REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 40 (2011) no. 2, pp. 107–119
ictp.acad.ro/jnaat

SEMILOCAL CONVERGENCE CONDITIONS FOR THE SECANT METHOD, USING RECURRENT FUNCTIONS

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Abstract. Using our new concept of recurrent functions, we present new sufficient convergence conditions for the secant method to a locally unique solution of a nonlinear equation in a Banach space. We combine Lipschitz and center–Lipschitz conditions on the divided difference operator to obtain the semilocal convergence analysis of the secant method. Our error bounds are tighter than earlier ones. Moreover, under our convergence hypotheses, we can expand the applicability of the secant method in cases not covered before [8], [9], [12]–[14], [16], [19]–[21]. Application and examples are also provided in this study.

MSC 2000. 65H10, 65B05, 65G99, 65N30, 47H17, 49M15.
Keywords. Recurrent functions, semilocal convergence, secant method, Banach space, majorizing sequence, divided difference, Fréchet derivative.

1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet–differentiable operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

The field of computational sciences has seen a considerable development in mathematics, engineering sciences, and economic equilibrium theory. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = T(x)$, for some suitable operator T, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations),

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vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative–when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework. We note that in computational sciences, the practice of numerical analysis for finding such solutions is essentially connected to variants of Newton's method.

We consider the secant method (SM) in the form

$$x_{n+1} = x_n - \delta F(x_{n-1}, x_n)^{-1}\, F(x_n) \quad (n \geq 0), \quad (x_{-1}, x_0 \in \mathcal{D}) \tag{1.2}$$

where, $\delta F(x, y) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ $(x, y \in \mathcal{D})$ is a consistent approximation of the Fréchet–derivative of F [4], [11]. $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ denotes the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$. (SM) is an alternative method of Newton's method (NM)

$$x_{n+1} = x_n - F'(x_n)^{-1}\, F(x_n), \quad (n \geq 0), \quad (x_0 \in \mathcal{D}).$$

Dennis [8], Potra [14]–[16], Bosarge and Falb [7], Hernández, M.J. Rubio and J.A. Ezquerro [9], Argyros [4], and others [10], [13], [20], have provided sufficient convergence conditions for (SM) based on Lipschitz–type conditions on δF (see, also relevant works in [1], [2], [5], [12], [17], [18], [19], [21]).

In the previous mentioned references, the conditions usually associated with the semilocal convergence of secant method (1.2) are:

($\mathcal{H}_1$) F is a nonlinear operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$;

($\mathcal{H}_2$) x_{-1} and x_0 are two points belonging to the interior $\mathcal{D}^0$ of $\mathcal{D}$ and satisfying the inequality

$$\| x_0 - x_{-1} \| \leq c;$$

($\mathcal{H}_3$) F is Fréchet–differentiable on $\mathcal{D}^0$, and there exists an operator $\delta F : \mathcal{D}^0 \times \mathcal{D}^0 \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$, such that the linear operator $A = \delta F(x_{-1}, x_0)$ is invertible, its inverse A^{-1} is bounded, and:

$$\| A^{-1}\, F(x_0) \| \leq \eta;$$

$$\| A^{-1}\, (\delta F(x, y) - F'(z)) \| \leq \ell\, (\| x - z \| + \| y - z \|), \quad \text{for all} \quad x, y, z \in \mathcal{D};$$

$$\overline{U}(x_0, r) = \{x \in \mathcal{X} : \| x - x_0 \| \leq r\} \subseteq \mathcal{D}^0,$$

for some $r > 0$ depending on ℓ, c, and η; and

$$\ell\, c + 2\, \sqrt{\ell\, \eta} \leq 1. \tag{1.3}$$

The sufficient convergence condition (1.3) is easily violated. Indeed, let $\ell = 1$, $\eta = .18$, and $c = .185$. Then, (1.3) does not holds, since

$$\ell\ c + 2\ \sqrt{\ell\ \eta} = 1.033528137.$$

Moreover, our recently found corresponding conditions are also violated [6] (see, Remark 4(c)). Hence, there is not guarantee that equation (1.1) under the information (ℓ, c, η) has a solution that can be found using (SM). In this study we are motivated by optimization considerations, and the above observation.

Here, using a combination of Lipschitz and center–Lipschitz conditions, we provide a semilocal convergence analysis for (SM). Our error bounds are tighter, and our convergence conditions hold in cases where the corresponding hypotheses in earlier references [8], [9], [12]–[14], [16], [19]–[21] are violated. Applications and examples are also provided in this study.

2. SEMILOCAL CONVERGENCE ANALYSIS OF (SM)

We need the following result on majorizing sequences for (SM).

LEMMA 1. *Let $\ell_0 > 0$, $\ell > 0$, $c \geq 0$, and $\eta \in (0, c]$ be given constants. Assume:*

$$(\ell + 2\ \ell_0)\ \eta + \ell_0\ c < 1; \tag{2.1}$$

for

$$\delta_0 = \frac{\ell\ (c+\eta)}{1-\ell_0\ (c+\eta)},$$

$$\delta = \frac{2\,(1-((\ell+2\,\ell_0)\,\eta+\ell_0\,c))}{(\ell+2\,\ell_0)\,\eta+\sqrt{((\ell+2\,\ell_0)\,\eta)^2+4\,\ell_0\,\eta\,(1-((\ell+2\,\ell_0)\,\eta+\ell_0\,c))}},$$

δ_1 the unique positive root of polynomial f in $(0, 1)$

$$f(t) = \ell_0\ t^3 + (\ell_0 + \ell)\ t^2 - \ell, \tag{2.2}$$

given by

$$\delta_1 = \frac{1}{2}\ \frac{-\ell+\sqrt{\ell^2+4\,\ell\,\ell_0}}{\ell_0}, \tag{2.3}$$

and

$$\delta_0 \leq \delta \leq \delta_1. \tag{2.4}$$

Then, scalar sequence $\{t_n\}$ $(n \geq -1)$ given by

$$t_{-1} = 0,\ t_0 = c,\ t_1 = c + \eta,\ t_{n+2} = t_{n+1} + \frac{\ell\ (t_{n+1}-t_{n-1})\ (t_{n+1}-t_n)}{1-\ell_0\ (t_{n+1}-t_0+t_n)} \tag{2.5}$$

is non–decreasing, bounded from above by

$$t^{\star\star} = \frac{\eta}{1-\delta} + c, \tag{2.6}$$

and converges to its unique least upper bound $t^\star$ such that

$$0 \leq t^\star \leq t^{\star\star}. \tag{2.7}$$

Moreover, the following estimates hold for all $n \geq 0$:

$$0 \leq t_{n+2} - t_{n+1} \leq \delta\ (t_{n+1} - t_n) \leq \delta^{n+1}\ \eta. \tag{2.8}$$

Proof. In view of (2.1), we have $\delta_0 \in [0,1)$, and $\delta > 0$.

We shall show using mathematical induction on $k \geq 0$

$$0 \leq t_{k+2} - t_{k+1} \leq \delta\ (t_{k+1} - t_k). \tag{2.9}$$

By (2.5) for $k = 0$, we must show

$$0 \leq \frac{\ell\ (t_1 - t_{-1})}{1 - \ell_0\ t_1} \leq \delta \quad \text{or} \quad 0 \leq \frac{\ell\ (c+\eta)}{1 - \ell_0\ (c+\eta)} \leq \delta,$$

which is true from (2.1), and the choice of $\delta \geq \delta_0$.

Let assume that (2.9) holds for $k \leq n+1$. The induction hypothesis gives

$$\begin{aligned} t_{k+2} &\leq t_{k+1} + \delta\ (t_{k+1} - t_k) \\ &\leq t_k + \delta\ (t_k - t_{k-1}) + \delta\ (t_{k+1} - t_k) \\ &\leq t_1 + \delta\ (t_1 - t_0) + \cdots + \delta\ (t_{k+1} - t_k) \\ &\leq c + \eta + \delta\ \eta + \cdots + \delta^{k+1}\ \eta \\ &= c + \frac{1-\delta^{k+2}}{1-\delta}\ \eta < \frac{\eta}{1-\delta} + c = t^{\star\star}. \end{aligned} \tag{2.10}$$

Moreover, we have:

(2.11)

$$\begin{aligned} &\ell\ (t_{k+2} - t_k) + \delta\ \ell_0\ (t_{k+2} - t_0 + t_{k+1}) \\ &\leq \ell\ \Big((t_{k+2} - t_{k+1}) + (t_{k+1} - t_k)\Big) + \delta\ \ell_0\ \left(\frac{1-\delta^{k+2}}{1-\delta} + \frac{1-\delta^{k+1}}{1-\delta}\right)\ \eta + \delta\ \ell_0\ c \\ &\leq \ell\ (\delta^k + \delta^{k+1})\ \eta + \frac{\delta\ \ell_0}{1-\delta}\ (2 - \delta^{k+1} - \delta^{k+2})\ \eta + \delta\ \ell_0\ c. \end{aligned}$$

We prove now (2.9). By (2.5), we can have estimate

$$\ell\ (\delta^k + \delta^{k+1})\ \eta + \frac{\delta\ \ell_0}{1-\delta}\ (2 - \delta^{k+1} - \delta^{k+2})\ \eta + \delta\ \ell_0\ c \leq \delta \tag{2.12}$$

or

(2.13)

$$\ell\ (\delta^{k-1} + \delta^k)\ \eta + \ell_0\ \Big((1 + \delta + \cdots + \delta^k) + (1 + \delta + \cdots + \delta^{k+1})\Big)\ \eta + \ell_0\ c - 1 \leq 0.$$

In view of (2.13), we are motivated to define (for $\delta = s$) the functions for $k \geq 1$

$$f_k(s) = \ell\ (s^{k-1} + s^k)\ \eta + \ell_0\ \Big(2\ (1 + s + \cdots + s^k) + s^{k+1}\Big)\ \eta + \ell_0\ c - 1. \tag{2.14}$$

We need the relationship between two consecutive functions f_k. Using (2.14), we obtain

(2.15)

$$\begin{aligned} f_{k+1}(s) &= \ell\ (s^k + s^{k+1})\ \eta + \ell_0\ \Big(2\ (1 + s + \cdots + s^{k+1}) + s^{k+2}\Big)\ \eta + \ell_0\ c - 1 \\ &= f_k(s) + \ell\ (s^{k+1} - s^{k-1})\ \eta + \ell_0\ (s^{k+1} + s^{k+2})\ \eta \\ &= f(s)\ s^{k-1}\ \eta + f_k(s). \end{aligned}$$

Note that δ is the unique positive root of polynomial f_1.

Instead of (2.13), we shall show

$$f_k(\delta) \leq 0 \qquad (k \geq 0). \tag{2.16}$$

Estimate (2.16) holds for $k=1$, as equality. Using (2.15), we get in turn

$$f_2(\delta) = f_1(\delta) + f(\delta)\,\delta\,\eta \leq 0,$$

since,

$$f_1(\delta) = 0, \quad f(\delta) \leq 0 \quad \text{(by (2.2), and } \delta_1 \geq \delta).$$

Assume (2.16) holds for $m \leq k$. Then, again by (2.15), we have:

$$f_{k+1}(\delta) = f_k(\delta) + f(\delta)\,\delta^{k-1}\,\eta \leq 0,$$

which completes the induction for (2.16).

Moreover, define function f_∞ on $[0,1)$ by

$$f_\infty(s) = \lim_{k\longrightarrow\infty} f_k(s). \tag{2.17}$$

Using (2.16), and (2.17), we obtain

$$f_\infty(\delta) = \lim_{k\longrightarrow\infty} f_k(\delta) \leq 0. \tag{2.18}$$

Hence, we showed sequence $\{t_n\}$ $(n \geq -1)$ is non–decreasing and bounded above from by $t^{\star\star}$, so that (2.8) holds. It follows that there exists $t^\star \in [0, t^{\star\star}]$, so that $\lim_{n\longrightarrow\infty} t_n = t^\star$.

That completes the proof of Lemma 1. □

We shall study (SM) for triplets (F, x_{-1}, x_0) belonging to the class $\mathcal{C}(\ell, \ell_0, \eta, c, \delta)$ defined as follows:

Definition 2. *Let ℓ, ℓ_0, η, c, δ be non–negative constants satisfying the hypotheses of Lemma 1. A triplet (F, x_{-1}, x_0) belongs to the class $\mathcal{C}(\ell, \ell_0, \eta, c, \delta)$ if:*

- $(\mathcal{A}_1)$ *F is a nonlinear operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$;*
- $(\mathcal{A}_2)$ *x_{-1} and x_0 are two points belonging to the interior $\mathcal{D}^0$ of $\mathcal{D}$ and satisfying the inequality*

  $$\| x_0 - x_{-1} \| \leq c;$$

- $(\mathcal{A}_3)$ *F is Fréchet–differentiable on $\mathcal{D}^0$, and there exists an operator $\delta F : \mathcal{D}^0 \times \mathcal{D}^0 \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$, such that, $A^{-1} = \delta F(x_{-1}, x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, and for all $x, y, z \in \mathcal{D}$, the following hold*

  $$\| A^{-1}\, F(x_0) \| \leq \eta,$$

  $$\| A^{-1}\, (\delta F(x,y) - F'(z)) \| \leq \ell\, (\| x - z \| + \| y - z \|),$$

  and

  $$\| A^{-1}\, (\delta F(x,y) - F'(x_0)) \| \leq \ell_0\, (\| x - x_0 \| + \| y - x_0 \|);$$

- $(\mathcal{A}_4)$ *$\overline{U}(x_0, t^\star) \subseteq \mathcal{D}_c = \{x \in \mathcal{D} \,:\, F$ is continuous at $x\} \subseteq \mathcal{D}$, where, $t^\star$ is given in Lemma 1;*

($\mathcal{A}_5$) *For*

$$\delta_2 = \frac{2\,(1-\ell_0\,(c+\eta))}{1-\ell_0\,c},$$

we have

$$\begin{cases} \delta \leq \delta_1 & \text{if} \quad \delta_2 < \delta_1 \\ \delta < \delta_2 & \text{if} \quad \delta_1 \leq \delta_2. \end{cases}$$

The semilocal convergence theorem for (SM) is as follows.

THEOREM 3. *If* $(F, x_{-1}, x_0) \in \mathcal{C}(\ell, \ell_0, \eta, c, \delta)$, *then, the sequence* $\{x_n\}$ $(n \geq -1)$ *generated by (SM) is well defined, remains in* $\overline{U}(x_0, t^\star)$ *for all* $n \geq 0$, *and converges to a unique solution* $x^\star \in \overline{U}(x_0, t^\star)$ *of* (1.1).

Moreover the following estimates hold for all $n \geq 0$

$$\| \, x_n - x_{n-1} \, \| \leq t_n - t_{n-1}, \tag{2.19}$$

and

$$\| \, x_n - x^\star \, \| \leq t^\star - t_n \tag{2.20}$$

where, $\{t_n\}$, $(n \geq 0)$ *is given by* (2.5).

Furthermore, if there exists $R > 0$, *such that*

$$R \geq t^\star - t_0 \quad \text{and} \quad \ell_0 \left(\frac{\eta}{1-\delta} + c + R \right) \leq 1, \tag{2.21}$$

then, the solution $x^\star$ *is unique in* $U(x_0, R)$.

Proof. First, we show that $L = \delta F(x_k, x_{k+1})$ is invertible for $x_k, x_{k+1} \in \overline{U}(x_0, t^\star)$. By (2.5), (2.6), ($\mathcal{A}_2$) and ($\mathcal{A}_3$), we have

$$\begin{aligned} \| \, I - A^{-1}\,L \, \| &= \| \, A^{-1}\,(L - A) \, \| \\ &\leq \| \, A^{-1}(L - F'(x_0)) \, \| + \| \, A^{-1}(F'(x_0) - A) \, \| \\ &\leq \ell_0\,(\| \, x_k - x_0 \, \| + \| \, x_{k+1} - x_0 \, \| + \| \, x_0 - x_{-1} \, \|) \\ &\leq \ell_0\,(t_k - t_0 + t_{k+1} - t_0 + c) \\ &\leq \ell_0\,(t^\star - t_0 + t^\star - t_0 + c) \\ &\leq \ell_0 \left(2 \left(\frac{\eta}{1-\delta} + c \right) - c \right) < 1, \end{aligned} \tag{2.22}$$

by ($\mathcal{A}_5$).

Using the Banach Lemma on invertible operators [4], [11], and (2.22), L is invertible and

$$\| \, L^{-1}\,A \, \| \leq \left(1 - \ell_0\,(\| \, x_k - x_0 \, \| + \| \, x_{k+1} - x_0 \, \| + c) \right)^{-1}. \tag{2.23}$$

By ($\mathcal{A}_3$), we have

$$\| \, A^{-1}\,(F'(u) - F'(v)) \, \| \leq 2\,\ell \, \| \, u - v \, \|, \quad u, v \in \mathcal{D}^0. \tag{2.24}$$

We can write the identity

$$F(x) - F(y) = \int_0^1 F'(y + t(x-y))\,\mathrm{d}t\,(x-y) \tag{2.25}$$

then, for all $x, y, u, v \in \mathcal{D}^0$, we obtain

$$\begin{aligned} &\| A_0^{-1}\,(F(x) - F(y) - F'(u)(x-y))\,\| \\ &\le \ell\,(\| x-u \| + \| y-u \|)\ \| x-y \| \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} &\| A_0^{-1}\,(F(x) - F(y) - \delta F(u,v)\,(x-y))\,\| \\ &\le \ell\,(\| x-v \| + \| y-v \| + \| u-v \|)\,\| x-y \|\,. \end{aligned} \tag{2.27}$$

By a continuity argument (2.24)–(2.27) remain valid if x and/or y belong to $\mathcal{D}_c$.

Now we show (2.19). If (2.19) holds for all $n \le k$ and if $\{x_n\}$ $(n \ge 0)$ is well defined for $n = 0, 1, 2, \cdots, k$, then

$$\| x_n - x_0 \| \le t_n - t_0 < t^\star - t_0, \quad n \le k. \tag{2.28}$$

That is (1.2) is well defined for $n = k+1$. For $n = -1$, and $n = 0$, (2.19) reduces to $\| x_{-1} - x_0 \| \le c$, and $\| x_0 - x_1 \| \le \eta$. Suppose (2.19) holds for $n = -1, 0, 1, \cdots, k$ $(k \ge 0)$. By (2.23), (2.27), and

$$F(x_{k+1}) = F(x_{k+1}) - F(x_k) - \delta F(x_{k-1}, x_k)\ (x_{k+1} - x_k) \tag{2.29}$$

we obtain the following estimate

$$\begin{aligned} \| x_{k+2} - x_{k+1} \| &= \| \delta F(x_k, x_{k+1})^{-1}\, F(x_{k+1}) \| \\ &\le \| \delta F(x_k, x_{k+1})^{-1}\, A \|\, \| A^{-1}\, F(x_{k+1}) \| \\ &\le \frac{\ell\,(\|x_{k+1}-x_k\| + \|x_k - x_{k-1}\|)}{1-\ell_0\,(\|x_{k+1}-x_0\| + \|x_k - x_0\| + c)}\ \| x_{k+1} - x_k \| \\ &\le \frac{\ell\,(t_{k+1}-t_k+t_k-t_{k-1})}{1-\ell_0\,(t_{k+1}-t_0+t_k-t_0+t_0-t_{-1})}\ (t_{k+1} - t_k) \\ &= t_{k+2} - t_{k+1}, \end{aligned} \tag{2.30}$$

and the induction for (2.19) is completed. It follows from (2.19), and Lemma 1 that$\{x_n\}$ $(n \ge -1)$ is Cauchy in a Banach space $\mathcal{X}$, and as such it converges to some $x^\star \in \overline{U}(x_0, t^\star)$ (since $\overline{U}(x_0, t^\star)$ is a closed set). By letting $k \to \infty$ in (2.30), we obtain $F(x^\star) = 0$. Estimate (2.20) follows from (2.19) by using standard majoration techniques [4], [5], [11].

Finally, for showing the uniqueness in $\overline{U}(x_0, t^\star)$, let $y^\star \in \overline{U}(x_0, t^\star)$ be a solution (1.1). Set

$$\mathcal{M} = \int_0^1 F'(y^\star + t\ (y^\star - x^\star))\ \mathrm{d}t.$$

It then follows by $(\mathcal{A}_3)$, and $(\mathcal{A}_5)$:

(2.31)

$$\begin{aligned} \| A^{-1}(A-\mathcal{M}) \| &= \ell_0 (\| y^\star - x_0 \| + \| x^\star - x_0 \| + \| x_0 - x_{-1} \|) \\ &\leq \ell_0 ((t^\star - t_0) + (t^\star - t_0) + t_0) \\ &\leq \ell_0 \left(2 \left(\frac{\eta}{1-\delta} + c \right) - c \right) = \ell_0 \left(\frac{2\,\eta}{1-\delta} + c \right) < 1. \end{aligned}$$

It follows from (2.31), and the Banach lemma on invertible operators that $\mathcal{M}^{-1}$ exists on $U(x_0, t^\star)$.

Using the identity:

$$F(x^\star) - F(y^\star) = \mathcal{M}\ (x^\star - y^\star) \tag{2.32}$$

we deduce $x^\star = y^\star$. Finally, we shall show uniqueness in $U(x_0, R)$. As in (2.31), we arrive at

$$\| A^{-1}\ (A-\mathcal{M}) \| < \ell_0 \left(\frac{\eta}{1-\delta} + c + R \right) \leq 1,$$

by (2.21)–(2.23).

That completes the proof of Theorem 3. □

Remark 4. (a) The point $t^{\star\star}$ given in closed form by (2.6) can replace $t^\star$ in Theorem 3.

(b) If we impose the condition $(5\,\ell_0 + 2\,\ell)\,\eta + \ell_0\,c > 1$, hence $\delta \in (0,1)$. Moreover, if

$$f(\delta) \leq 0,$$

then,

$$\delta \leq \delta_1$$

(see also part (c) that follows).

(c) Returning back to the example given in the introduction, say $\ell_0 = .9$, we obtain $\delta_0 = .543559196$, $\delta = .554824435$, $\delta_1 = .6359783661$, whereas (2.1) holds, since $.6705 < 1$. That is our results can apply, whereas the ones using (1.3) cannot.

Let us define δ_∞ by

$$\delta_\infty = \frac{1-\ell_0\,(c+2\,\eta)}{1-\ell_0\,c}.$$

The corresponding to (2.4) condition in [6] is given by

$$\delta_0 \leq \delta_1 \leq \delta_\infty.$$

But we have

$$\delta_\infty = .611277744 < \delta_1 = .6359783661.$$

Hence, again the results in this study apply, but not the ones in our study [6].

That is the sufficient convergence conditions in this study are different from ones in [6]. We conclude that as far as the convergence domains go, in practice we shall test all of them, to see which one (if any) applies.

□

REMARK 5. (a) Let us define the majoring sequence $\{w_n\}$ used in [8], [9], [12]–[14], [16], [19]–[21] (under condition (1.3)):

$$w_{-1} = 0,\ w_0 = c,\ w_1 = c+\eta,\ w_{n+2} = w_{n+1} + \frac{\ell\,(w_{n+1}-w_{n-1})\,(w_{n+1}-w_n)}{1-\ell\,(w_{n+1}-w_0+w_n)}. \tag{2.33}$$

Note that in general

$$\ell_0 \le \ell \tag{2.34}$$

holds , and $\frac{\ell}{\ell_0}$ can be arbitrarily large [2], [4]. In the case $\ell_0 = \ell$, then $t_n = w_n$ $(n \ge -1)$. Otherwise:

$$t_n < w_n, \quad t_{n+1} - t_n \le w_{n+1} - w_n, \tag{2.35}$$

and

$$0 \le t^\star - t_n \le w^\star - w_n, \quad w^\star = \lim_{n\longrightarrow\infty} w_n. \tag{2.36}$$

Note also that strict inequality holds in (2.35) for $n \ge 1$, if $\ell_0 < \ell$.

The proof of (2.35), (2.36) can be found in [4]. Note that the only difference in the proofs is that the conditions of Lemma 1 are used here, instead of the ones in [3]. However this makes no difference between the proofs.

Finally, note that (1.3) is the sufficient convergence condition for sequence (2.33).

(b) It turns out from the proof of Theorem 3 that $\{v_n\}$ given by

$$v_{-1} = 0,\ v_0 = c,\ v_1 = c+\eta,\ v_{n+2} = v_{n+1} + \frac{\ell_1\,(v_{n+1}-v_{n-1})\,(v_{n+1}-v_n)}{1-\ell_0\,(v_{n+1}-v_0+v_n)}, \tag{2.37}$$

where,

$$\ell_1 = \begin{cases} \ell_0 & \text{if} \quad n = 0 \\ \ell & \text{if} \quad n > 0 \end{cases}$$

is a finer majorizing sequence for $\{x_n\}$ than $\{t_n\}$, if $\ell_0 < \ell$.

Moreover, we have

$$v_n < t_n, \quad v_{n+1} - v_n < t_{n+1} - t_n, \tag{2.38}$$

and

$$0 \le v^\star - v_n \le t^\star - t_n, \quad v^\star = \lim_{n\longrightarrow\infty} v_n. \tag{2.39}$$

□

3. EXAMPLES

In this section, we present some numerical examples.

Example 6. In the following table, we validate our Remark 4(c) and 5(b).

Comparison table.

n	(2.5) $t_{n+1} - t_n$	(2.33) $w_{n+1} - w_n$	(2.37) $v_{n+1} - v_n$
1	.0978406552	.1034645669	.0880565897
2	.0645024008	.0834298221	.0548615926
3	.0380319604	.0947064307	.0259951222
4	.0213029402	-1.250121362	.0091844655
5	.0097492108	.0578542134	.0016385499
6	.0029765775	.0165794721	.0000946072
7	.0004197050	-.1376680823	8.821×10^{-7}
8	.0000163476	-.5014996853	5×10^{-10}
9	8.21×10^{-8}	.5290112928	0

The table shows that our error bounds $v_{n+1} - v_n$, and $t_{n+1} - t_n$ are finer than $w_{n+1} - w_n$ given in [8], [9], [12]–[14], [16], [19]–[21]. □

Example 7. Define the scalar function F by $F(x) = c_0\, x + c_1 + c_2 \sin e^{c_3\, x}$, $x_0 = 0$, where c_i, $i = 0, 1, 2, 3$ are given parameters. Define linear operator $\delta F(x, y)$ by

$$\delta F(x,y) = \int_0^1 F'(y + t\,(x-y))\,\mathrm{d}t = c_0 + c_2\, \tfrac{\sin e^{c_3\,x} - \sin e^{c_3\,y}}{x-y}.$$

Then it can easily be seen that for c_3 large and c_2 sufficiently small, $\frac{\ell}{\ell_0}$ can be arbitrarily large. That is (2.4) may be satisfied but not (1.3). □

Example 8. [4] **(Newton's method case)** Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0,1]$ be the space of real–valued continuous functions defined on the interval $[0,1]$, equipped with the max–norm $\| \,.\, \|$. Let $\theta \in [0,1]$ be a given parameter. Consider the "Cubic" Chandrasekhar integral equation

$$u(s) = u^3(s) + \lambda\, u(s) \int_0^1 q(s,t)\, u(t)\,\mathrm{d}t + y(s) - \theta. \tag{3.1}$$

Here the kernel $q(s,t)$ is a continuous function of two variables defined on $[0,1] \times [0,1]$. The parameter λ in (3.1) is a real number called the "albedo" for scattering, and $y(s)$ is a given continuous function defined on $[0,1]$ and $x(s)$ is the unknown function sought in $\mathcal{C}[0,1]$. For simplicity, we choose $u_0(s) = y(s) = 1$, and $q(s,t) = \frac{s}{s+t}$, for all $s \in [0,1]$, and $t \in [0,1]$, with $s + t \neq 0$. If we let $\mathcal{D} = U(u_0, 1-\theta)$, and define the operator F on $\mathcal{D}$ by

$$F(x)(s) = x^3(s) - x(s) + \lambda\, x(s) \int_0^1 q(s,t)\, x(t)\,\mathrm{d}t + y(s) - \theta, \tag{3.2}$$

for all $s \in [0,1]$, then every zero of F satisfies equation (3.1).

We have the estimates

$$\max_{0\le s\le 1} \Big|\int_0^1 \tfrac{s}{s+t}\,\mathrm{d}t\Big| = \ln 2.$$

Therefore, if we set $\xi = \| F'(u_0)^{-1} \|$, then the hypotheses of Theorem 3 (see $(\mathcal{A}_3)$) correspond to the usual Lipschitz and center–Lipschitz conditions for (NM) (see [6, Theorem 3.4]), such that

$$\eta = \xi\,(|\lambda|\,\ln 2 + 1 - \theta),$$

$$\ell = 2\,\xi\,\left(|\lambda|\,\ln 2 + 3\,(2-\theta)\right) \quad \text{and} \quad \ell_0 = \xi\,\left(2\,|\lambda|\,\ln 2 + 3\,(3-\theta)\right).$$

It follows from an equivalent Theorem for (NM) to Theorem 3 that if condition

$$h_A = \tfrac{1}{8}\left(\ell + 4\,\ell_0 + \sqrt{\ell^2 + 8\,\ell\,\ell_0}\right)\eta \le \tfrac{1}{2},$$

holds, then problem (3.1) has a unique solution near u_0. This assumption is weaker than the one given before using the Newton–Kantorovich hypothesis. Note also that $\ell_0 < \ell$ for all $\theta \in [0,1]$. □

Example 9. **(secant method case)** Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0,1]$, equipped with the norm $\| x \| = \max_{0\le s\le 1} |x(s)|$. Consider the following nonlinear boundary value problem [4]

$$\begin{cases} u'' = -u^3 - \gamma\, u^2 \\ u(0) = 0, \quad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$u(s) = s + \int_0^1 Q(s,t)\,\left(u^3(t) + \gamma\, u^2(t)\right)\mathrm{d}t \tag{3.3}$$

where, Q is the Green function:

$$Q(s,t) = \begin{cases} t\,(1-s), & t \le s \\ s\,(1-t), & s < t. \end{cases}$$

We observe that

$$\max_{0\le s\le 1} \int_0^1 |Q(s,t)|\,\mathrm{d}t = \tfrac{1}{8}.$$

Then problem (3.3) is in the form (1.1), where, $F : \mathcal{D} \longrightarrow \mathcal{Y}$ is defined as

$$[F(x)]\,(s) = x(s) - s - \int_0^1 Q(s,t)\,\left(x^3(t) + \gamma\, x^2(t)\right)\mathrm{d}t.$$

It is easy to verify that the Fréchet derivative of F is defined in the form

$$[F'(x)v]\,(s) = v(s) - \int_0^1 Q(s,t)\,\left(3\,x^2(t) + 2\,\gamma\,x(t)\right)v(t)\,\mathrm{d}t.$$

Let

$$\delta F(x,y) = \int_0^1 F'(y + t\,(x-y))\,\mathrm{d}t.$$

If we set $u_0(s) = s$, and $\mathcal{D} = U(u_0, R)$, then since $\| u_0 \| = 1$, it is easy to verify that $U(u_0, R) \subset U(0, R+1)$. It follows that $2\,\gamma < 5$, then (see [4])

$$\| I - F'(u_0) \| \leq \frac{3+2\,\gamma}{8}, \quad \| F'(u_0)^{-1} \| \leq \frac{8}{5-2\,\gamma},$$

$$\| F(u_0) \| \leq \frac{1+\gamma}{8}, \quad \| F(u_0)^{-1}\, F(u_0) \| \leq \frac{1+\gamma}{5-2\,\gamma}.$$

On the other hand, for $x, y \in \mathcal{D}$, we have

$$[(F'(x) - F'(y))v]\,(s) = -\int_0^1 Q(s,t)\,(3\,x^2(t) - 3\,y^2(t) + 2\,\gamma\,(x(t) - y(t)))\,v(t)\,\mathrm{d}t.$$

Consequently (see [4]),

$$\| F'(x) - F'(y) \| \leq \frac{\gamma+6\,R+3}{4}\ \| x - y \|,$$

and

$$\| F'(x) - F'(u_0) \| \leq \frac{2\,\gamma+3\,R+6}{8}\ \| x - u_0 \|\ .$$

Define linear operator $\delta F(x, y)$ by

$$\delta F(x, y) = \int_0^1 F'(y + t\,(x-y))\,\mathrm{d}t.$$

Then, conditions of Theorem 3 hold with

$$\eta = \frac{1+\gamma}{5-2\,\gamma}, \quad \ell = \frac{\gamma+6\,R+3}{8}, \quad \ell_0 = \frac{2\,\gamma+3\,R+6}{16}.$$

Note also that $\ell_0 < \ell$. □

CONCLUSION

We provided new sufficient convergence conditions for (SM) to a locally unique solution of a nonlinear equation in a Banach space. Using our new concept of recurrent functions, and combining Lipschitz and center–Lipschitz conditions on the divided difference operator, we obtained the semilocal convergence analysis of (SM). Our error bounds are more precise than earlier ones, and under our convergence hypotheses we can cover cases where earlier conditions are violated [8], [9], [12]–[14], [16], [19]–[21]. Applications, and numerical examples are also provided in this study.

REFERENCES

[1] I.K. Argyros, *On the Newton–Kantorovich hypothesis for solving equations*, J. Comput. Appl. Math., **169**, pp. 315–332, 2004.

[2] I.K. Argyros, *A unifying local–semilocal convergence analysis and applications for two–point Newton–like methods in Banach space*, J. Math. Anal. Appl., **298**, pp. 374–397, 2004.

[3] I.K. Argyros, *New sufficient convergence conditions for the Secant method*, Chechoslovak Math. J., **55**, pp. 175–187, 2005.

[4] I.K. Argyros, *Convergence and applications of Newton–type iterations*, Springer–Verlag Publ., New–York, 2008.

[5] I.K. Argyros and S. Hilout, *Efficient methods for solving equations and variational inequalities*, Polimetrica Publisher, Milano, Italy, 2009.

[6] I.K. Argyros and S. Hilout, *Convergence conditions for Secant–type methods*, Chechoslovak Math. J., **60**, pp. 253–272. 2010.

[7] W.E. Bosarge and P.L. Falb, *A multipoint method of third order*, J. Optimiz. Th. Appl., **4**, pp. 156–166, 1969.

[8] J.E. Dennis, *Toward a unified convergence theory for Newton–like methods*, in Nonlinear Functional Analysis and Application, (L.B. Rall, ed.), Academic Press, New York, pp. 425–472, 1971).

[9] M.A. Hernández, M.J. Rubio and J.A. Ezquerro, *Solving a special case of conservative problems by Secant–like method*, Appl. Math. Cmput., **169**, pp. 926–942, 2005.

[10] M.A. Hernández, M.J. Rubio and J.A. Ezquerro, *Secant–like methods for solving nonlinear integral equations of the Hammerstein type*, J. Comput. Appl. Math., **115**, pp. 245–254, 2000.

[11] L.V. Kantorovich and G.P. Akilov, *Functional Analysis*, Pergamon Press, Oxford, 1982.

[12] P. Laasonen, *Ein überquadratisch konvergenter iterativer algorithmus*, Ann. Acad. Sci. Fenn. Ser I, **450**, pp. 1–10, 1969.

[13] J.M. Ortega and W.C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, 1970.

[14] F.A. Potra, *On the convergence of a class of Newton–like methods. Iterative solution of nonlinear systems of equations*, (Oberwolfach, 1982), Lecture Notes in Math., **953**, Springer, Berlin–New York, pp. 125–137, 1982.

[15] F.A. Potra, *On an iterative algorithm of order* $1.839\cdots$ *for solving nonlinear operator equations*, Numer. Funct. Anal. Optim., 7(1), pp. 75–106, 1984/85.

[16] F.A. Potra, *Sharp error bounds for a class of Newton–like methods*, Libertas Mathematica, **5**, pp. 71–84, 1985.

[17] P.D. Proinov, *General local convergence theory for a class of iterative processes and its applications to Newton's method*, J. Complexity, **25**, pp. 38–62, 2009.

[18] P.D. Proinov, *New general convergence theory for iterative processes and its applications to Newton–Kantorovich type theorems*, J. Complexity, **26**, pp. 3–42, 2010.

[19] J.W. Schmidt, *Untere Fehlerschranken fur Regula–Falsi Verhafren*, Period. Hungar., **9**, pp. 241–247, 1978.

[20] T. Yamamoto, *A convergence theorem for Newton–like methods in Banach spaces*, Numer. Math., **51**, pp. 545–557, 1987.

[21] M.A. Wolfe, *Extended iterative methods for the solution of operator equations*, Numer. Math., **31**, pp. 153–174, 1978.

Received by the editors: October 1, 2010.