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A VORONOVSKAJA-TYPE FORMULA FOR THE q-MEYER-KÖNIG AND ZELLER OPERATORS

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Abstract. A Voronovskaja-type formula for the q-Meyer-König and Zeller operators is presented.

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1. INTRODUCTION AND NOTATION

Starting from the identity

$$(1-x)^{n+1}\sum_{k=0}^{\infty}\binom{n+k}{k}x^k = 1 \quad \text{for all } x\in[0,1),$$

W. Meyer-König and K. Zeller [12] defined a sequence of linear positive operators associating with each continuous real-valued function defined on $[0,1]$ a so-called "Bernstein power series". In the slight modification by E. W. Cheney and A. Sharma [4], the Meyer-König and Zeller operators are defined for every $n\in\mathbb{N}$ (the set of all positive integers) and every $f\in C[0,1]$ by

$$\begin{aligned} M_n f(x) &:= \sum_{k=0}^{\infty} f\left(\tfrac{k}{n+k}\right) m_{n,k}(x) \quad \text{if } x\in[0,1),\\ M_n f(1) &:= f(1), \end{aligned}$$

where

$$m_{n,k}(x) := \binom{n+k}{k}x^k(1-x)^{n+1}.$$

Let $q>0$. For every $n\in\{0,1,2,\dots\}$ the *q-integer* $[n]_q$ is defined by

$$[0]_q := 0 \quad \text{and} \quad [n]_q := 1+q+\cdots+q^{n-1} \quad \text{if } n\geq 1.$$

The *q-factorial* $[n]_q!$ is defined by

$$[0]_q! := 1 \quad \text{and} \quad [n]_q! := [1]_q[2]_q\cdots[n]_q \quad \text{if } n\geq 1.$$

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For all nonnegative integers n and k with $n \geq k$, the *Gaussian binomial coefficient* (or *q-binomial coefficient*) $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} .$$

Clearly, when $q = 1$ we have

$$[n]_1 = n, \quad [n]_1! = n! \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}.$$

Throughout the rest of the paper q denotes a positive real number such that $0 < q < 1$. In order to simplify the notation, whenever it is not necessary to mention explicitly q, we write $[n]$, $[n]!$ and $\begin{bmatrix} n \\ k \end{bmatrix}$ instead of $[n]_q$, $[n]_q!$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q$, respectively. Likewise, in order to emphasize the analogy with the classical M_n-operators, we make use (for any positive integer n) of the following notation:

$$(a + b)_q^n := (a + b)(a + qb) \cdots (a + q^{n-1}b).$$

We set also $(a + b)_q^0 := 1$.

For every $n \in \mathbb{N}$ one has (see, for instance, G. E. Andrews, R. Askey, R. Roy [2, Corollary 10.2.2])

$$(1 - x)_q^{n+1} \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix} x^k = 1 \quad \text{for all } x \in [0, 1). \tag{1}$$

Starting from this identity, L. Lupaş [9] introduced a q-generalization of the M_n-operators. The q-Meyer-König and Zeller operators are defined for every $n \in \mathbb{N}$ and every $f \in C[0, 1]$ by

$$\begin{aligned} M_{n,q}f(x) &:= \sum_{k=0}^{\infty} f\left(\frac{[k]}{[n+k]}\right) m_{n,k,q}(x) \quad \text{if } x \in [0, 1), \\ M_{n,q}f(1) &:= f(1), \end{aligned}$$

where

$$m_{n,k,q}(x) := \begin{bmatrix} n+k \\ k \end{bmatrix} x^k (1 - x)_q^{n+1}.$$

The $M_{n,q}$-operators and other similar ones have been intensively investigated in the last decade by many authors (see, for instance, [5], [6], [7], [8], [10], [13], [14], [15], [16], [19], [20], [21]). We merely mention here that in [19, Lemma 2.1] it has been proved that for all $n \in \mathbb{N}$, $n \geq 3$ and all $x \in [0, 1]$ one has

$$M_{n,q}e_2(x) = x^2 + \frac{x(1-x)(1-q^n x)}{[n-1]} - R_{n,q}(x),$$

where

$$0 \leq R_{n,q}(x) \leq \frac{q^{n-1}(1+q)}{[n-1][n-2]} x(1 - x)(1 - qx)(1 - q^n x).$$

Here e_k $(k = 0, 1, 2, \ldots)$ denotes, as usual, the monomial $e_k(t) := t^k$.

2. THE SECOND MOMENT FOR THE Q-MEYER-KÖNIG AND ZELLER OPERATORS

J. A. H. Alkemade [1] was the first who derived an explicit expression for $M_n e_2$ in terms of a hypergeometric series. More precisely, he proved that

$$M_n e_2(x) = x^2 + \tfrac{x(1-x)^2}{n+1}\, {}_2F_1(1,2;n+2;x) \tag{2}$$

for all $n \in \mathbb{N}$ and all $x \in [0,1)$. Moreover, (2) holds also for $x = 1$ if $n \geq 2$. In (2) the notation ${}_2F_1(a,b;c;x)$ is used for the sum of the hypergeometric series

$$\sum_{k=0}^{\infty} \tfrac{(a)_k (b)_k}{(c)_k k!}\, x^k, \tag{3}$$

where

$$(y)_0 := 1 \quad \text{and} \quad (y)_k := y(y+1)\cdots(y+k-1) \quad \text{if } k \geq 1.$$

The series (3) converges for $|x| < 1$ and if $c - a - b > 0$ also for $x = 1$.

The q-analogue of the hypergeometric series (3) is the basic q-hypergeometric series

$$\sum_{k=0}^{\infty} \tfrac{(\alpha;q)_k (\beta;q)_k}{(\gamma;q)_k (q;q)_k}\, x^k, \tag{4}$$

where

$$(y;q)_k := (1-y)(1-qy)\cdots(1-q^{k-1}y) = (1-y)_q^k.$$

The series (4) converges for $|x| < 1$. Its sum is usually denoted by

$${}_2\phi_1\left(\begin{matrix}\alpha,\,\beta\\ \gamma\end{matrix}\;;\,q,\,x\right).$$

Set $[a] := [a]_q := \frac{1-q^a}{1-q}$ for every real number a. Set also

$$[a]_0 := [a]_{0,q} := 1 \quad \text{and} \quad [a]_k := [a]_{k,q} := [a][a+1]\cdots[a+k-1] \quad \text{if } k \geq 1,$$

and note that

$$(q^a;q)_k = (1-q)^k [a]_k$$

for every real number a and every nonnegative integer k. Therefore one has

$${}_2\phi_1\left(\begin{matrix}q^a,\,q^b\\ q^c\end{matrix}\;;\,q,\,x\right) = \sum_{k=0}^{\infty} \tfrac{[a]_k [b]_k}{[c]_k [k]!}\, x^k \quad \text{for all } x \in (-1,1).$$

H. Wang [21, Theorem 1] proved that for every $n \in \mathbb{N}$ and all $x \in [0,1]$ one has

$$M_{n,q} e_2(x) = x^2 + \tfrac{x(1-x)}{[n+1]}\left(1 - \tfrac{q^{n+2}[n]x}{[n+2]}\, {}_2\phi_1\left(\begin{matrix}q,\,q^2\\ q^{n+3}\end{matrix}\;;\,q,\,q^{n+1}x\right)\right). \tag{5}$$

But a simple computation shows that

$$1 - \tfrac{q^{n+2}[n]x}{[n+2]}\, {}_2\phi_1\left(\begin{matrix}q,\,q^2\\ q^{n+3}\end{matrix}\;;\,q,\,q^{n+1}x\right) = (1-q^n x)\, {}_2\phi_1\left(\begin{matrix}q,\,q^2\\ q^{n+2}\end{matrix}\;;\,q,\,q^n x\right). \tag{6}$$

By (5) and (6) we get

$$M_{n,q}e_2(x) = x^2 + \tfrac{x(1-x)(1-q^n x)}{[n+1]}\, {}_2\phi_1\left(\begin{matrix} q,\, q^2 \\ q^{n+2} \end{matrix} ;\, q,\, q^n x\right), \tag{7}$$

a formula which is closer to (2) than (5).

By using (7), one can easily derive estimates for the second moment $M_{n,q}e_2(x) - x^2$. Set

$$\Phi_m(x) := \sum_{k=m}^{\infty} \tfrac{[2]_k}{[n+2]_k}\, x^k$$

for every $m \in \mathbb{N}$ and all $x \in [0,1)$.

THEOREM 1. *For all $n, m \in \mathbb{N}$ with $n \geq 2$ and all $x \in [0, q^{n-1}]$ one has*

$$\Phi_m(x) \leq \tfrac{[m+1]!}{[n-1][n+2]_{m-1}}\, x^m \tag{8}$$

and

$$\sum_{k=0}^{m} \tfrac{[2]_k}{[n+2]_k}\, x^k \leq {}_2\phi_1\left(\begin{matrix} q,\, q^2 \\ q^{n+2} \end{matrix} ;\, q,\, x\right) \leq \tag{9}$$

$$\leq \sum_{k=0}^{m-1} \tfrac{[2]_k}{[n+2]_k}\, x^k + \tfrac{[m+1]!}{[n-1][n+2]_{m-1}}\, x^m.$$

Proof. We have

$$\begin{aligned} \Phi_m(x) &= \tfrac{[m+1]!}{[n+2]_m}\, x^m \sum_{k=m}^{\infty} \tfrac{[m+2]_{k-m}}{[n+m+2]_{k-m}}\, x^{k-m} \\ &= \tfrac{[m+1]!}{[n+2]_m}\, x^m \sum_{k=0}^{\infty} \tfrac{[m+2]_k}{[n+m+2]_k}\, x^k \\ &= \tfrac{[m+1]!}{[n+2]_m}\, x^m\, {}_2\phi_1\left(\begin{matrix} q,\, q^{m+2} \\ q^{n+m+2} \end{matrix} ;\, q,\, x\right) \\ &\leq \tfrac{[m+1]!}{[n+2]_m}\, x^m\, {}_2\phi_1\left(\begin{matrix} q,\, q^{m+2} \\ q^{n+m+2} \end{matrix} ;\, q,\, q^{n-1}\right). \end{aligned}$$

But, for $|\gamma/\alpha\beta| < 1$ it holds that (see [2, Corollary 10.9.2])

$${}_2\phi_1\left(\begin{matrix} \alpha,\, \beta \\ \gamma \end{matrix} ;\, q,\, \gamma/\alpha\beta\right) = \tfrac{(\gamma/\alpha;q)_\infty (\gamma/\beta;q)_\infty}{(\gamma;q)_\infty (\gamma/\alpha\beta;q)_\infty},$$

where

$$(y;q)_\infty := \prod_{k=0}^{\infty} (1 - q^k y).$$

Taking this into account, we have

$${}_2\phi_1\left(\begin{matrix} q,\, q^{m+2} \\ q^{n+m+2} \end{matrix} ;\, q,\, q^{n-1}\right) = \tfrac{(q^{n+m+1};q)_\infty (q^n;q)_\infty}{(q^{n+m+2};q)_\infty (q^{n-1};q)_\infty} = \tfrac{[n+m+1]}{[n-1]}.$$

Consequently,

$$\Phi_m(x) \le \tfrac{[m+1]!}{[n+2]_m} \cdot \tfrac{[n+m+1]}{[n-1]}\, x^m = \tfrac{[m+1]!}{[n-1][n+2]_{m-1}}\, x^m.$$

The left inequality in (9) is obvious, while the right one follows immediately by (8). □

By means of Theorem 1 we deduce the following estimates of the second moment for the $M_{n,q}$-operators. These estimates are quite similar to those obtained by M. Becker and R. J. Nessel [3] for the classical M_n-operators.

COROLLARY 2. *For every $n \in \mathbb{N}$, $n \ge 2$ and all $x \in [0,1]$ one has*

$$\tfrac{x(1-x)(1-q^n x)}{[n+1]}\left(1 + \tfrac{(1+q)q^n x}{[n+2]}\right) \le M_{n,q}e_2(x) - x^2 \le$$
$$\le \tfrac{x(1-x)(1-q^n x)}{[n+1]}\left(1 + \tfrac{(1+q)q^n x}{[n-1]}\right).$$

Proof. By Theorem 1 with $m = 1$ it follows that

$$1 + \tfrac{1+q}{[n+2]}\, q^n x \le {}_2\phi_1\left(\begin{matrix} q,\, q^2 \\ q^{n+2} \end{matrix}\, ;\, q,\, q^n x\right) \le 1 + \tfrac{1+q}{[n-1]}\, q^n x.$$

This inequality and (7) yield the conclusion. □

3. A VORONOVSKAJA-TYPE FORMULA FOR THE Q-MEYER-KÖNIG AND ZELLER OPERATORS

The goal of this section is to establish a Voronovskaja-type formula for the $M_{n,q}$-operators. Such a formula is lacking in the literature. In order to derive it, we need the following auxiliary results whose proofs are postponed to the end of the section.

LEMMA 3. *For every $n \in \mathbb{N}$, $n \ge 3$ and all $x \in [0,1]$ one has*

$$M_{n,q}e_3(x) = xM_{n,q}e_2(x) + \tfrac{2qx^2(1-x)(1-q^n x)}{[n-1]} + R_{n,q}(x), \tag{10}$$

where

$$|R_{n,q}(x)| \le \tfrac{9}{[n-1][n+2]}\,. \tag{11}$$

LEMMA 4. *For every $n \in \mathbb{N}$, $n \ge 3$ and all $x \in [0,1]$ one has*

$$M_{n,q}e_4(x) = xM_{n,q}e_3(x) + \tfrac{3q^2x^3(1-x)(1-q^n x)}{[n-1]} + \widetilde{R}_{n,q}(x),$$

where

$$|\widetilde{R}_{n,q}(x)| \le \tfrac{C}{[n-1][n+3]}\,,$$

C being an absolute constant (i.e., not depending on n, q or x).

We notice that, for a fixed $q \in (0,1)$, the sequence $(M_{n,q}f)_{n\ge1}$ does not converge to f for every $f \in C[0,1]$. For instance, by Corollary 2 it follows that

$$M_{n,q}e_2(x) \to x^2 + (1-q)x(1-x) \quad \text{as } n \to \infty.$$

In order to obtain a convergent sequence of q-Meyer-König and Zeller operators we must replace q by a sequence $(q_n)_{n\geq 1}$ of numbers in $(0,1)$. If $(q_n)_{n\geq 1}$ satisfies

$$(12)\qquad q_n \to 1 \quad \text{and} \quad [n]_{q_n} = 1 + q_n + \cdots + q_n^{n-1} \to \infty \quad \text{as } n \to \infty,$$

then the sequence $(M_{n,q_n} f)_{n\geq 1}$ converges uniformly to f on $[0,1]$ for all $f \in C[0,1]$ (see [9, Theorem 2] or [19, Theorem 2.2]).

In order to obtain a Voronovskaja-type formula for the M_{n,q_n}-operators, $(q_n)_{n\geq 1}$ must satisfy an additional condition, namely

$$(13)\qquad \text{there exists } \lim_{n\to\infty} q_n^n =: \alpha \in [0,1].$$

It is not difficult to construct a sequence $(q_n)_{n\geq 1}$, satisfying both (12) and (13). Indeed, it suffices to take q_n such that

$$1 - \tfrac{1}{n} \leq q_n \leq 1 - \tfrac{1}{n-1} \quad \text{for all } n \geq 3.$$

Then clearly $q_n \to 1$ and $q_n^n \to e^{-1}$ as $n \to \infty$. Moreover, since

$$1 - \tfrac{r}{n} \leq q_n^r \quad \text{for } 1 \leq r \leq n-1,$$

we have

$$[n]_{q_n} \geq n - \tfrac{n(n-1)}{2n} = \tfrac{n+1}{2} \quad \text{for all } n \in \mathbb{N}.$$

We notice also that, if $(q_n)_{n\geq 1}$ is a sequence in $(0,1)$ satisfying (13), then

$$(14)\qquad \lim_{n\to\infty} \tfrac{[n]_{q_n}}{[n-1]_{q_n}} = \lim_{n\to\infty} \tfrac{[n]_{q_n}}{[n+1]_{q_n}} = 1.$$

Indeed, if $\alpha \in [0,1)$, then

$$\tfrac{[n]_{q_n}}{[n-1]_{q_n}} = \tfrac{1-q_n^n}{1-q_n^{n-1}} \to 1 \quad \text{as } n \to \infty$$

and, analogously, $[n]_{q_n}/[n+1]_{q_n} \to 1$ as $n \to \infty$. On the other hand, if $\alpha = 1$, then it is easily seen that

$$\lim_{n\to\infty} \tfrac{[n]_{q_n}}{n} = \lim_{n\to\infty} \tfrac{[n-1]_{q_n}}{n} = \lim_{n\to\infty} \tfrac{[n+1]_{q_n}}{n} = 1,$$

whence (14) holds also in this case.

THEOREM 5. *Let $(q_n)_{n\geq 1}$ be a sequence in $(0,1)$ satisfying (12) and (13). Then for every $x \in [0,1]$ and every function $f \in C[0,1]$ which is twice differentiable at x one has*

$$\lim_{n\to\infty} [n]_{q_n} \Big(M_{n,q_n} f(x) - f(x) \Big) = \tfrac{x(1-x)(1-\alpha x)}{2} f''(x).$$

Proof. For all $n \in \mathbb{N}$ we have

$$M_{n,q_n} e_k(x) = e_k(x), \quad k = 0, 1.$$

By Corollary 2, (12), (13) and (14) it follows that

$$\lim_{n\to\infty} [n]_{q_n} \Big(M_{n,q_n} e_2(x) - x^2 \Big) = x(1-x)(1-\alpha x).$$

Further, let ψ_x be the function defined by $\psi_x(t) := t - x$. We have

$$\left(M_{n,q_n}\psi_x^4\right)(x) = M_{n,q_n}e_4(x) - 4xM_{n,q_n}e_3(x) + 6x^2M_{n,q_n}e_2(x) - 3x^4.$$

By Lemma 3, Lemma 4, (12), (13) and (14) it follows that

$$\lim_{n\to\infty}[n]_{q_n}\left(M_{n,q_n}\psi_x^4\right)(x) = 0.$$

Now the conclusion of the theorem is an immediate consequence of a standard result (for instance [11, Theorem 3] or [17, Theorem 1]). □

Proof of Lemma 3. Clearly, the assertion holds for $x = 1$. For $x \in [0,1)$ we have

$$\begin{aligned} M_{n,q}e_3(x) &= (1-x)_q^{n+1}\sum_{k=1}^{\infty}\begin{bmatrix}n+k\\k\end{bmatrix}x^k\tfrac{[k]^3}{[n+k]^3} \\ &= x(1-x)_q^{n+1}\sum_{k=0}^{\infty}\begin{bmatrix}n+k\\k\end{bmatrix}x^k\tfrac{[k+1]^2}{[n+k+1]^2}\,. \end{aligned}$$

Following P. C. Sikkema [18] in his proof of Theorem 3, we notice that

$$\tfrac{[k+1]^2}{[n+k+1]^2} = \tfrac{[k]^2}{[n+k]^2} + \tfrac{2q^k[n][k]}{[n+k]^2[n+k+1]} + \tfrac{q^{2k}[n]^2}{[n+k]^2[n+k+1]^2}\,.$$

Taking this into account, we deduce that

$$M_{n,q}e_3(x) = xM_{n,q}e_2(x) + S_{n,q}(x) + T_{n,q}(x), \tag{15}$$

where

$$\begin{aligned} S_{n,q}(x) &:= 2[n]x(1-x)_q^{n+1}\sum_{k=1}^{\infty}\begin{bmatrix}n+k\\k\end{bmatrix}(qx)^k\tfrac{[k]}{[n+k]^2[n+k+1]}\,, \\ T_{n,q}(x) &:= x(1-x)_q^{n+1}\sum_{k=0}^{\infty}\begin{bmatrix}n+k\\k\end{bmatrix}(q^2x)^k\tfrac{[n]^2}{[n+k]^2[n+k+1]^2}\,. \end{aligned}$$

Since

$$S_{n,q}(x) = 2[n]qx^2(1-x)_q^{n+1}\sum_{k=0}^{\infty}\begin{bmatrix}n+k\\k\end{bmatrix}(qx)^k\tfrac{1}{[n+k+1][n+k+2]}$$

and

$$\tfrac{1}{[n+k+1]} = \tfrac{1}{[n+k]} - \tfrac{q^{n+k}}{[n+k][n+k+1]}\,,$$

it follows that

$$S_{n,q}(x) = U_{n,q}(x) - V_{n,q}(x), \tag{16}$$

where

$$U_{n,q}(x) := 2qx^2(1-x)_q^{n+1}\sum_{k=0}^{\infty}\left[{n+k \atop k}\right]\frac{(qx)^k\,[n]}{[n+k][n+k+2]}\,,$$

$$V_{n,q}(x) := 2q^{n+1}x^2(1-x)_q^{n+1}\sum_{k=0}^{\infty}\left[{n+k \atop k}\right]\frac{(q^2x)^k\,[n]}{[n+k][n+k+1][n+k+2]}\,.$$

Since

$$U_{n,q}(x) = 2qx^2(1-x)_q^{n+1}\sum_{k=0}^{\infty}\left[{n-1+k \atop k}\right](qx)^k\frac{1}{[n+k+2]}$$

and

$$\frac{1}{[n+k+2]} = \frac{1}{[n-1+k]} - \frac{[3]q^{n-1+k}}{[n-1+k][n+k+2]}\,,$$

we deduce that

$$(17)\qquad \begin{aligned} U_{n,q}(x) &= \frac{2qx^2(1-x)(1-q^nx)(1-qx)_q^{n-1}}{[n-1]}\sum_{k=0}^{\infty}\left[{n-2+k \atop k}\right](qx)^k \\ &\quad -W_{n,q}(x) \\ &= \frac{2qx^2(1-x)(1-q^nx)}{[n-1]} - W_{n,q}(x), \end{aligned}$$

where

$$W_{n,q}(x) := 2[3]q^nx^2(1-x)_q^{n+1}\sum_{k=0}^{\infty}\left[{n-1+k \atop k}\right]\frac{(q^2x)^k}{[n-1+k][n+k+2]}\,.$$

By (15), (16) and (17) we get

$$(18)\qquad \begin{aligned} M_{n,q}e_3(x) &= xM_{n,q}e_2(x) + \frac{2qx^2(1-x)(1-q^nx)}{[n-1]} \\ &\quad -W_{n,q}(x) - V_{n,q}(x) + T_{n,q}(x). \end{aligned}$$

Now we have

$$(19)\qquad \begin{aligned} 0 \le W_{n,q}(x) &\le \frac{6}{[n-1][n+2]}\,(1-x)_q^{n+1}\sum_{k=0}^{\infty}\left[{n-1+k \atop k}\right](q^2x)^k \\ &\le \frac{6}{[n-1][n+2]}\,(1-x)_q^{n}\sum_{k=0}^{\infty}\left[{n-1+k \atop k}\right]x^k \\ &= \frac{6}{[n-1][n+2]}\,, \end{aligned}$$

$$(20)\qquad \begin{aligned} 0 \le V_{n,q}(x) &\le \frac{2}{[n+1][n+2]}\,(1-x)_q^{n+1}\sum_{k=0}^{\infty}\left[{n-1+k \atop k}\right](q^2x)^k \\ &\le \frac{2}{[n+1][n+2]}\,(1-x)_q^{n}\sum_{k=0}^{\infty}\left[{n-1+k \atop k}\right]x^k \\ &= \frac{2}{[n+1][n+2]}\,, \end{aligned}$$

and

$$0 \le T_{n,q}(x) \le \frac{1}{[n+1]^2}(1-x)_q^{n+1}\sum_{k=0}^{\infty}\begin{bmatrix} n+k \\ k \end{bmatrix}(q^2x)^k \tag{21}$$
$$\le \frac{1}{[n+1]^2}(1-x)_q^{n+1}\sum_{k=0}^{\infty}\begin{bmatrix} n+k \\ k \end{bmatrix}x^k$$
$$= \frac{1}{[n+1]^2}\,.$$

By (18), (19), (20) and (21) we conclude that (10) and (11) hold. □

Proof of Lemma 4. Since this proof is similar to that of Lemma 3 and follows the same lines, we omit it. □

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