}(f^y) =& f\left(\tfrac{k}{m+k}, y\right), \\ B_{n,j}(f_x) =& (n+1)\int_0^1 p_{n,j}(t)f(x,t)\mathrm{d}t, \\ A_{m,n,k,j}(f) =& (n+1)\int_0^1 p_{n,j}(t)f\left(\tfrac{k}{m+k}, t\right)\mathrm{d}t \end{aligned}$$

for any $(x, y) \in [0, 1] \times [0, 1]$, $m, n \in \mathbb{N}$, $k \in \mathbb{N}_0$, $j \in \{0, 1, \ldots, n\}$ *and* $f \in E([0, 1] \times [0, 1])$. *Then with* $(Z_m)_{m\geq 1}$ *the Meyer-König and Zeller operators and* $(M_n)_{n\geq 1}$ *the Durrmeyer operators, we construct the operators* $\left(L^*_{m,n}\right)_{m,n\geq 1}$ *and* $\left(UL^*_{m,n}\right)_{m,n\geq 1}$ *defined for any function* $f \in E([0, 1] \times [0, 1])$, *any* $(x, y) \in [0, 1] \times [0, 1]$ *and* $m, n \in \mathbb{N}$ *by*

(58)

$$\begin{aligned} &\left(L^*_{m,n}f\right)(x, y) = \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{n}(n+1)\binom{m+k}{k}\binom{n}{j}(1-x)^{m+1}x^k y^j(1-y)^{n-j}\int_0^1 p_{n,j}(t)f\left(\tfrac{k}{m+k}, t\right)\mathrm{d}t, \end{aligned}$$

$$\left(UL^*_{m,n}f\right)(x, y) = (M_n f_x)(y) + (Z_m f^y)(x) - \left(L^*_{m,n}f\right)(x, y). \tag{59}$$

In this case $\alpha_2 = \beta_2 = 1$, $\delta_0 = 1$, $K_1 = K_2 = [0, 1]$, $a_2 = 2$, $b_2 = \frac{3}{2}$ *and* $m(0) = 3$ (see [2] and [3]).

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