REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 40 (2011) no. 2, pp. 120–127
ictp.acad.ro/jnaat

EXACT INEQUALITIES INVOLVING POWER MEAN, ARITHMETIC MEAN AND IDENTRIC MEAN*

YU-MING CHU†, MING-YU SHI‡ and YUE-PING JIANG‡

Abstract. For $p \in \mathbb{R}$, the power mean $M_p(a,b)$ of order p, identric mean $I(a,b)$ and arithmetic mean $A(a,b)$ of two positive real numbers a and b are defined by

$$M_p(a,b) = \begin{cases} \left(\frac{a^p+b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases} \quad I(a,b) = \begin{cases} \frac{1}{e}\left(b^b/a^a\right)^{1/(b-a)}, & a \neq b, \\ a, & a = b, \end{cases}$$

and $A(a,b) = (a+b)/2$, respectively.

In the article, we answer the questions: What are the least values p, q and r, such that inequalities $A^{1/2}(a,b)I^{1/2}(a,b) \leq M_p(a,b)$, $A(a,b)^{1/3}I^{2/3}(a,b) \leq M_q(a,b)$ and $A^{2/3}(a,b)I^{1/3}(a,b) \leq M_r(a,b)$ hold for all $a,b > 0$?

MSC 2000. 26E60.

Keywords. Power mean, identric mean, arithmetic mean.

1. INTRODUCTION

For $p \in \mathbb{R}$, the power mean $M_p(a,b)$ of order p and the identric mean $I(a,b)$ of two positive real numbers a and b are defined by

$$M_p(a,b) = \begin{cases} \left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}, & p \neq 0, \\ \sqrt{ab}, & p = 0 \end{cases} \tag{1.1}$$

and

$$I(a,b) = \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & a \neq b, \\ a, & a = b, \end{cases} \tag{1.2}$$

respectively.

† Department of Mathematics, Huzhou Teachers College, Huzhou 313000, Zhejiang, China, e-mail: chuyuming@hutc.zj.cn.

‡College of Mathematics and Econometrics, Hunan University, Changsha, 410082, Hunan, China, e-mail: mingyulj08@163.com, ypjiang731@163.com.

*The work of the first author has been supported by the Natural Science Foundation (Grant no. 11071069) and the Innovation Team Foundation of the Department of Education of Zhejiang Province (Grant no. T200924).

It is well-known that $M_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Recently, both mean values have been the subject of intensive research. In particular, many remarkable inequalities for $M_p(a,b)$ and $I(a,b)$ can be found in literature [1-24].

Let $A(a,b) = (a+b)/2$, $L(a,b) = (b-a)/(\log b - \log a)(b \neq a)$ and $L(a,a) = a$, $G(a,b) = \sqrt{ab}$ and $H(a,b) = 2ab/(a+b)$ be the arithmetic, logarithmic, geometric and harmonic means of two positive numbers a and b, respectively. Then

$$\begin{aligned} \min\{a,b\} \leq H(a,b) = M_{-1}(a,b) \leq G(a,b) = M_0(a,b) \leq L(a,b) \\ \leq I(a,b) \leq A(a,b) = M_1(a,b) \leq \max\{a,b\}. \end{aligned} \tag{1.3}$$

In [25], Alzer and Janous established the following sharp double inequality

$$M_{\log 2/\log 3}(a,b) \leq \tfrac{2}{3}A(a,b) + \tfrac{1}{3}G(a,b) \leq M_{2/3}(a,b)$$

for all $a, b > 0$.

In [26-28], the authors presented the bounds for L and I in terms of A and G as follows

$$G^{2/3}(a,b)A^{1/3}(a,b) \leq L(a,b) \leq \tfrac{2}{3}G(a,b) + \tfrac{1}{3}A(a,b)$$

and

$$\tfrac{1}{3}G(a,b) + \tfrac{2}{3}A(a,b) \leq I(a,b)$$

for all $a, b > 0$.

The following companion of (1.3) provides inequalities for the geometric and arithmetic means of L and I, the proof can be found in [29].

$$\begin{aligned} G^{1/2}(a,b)A^{1/2}(a,b) \leq L^{1/2}(a,b)I^{1/2}(a,b) \leq \tfrac{1}{2}L(a,b) + \tfrac{1}{2}I(a,b) \\ \leq \tfrac{1}{2}G(a,b) + \tfrac{1}{2}A(a,b) \end{aligned}$$

for all $a, b > 0$.

The following sharp bounds for L, I, $(LI)^{1/2}$, and $(L+I)/2$ in terms of power means $M_p(a,b)$ are proved in [29-35].

$$L(a,b) \leq M_{1/3}(a,b), \quad M_{2/3}(a,b) \leq I(a,b) \leq M_{\log 2}(a,b),$$

$$M_0(a,b) \leq \sqrt{L(a,b)I(a,b)} \leq M_{1/2}(a,b),$$

and

$$\tfrac{1}{2}\left(L(a,b) + I(a,b)\right) < M_{1/2}(a,b)$$

for all $a, b > 0$.

Alzer and Qiu [36] proved

$$M_c(a,b) \leq \tfrac{1}{2}L(a,b) + \tfrac{1}{2}I(a,b)$$

for all a, $b > 0$ with the best possible parameter $c = \log 2/(1 + \log 2)$.

The main purpose of this paper is to answer the questions: What are the least values p, q and r, such that inequalities $A^{1/2}(a,b)I^{1/2}(a,b) \leq M_p(a,b)$, $A(a,b)^{1/3}I^{2/3}(a,b) \leq M_q(a,b)$ and $A^{2/3}(a,b)I^{1/3}(a,b) \leq M_r(a,b)$ hold for all a, $b > 0$?

2. LEMMAS

In order to establish our main results, we need a lemma, which we present in this section.

LEMMA 2.1. *Let*

$$g(t) = (1-r)(t^{p+1}+t^p+t+1)\log t + (2r-1)t^{p+1} - \\ - 2rt^p + t^{p-1} - t^2 + 2rt + 1 - 2r.$$

If $(r,p) = \{(\frac{1}{3},\frac{7}{9}),(\frac{2}{3},\frac{8}{9}),(\frac{1}{2},\frac{5}{6})\}$, *then there exists* $\lambda \in (1,+\infty)$, *such that* $g(t) > 0$ *for* $t \in (1,\lambda)$ *and* $g(t) < 0$ *for* $t \in (\lambda,+\infty)$.

Proof. Let $g_1(t) = t^{1-p}g'(t)$, $g_2(t) = t^p g_1'(t)$, $g_3(t) = t^{1-p}g_2'(t)$, $g_4(t) = t^3 g_3'(t)$, $g_5(t) = t^{p-2}g_4'(t)$, $g_6(t) = t^3 g_5'(t)$, $g_7(t) = t^{1-p}g_6'(t)$, and $g_8(t) = t^p g_7'(t)$. Then simple computations lead to

$$g(1) = 0, \tag{2.1}$$

$$\lim_{t\to+\infty} g(t) = -\infty, \tag{2.2}$$

$$g_1(t) = (1-r)[t^{1-p}+(1+p)t+p]\log t - 2t^{2-p} + (1+r)t^{1-p} \\ +(1-r)t^{-p} + (2pr-p+r)t - (1-p)t^{-1} - 2pr - r + 1,$$

$$g_1(1) = 0, \tag{2.3}$$

$$\lim_{t\to+\infty} g_1(t) = -\infty, \tag{2.4}$$

$$g_2(t) = (1-r)[(1+p)t^p + 1 - p]\log t + (pr+1)t^p + p(1-r)t^{p-1} \\ +(1-p)t^{p-2} - 2(2-p)t - p(1-r)t^{-1} - pr - p + 2,$$

$$g_2(1) = 0, \tag{2.5}$$

$$\lim_{t\to+\infty} g_2(t) = -\infty. \tag{2.6}$$

$$g_3(t) = p(1+p)(1-r)\log t - 2(2-p)t^{1-p} + (1+p)(1-r)t^{-p} \\ +p(1-r)t^{-1-p} - p(1-p)(1-r)t^{-1} - (1-p)(2-p)t^{-2} \\ +p^2 r - pr + 2p - r + 1,$$

$$g_3(1) = 6p - 4 - 2r = 0, \tag{2.7}$$

$$\lim_{t\to+\infty} g_3(t) = -\infty, \tag{2.8}$$

$$g_4(t) = p(1-r)[(1+p)t^2 + (1-p)t - (1-p)t^{2-p} - (1+p)t^{1-p}] \\ -2(1-p)(2-p)(t^{3-p} - 1),$$

$$g_4(1) = 0, \tag{2.9}$$

$$\lim_{t\to+\infty} g_4(t) = -\infty, \tag{2.10}$$

$$g_5(t) = p(1-r)[2(1+p)t^{p-1} + (1-p)t^{p-2} - (1-p)(2-p)t^{-1} \\ -(1+p)(1-p)t^{-2}] - 2(1-p)(2-p)(3-p),$$

$$g_5(1) = 4(1-r)p^2 - 2(1-p)(2-p)(3-p), \tag{2.11}$$

$$\lim_{t\to+\infty} g_5(t) = -2(1-p)(2-p)(3-p) < 0, \tag{2.12}$$

$$g_6(t) = p(1-r)[-2(1+p)(1-p)t^{p+1} - (1-p)(2-p)t^p + (1-p)(2-p)t + 2(1+p)(1-p)],$$

$$g_6(1) = 0, \tag{2.13}$$

$$g_7(t) = p(1-p)(1-r)[(2-p)t^{1-p} - 2(1+p)^2 t - p(2-p)],$$

$$g_7(1) = -p^2(1-p)(7+p)(1-r) < 0, \tag{2.14}$$

$$g_8(t) = p(1-p)(1-r)[-2(1+p)^2 t^p + (1-p)(2-p)], \tag{2.15}$$

and

$$g_8(1) = -p^2(1-p)(7+p)(1-r) < 0. \tag{2.16}$$

From (2.11) we know that $g_5(1) = \frac{296}{729}$ if $(r,p) = (\frac{1}{3}, \frac{7}{9})$, $g_5(1) = \frac{388}{729}$ if $(r,p) = (\frac{2}{3}, \frac{8}{9})$, and $g_5(1) = \frac{119}{729}$ if $(r,p) \in (\frac{1}{2}, \frac{5}{6})$. Therefore

$$g_5(1) > 0 \tag{2.17}$$

for $(r,p) = \{(\frac{1}{3}, \frac{7}{9}), (\frac{2}{3}, \frac{8}{9}), (\frac{1}{2}, \frac{5}{6})\}$.

From (2.15) we clearly see that $g_8(t)$ is strictly decreasing in $[1,+\infty)$, then (2.16) implies that $g_8(t) < 0$ for $t \in [1,+\infty)$. Hence that $g_7(t)$ is strictly decreasing in $[1,+\infty)$.

From (2.14) and the monotonicity of $g_7(t)$ we know that $g_7(t) < 0$ for $t \in [1,+\infty)$. Hence $g_6(t)$ is strictly decreasing in $[1,+\infty)$.

From (2.13) and the monotonicity of $g_6(t)$ we know that $g_6(t) < 0$ for $t \in [1,+\infty)$. Hence that $g_5(t)$ is strictly decreasing in $[1,+\infty)$.

Inequalities (2.12) and (2.17) together the monotonicity of $g_5(t)$ imply that there exists $t_0 \in (1,+\infty)$, such that $g_5(t) > 0$ for $t \in (1,t_0)$ and $g_5(t) < 0$ for $t \in (t_0,+\infty)$. Hence $g_4(t)$ is strictly increasing in $[1,t_0]$ and strictly decreasing in $[t_0,+\infty)$.

From equations (2.9) and (2.10) together with the monotonicity of $g_4(t)$ we clearly see that there exists $t_1 \in (1,+\infty)$, such that $g_4(t) > 0$ for $t \in (1,t_1)$ and $g_4(t) < 0$ for $t \in (t_1,+\infty)$. Hence $g_3(t)$ is strictly increasing in $[1,t_1]$ and strictly decreasing in $[t_1,+\infty)$.

Equations (2.7) and (2.8) together with the monotonicity of $g_3(t)$ imply that there exists $t_2 \in (1,+\infty)$, such that $g_3(t) > 0$ for $t \in (1,t_2)$ and $g_3(t) < 0$ for $t \in (t_2,+\infty)$. Hence $g_2(t)$ is strictly increasing in $[1,t_2]$ and strictly decreasing in $[t_2,+\infty)$.

It follows from equations (2.5) and (2.6) together with the monotonicity of $g_2(t)$ that there exists $t_3 \in (1,+\infty)$, such that $g_2(t) > 0$ for $t \in (1,t_3)$ and $g_2(t) < 0$ for $t \in (t_3,+\infty)$. Hence $g_1(t)$ is strictly increasing in $[1,t_3]$ and strictly decreasing in $[t_3,+\infty)$.

Equations (2.3) and (2.4) together with the monotonicity of $g_1(t)$ lead to the conclusion that there exists $\lambda_5 \in (1,+\infty)$, such that $g_1(t) > 0$ for $t \in (1,t_4)$ and $g_1(t) < 0$ for $t \in (t_4,+\infty)$. Hence $g(t)$ is strictly increasing in $[1,t_4]$ and strictly decreasing in $[t_4,+\infty)$.

Therefore, Lemma 2.1 follows from equations (2.1) and (2.2) together with the monotonicity of $g(t)$. □

3. MAIN RESULTS

THEOREM 3.1. *For all $a, b > 0$ one has*

$$A^{1/3}(a,b)I^{2/3}(a,b) \leq M_{7/9}(a,b),$$

$$A^{2/3}(a,b)I^{1/3}(a,b) \leq M_{8/9}(a,b),$$

and

$$A^{1/2}(a,b)I^{1/2}(a,b) \leq M_{5/6}(a,b),$$

each inequality holds equality if and only if $a = b$, and the parameters $7/9$, $8/9$ and $5/6$ in each inequality cannot be improved.

Proof. If $a = b$, then we clearly see that

$$A^{1/2}(a,b)I^{1/2}(a,b) = M_{5/6}(a,b) = A^{1/3}(a,b)I^{2/3}(a,b) = M_{7/9}(a,b)$$
$$= A^{2/3}(a,b)I^{1/3}(a,b) = M_{8/9}(a,b) = a.$$

If $a \neq b$, without loss of generality, we assume that $a > b$. Let $(\alpha, p) = \{(1/3, 7/9), (2/3, 8/9), (1/2, 5/6)\}$ and $t = a/b > 1$, then (1.1) and (1.2) lead to

$$M_p(a,b) - A^{\alpha}(a,b)I^{1-\alpha}(a,b) = b\left[\left(\tfrac{t^p+1}{2}\right)^{1/p} - \left(\tfrac{t+1}{2}\right)^{\alpha}\left(\tfrac{1}{e}\cdot t^{\frac{t}{t-1}}\right)^{1-\alpha}\right]. \tag{3.1}$$

Let

$$f(t) = \tfrac{1}{p}\log\tfrac{1+t^p}{2} - \alpha\log\tfrac{t+1}{2} - (1-\alpha)\tfrac{t}{t-1}\log t + (1-\alpha),$$

then

$$\lim_{t\to 1^+} f(t) = 0, \tag{3.2}$$

$$\lim_{t\to\infty} f(t) = (1-\alpha) + (\alpha - \tfrac{1}{p})\log 2 \tag{3.3}$$

and

$$f'(t) = \tfrac{g(t)}{(t+1)(t-1)^2(t^p+1)}, \tag{3.4}$$

where

$$g(t) = (1-\alpha)(t^{p+1} + t^p + t + 1)\log t$$
$$+ (2\alpha - 1)t^{p+1} - 2\alpha t^p + t^{p-1} - t^2 + 2\alpha t + 1 - 2\alpha.$$

From (3.3) we know that $\lim\limits_{t\to\infty} f(t) = 2(7 - 10\log 2)/21 > 0$ if $(\alpha, p) = (\frac{1}{3}, \frac{7}{9})$, $\lim\limits_{t\to\infty} f(t) = (8 - 11\log 2)/24 > 0$ if $(\alpha, p) = (\frac{2}{3}, \frac{8}{9})$, and $\lim\limits_{t\to\infty} f(t) = (5 - 7\log 2)/10 > 0$ if $(\alpha, p) = (\frac{1}{2}, \frac{5}{6})$. Hence we get

$$\lim_{t\to\infty} f(t) > 0. \tag{3.5}$$

Equation (3.4) and Lemma 2.1 lead to the conclusion that there exists $\lambda \in (1, +\infty)$, such that $f(t)$ is strictly increasing in $[1, \lambda]$ and strictly decreasing in $[\lambda, +\infty)$. Therefore, $A^{1/3}(a,b)I^{2/3}(a,b) < M_{7/9}(a,b)$, $A^{2/3}(a,b)I^{1/3}(a,b) < M_{8/9}(a,b)$, and $A^{1/2}(a,b)I^{1/2}(a,b) < M_{5/6}(a,b)$ for all $a, b > 0$ with $a \neq b$ follow from (3.1), (3.2), (3.5) and the monotonicity of $f(t)$.

Next, we prove that the parameters 7/9, 8/9 and 5/6 in each inequality cannot be improved.

For any $0 < \varepsilon < 7/9$, $0 < x < 1$ and $x \to 0$, making use of Taylor expansion one has

$$\begin{aligned}(3.6)\qquad & \log\left[A^{1/2}(1,1+x)I^{1/2}(1,1+x)\right] - \log M_{5/6-\varepsilon}(1,1+x)\\ &= \tfrac{1}{2}\log(1+\tfrac{x}{2}) + \tfrac{1+x}{2x}\log(1+x) - \tfrac{1}{2} - \tfrac{6}{5-6\varepsilon}\log\tfrac{1+(1+x)^{5/6-\varepsilon}}{2}\\ &= \tfrac{\varepsilon}{8}x^2 + o(x^2),\end{aligned}$$

$$\begin{aligned}(3.7)\qquad & \log\left[A^{1/3}(1,1+x)I^{2/3}(1,1+x)\right] - \log M_{7/9-\varepsilon}(1,1+x)\\ &= \tfrac{1}{3}\log(1+\tfrac{x}{2}) + \tfrac{2(1+x)}{3x}\log(1+x) - \tfrac{2}{3} - \tfrac{9}{7-9\varepsilon}\log\tfrac{1+(1+x)^{7/9-\varepsilon}}{2}\\ &= \tfrac{\varepsilon}{8}x^2 + o(x^2)\end{aligned}$$

and

$$\begin{aligned}(3.8)\qquad & \log\left[A^{2/3}(1,1+x)I^{1/3}(1,1+x)\right] - \log M_{8/9-\varepsilon}(1,1+x)\\ &= \tfrac{2}{3}\log(1+\tfrac{x}{2}) + \tfrac{1+x}{3x}\log(1+x) - \tfrac{1}{3} - \tfrac{9}{8-9\varepsilon}\log\tfrac{1+(1+x)^{8/9-\varepsilon}}{2}\\ &= \tfrac{\varepsilon}{8}x^2 + o(x^2).\end{aligned}$$

Equations (3.6)–(3.8) imply that for any $0 < \varepsilon < 7/9$, there exists $0 < \delta = \delta(\varepsilon) < 1$, such that

$$A^{1/2}(1,1+x)I^{1/2}(1,1+x) > M_{5/6-\varepsilon}(1,1+x),$$
$$A^{1/3}(1,1+x)I^{2/3}(1,1+x) > M_{7/9-\varepsilon}(1,1+x)$$

and

$$A^{2/3}(1,1+x)I^{1/3}(1,1+x) > M_{8/9-\varepsilon}(1,1+x)$$

for $x \in (0, \delta)$. □

REFERENCES

[1] H. ALZER, *Ungleichungen für* $(e/a)^a(e/b)^b$, Elem. Math., **40**, pp. 120–123, 1985.
[2] H. ALZER, *Ungleichungen für Mittlewerte*, Arch. Math., **47**(5), pp. 422–426, 1986.
[3] H. ALZER and W. JANOUS, *Solution of problem 8**, Crux. Math., **13**, pp. 173–178, 1987.
[4] H. ALZER and S.-L. QIU, *Inequalities for means in two variables*, Arch. Math., **80**(2), pp. 201–215, 2003.
[5] F. BURK, *The geometric, logarithmic, and arithmetic mean inequalities*, Amer. Math. Monthly, **94**(6), pp. 527–528, 1987.
[6] B. C. CARLSON, *The logarithmic mean*, Amer. Math. Monthly, **79**, pp. 615–618, 1972.

[7] Y.-M. Chu, Y.-F. Qiu and M.-K. Wang, *Sharp power mean bounds for the combination of Seiffert and geometric means*, Abstr. Appl. Anal., Article ID 108920, 12 pages, 2010.

[8] Y.-M. Chu and W.-F. Xia, *Two sharp inequalities for power mean, geometric mean, and harmonic mean*, J. Inequal. Appl., Article ID 741923, 6 pages, 2009.

[9] Y.-M. Chu and W.-F. Xia, *Two optimal double inequalities between power mean and logarithmic mean*, Comput. Math. Appl., **60**(1), pp. 83–89, 2010.

[10] Y.-M. Chu and B.-O. Long, *Sharp inequalities between means*, Math. Inequal. Appl., **14**(3), pp. 647–655, 2011.

[11] Y.-M. Chu, S.-S. Wang and C. Zong, *Optimal lower power mean bound for the convex combination of harmonic and logarithmic means*, Abstr. Appl. Anal., Article ID 520648, 9 pages, 2011.

[12] Y.-M. Chu and B.-Y. Long, *A sharp double inequality between generalized Heronian and power means*, Pac. J. Appl. Math., **3**(4), pp. 39–48, 2011.

[13] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, *A sharp double inequality between harmonic and identric means*, Abstr. Appl. Anal., Article ID 657935, 7 pages, 2011.

[14] E. B. Leach and M. C. Sholander, *Extended mean values II*, J. Math. Anal. Appl., **92**(1), pp. 207–233, 1983.

[15] Y.-M. Li, B.-Y. Long, Y.-M. Chu and W.-M. Gong, *Optimal inequalities for power means*, J. Appl. Math., Article ID 182905, 8 pages, 2012.

[16] T. P. Lin, *The power mean and the logarithmic mean*, Amer. Math. Monthly, **81**, pp. 879–883, 1974.

[17] B.-O. Long and Y.-M. Chu, *Optimal power mean bounds for the weighted geometric mean of classical means*, J. Inequal. Appl., Article ID 905679, 2010.

[18] B.-O. Long, W.-F. Xia and Y.-M. Chu, *An optimal inequality for power mean, geometric mean and harmonic mean*, Int. J. Mod. Math., **5**(2), pp. 149–155, 2010.

[19] A. O. Pittenger, *Inequalities between arithmetic and logarithmic means*, Univ. Beogard. Publ. Elecktrotehn. Fak. Ser. Mat. Fiz., **678–715**, pp. 15–18, 1981.

[20] A. O. Pittenger, *The symmetric, logarithmic and power means*, Univ. Beogard. Publ. Elecktrotehn. Fak. Ser. Mat. Fiz., **678–715**, pp. 19–23, 1981.

[21] Y.-F. Qiu, M.-K. Wang, Y.-M. Chu and G.-D. Wang, *Two sharp inequalities for Lehmer mean, idnetric mean and logarithmic mean*, J. Math. Inequal., **5**(3), pp. 301–306, 2011.

[22] S.-L. Qiu, Y.-F. Qiu, M.-K. Wang and Y.-M. Chu, *Hölder mean inequalities for the generalized Grötzsch ring and Hersch-Pfluger distortion functions*, Math. Inequal. Appl., **15**(1), pp. 237–245, 2012.

[23] Y.-F. Qiu, M.-K. Wang and Y.-M. Chu, *The optimal generalized Heronian mean bounds for the identric mean*, Int. J. Pure Appl. Math., **72**(1), pp. 19–26, 2011.

[24] J. Sándor, *A note on some inequalities for means*, Arch. Math., **56**(5), pp. 471–473, 1991.

[25] M.-Y. Shi, Y.-M. Chu and Y.-P. Jiang, *Optimal inequalities among various means of two arguments*, Abstr. Appl. Anal., Article ID 694394, 10 pages, 2009.

[26] M.-Y. Shi, Y.-M. Chu and Y.-P. Jiang, *Three best inequalities for means in two variables*, Int. Math. Forum, **5**(21-24), pp. 1059–1066, 2010.

[27] M.-Y. Shi, Y.-M. Chu and Y.-P. Jiang, *Optimal inequalities related to the power, harmonic and identric means*, Acta Math. Sci, **31A**(5), pp. 1377–1384, 2011 (Chinese).

[28] K. B. Stolarsky, *The power and gerneralized logarithmic means*, Amer. Math. Monthly, **87**(7), pp. 545–548, 1980.

[29] M.-K. Wang, Y.-M. Chu and Y.-F. Qiu, *Some comparison inequalities for generalized Muirhead and identric means*, J. Inequal. Appl., Article ID 295620, 10 pages, 2010.

[30] M.-K. Wang, Y.-M. Chu, Y.-F. Qiu and S.-L. Qiu, *An optimal power mean inequality for the complete elliptic integrals*, Appl. Math. Lett., **24**(6), pp. 887–890, 2011.

[31] G.-D. Wang, X.-H. Zhang and Y.-M. Chu, *A power mean inequality for the Grötzsch ring function*, Math. Inequal. Appl., **14**(4), pp. 833–837, 2011.

[32] M.-K. Wang, Z.-K. Wang and Y.-M. Chu, *An optimal double inequality between geometric and identric means*, Appl. Math. Lett., **25**(3), pp. 471–475, 2012.

[33] M.-K. Wang, Y.-M. Chu, S.-L. Qiu and Y.-P. Jiang, *Convexity of the complete elliptic integrals of the first kind with respect to Hölder means*, J. Math. Anal. Appl., **388**(2), pp. 1141–1146, 2012.

[34] W.-F. Xia, Y.-M. Chu and G.-D. Wang, *The optimal upper and lower power mean bounds for a convex combination of the arithmetic and logarithmic means*, Abstr. Appl. Anal., Article ID 604804, 2010.

[35] W.-F. Xia and Y.-M. Chu, *Optimal inequalities related to the logarithmic, identric, arithmetic and harmonic means*, Rev. Anal. Numér. Théor. Approx., **39**(2), pp. 176–193, 2010.

[36] C. Zong and Y.-M. Chu, *An inequality among identric, geometric and Seiffert's means*, Int. Math. Forum, **5**(25-28), pp. 1297–1302, 2010.

Received by the editors: October 10, 2009.