REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 40 (2011) no. 2, pp. 128–148
ictp.acad.ro/jnaat

OPTIMAL INEQUALITY FACTOR FOR DURAND–KERNER'S AND TANABE'S METHODS

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Abstract. The convergence condition for the simultaneous inclusion methods is $w^{(0)} < c(a,b,n)d^{(0)}$, where $w^{(0)}$ is the maximum Weierstrass factor $W_k^{(0)}$, $k \in I_n$, and $d^{(0)}$ is the minimum distance between $z_1^{(0)}$, $z_2^{(0)}$, $\dots z_n^{(0)}$, the distinct approximations of the simple roots of the polynomial $\zeta_1, \zeta_2, \dots \zeta_n$. The *i-factor* (inequality-factor) is the positive real function $c(a,b,n) = \frac{1}{an+b}$. The article presents the optimum *i-factor* for the simultaneous inclusion methods Durand–Kerner and Tanabe.

MSC 2000. 65H10.
Keywords. Root-finding methods; Polynomial zeros; Simultaneous inclusion methods; Durand–Kerner's method, Tanabe's method, Convergence; Computational efficiency.

1. INTRODUCTION

Let

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0,\ a_k \in \mathbb{C}\ ,\ k \in I_{n-1}^*\ , \tag{1}$$

be a polynomial, where $I_{n-1}^* = \{0, 1, \dots, n-1\}$ and let

$$P(z) = 0\ , \tag{2}$$

be its attached algebraic equation.

Abel's impossibility theorem states that: "*In general, polynomial equations higher than fourth degree are incapable of algebraic solution in terms of a finite number of additions, subtractions, multiplications, divisions, and root extractions*", [1]. This was also shown by Ruffini in 1813, [11, pp. 59].

Galois theorem states the same thing, namely that: "*An algebraic equation is algebraically solvable necessary and sufficient its group is solvable. In order that an irreducible equation of prime degree be solvable by radicals, it is necessary and sufficient that all its roots be rational functions of two roots*", [18, 27].

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Modern softwares such as *Maple*, *Mathematica*, *Mathcad* or *Matlab* solve the algebraic equation of degree 2, 3 and 4 using symbolic computation and classical formulas for the second degree equation or Cardano's formulas, [14, pp. 282-284] and Ferrari–Cardano, [14, pp. 284-287] for the 3^{rd} and 4^{th} degree equations. This methods always give the exact solutions but have the disadvantage that, sometimes, the solutions have a complicated representation.

Arbitrary-precision calculus in *Mathcad* prints the solution with up to 250 decimals, while *Mathematica*, [33] permits the display of any number of decimals for a result. Computing a result with n exact decimals is equivalent to finding the solution of the equation (2) with a numerical method with the precision 10^{-n}.

Therefore obtaining the solutions for an algebraic equation of degree 2, 3 and 4 is a solved problem. According to Abel's impossibility theorem, in general for equation of degree greater than 4, we need to apply a numerical method to approximate the solutions.

Let us denote by $\mathbf{d}_n = \{n, n+1, \ldots\}$, $n \in \mathbb{N}$ the polynomial degree greater or equal n.

2. SIMULTANEOUS METHODS WITH CORRECTIONS

Since $k \in I_n = \{1, 2, \ldots, n\}$ we denote

$$\sum \tfrac{W_j}{\widehat{z}_k - z_j} = \sum_{j=1}^{n} \tfrac{W_j}{\widehat{z}_k - z_j}, \quad \prod_{j \neq k} (z_k - z_j) = \prod_{\substack{j=1\\k\neq j}}^{n} (z_k - z_j),$$

$$\max |W_k| = \max_{k \in I_n} |W_k|, \quad \min_{k<j} |z_k - z_j| = \min_{\substack{k,j \in I_n\\k<j}} |z_k - z_j|,$$

Weierstrass correction factor is denoted by

$$W_k = W(z_k) = \frac{P(z_k)}{\prod\limits_{j \neq k} (z_k - z_j)}, \tag{3}$$

where z_k is the approximation for the simple zero ζ_k of the polynomial (1), w is the absolute maximum value of the Weierstrass correction factors and d the minimum distance between two approximations z_1, z_2, …, z_n.

$$w = \max |W_k| \ , \quad d = \min_{k<j} |z_k - z_j| .$$

We will also note by z, w, d, W the current iteration $z^{(m)}$, $w^{(m)}$, $d^{(m)}$, $W^{(m)}$ and by $\widehat{z}$, $\widehat{w}$, $\widehat{d}$, $\widehat{W}$ the next iteration $z^{(m+1)}$, $w^{(m+1)}$, $d^{(m+1)}$, $W^{(m+1)}$.

We will consider a class C of simultaneous method with corrections, $C(z) = P(z)/F(z)$, with $F(z) \neq 0$ for any zero ζ_k with $k \in I_n$, of polynomial P, and for any approximation $z_k^{(m)}$ with $k \in I_n$ and $m = 0, 1, \ldots$ obtained by the iterative process. We will denote by

$$C_k^{(m)} = C_k(z_1^{(m)}, z_2^{(m)}, \ldots, z_k^{(m)}), \tag{4}$$

the correction factor for the m^{th} iteration. The vast majority of the iterative methods that simultaneously approximate the simple zeros of an polynomial can be expressed as:

$$z_k^{(m+1)} = z_k^{(m)} - C_k^{(m)}, \quad k \in I_n \;\; m = 0, 1, \ldots, \tag{5}$$

where $z_1^{(m)}$, $z_2^{(m)}$, ..., $z_n^{(m)}$ are distinct approximations of the simple zeros ζ_1, ζ_2, ..., ζ_n.

We try to find a convergence condition of the form $w < c(n)d$, where $c(n)$ is a real positive function that depends on n (the polynomial degree) asymptotic to the function $\phi(n) = 1/(n + 2\sqrt{n-1})$ or $\psi(n) = 1/(2n)$. We consider the initial condition satisfied

$$w^{(0)} < c(n)d^{(0)}. \tag{6}$$

The function $c(n)$ is called the *i-factor* (inequality factor) in [20], [31] and many other works, and it depends on the degree n of the polynomial. For the choice of $c(n)$, Petković and its collaborators propose that $c(n)$ has the form:

$$c(n) = c(a, b, n) = \tfrac{1}{an+b} \quad \text{with } a, b \in \mathbb{R}_+. \tag{7}$$

THEOREM 1. *Let $\eta_k = z_k - W_k \in \mathbb{C} \setminus \{z_1, z_2, \ldots, z_n\}$ and set*

$$\gamma_k = |W_k| \cdot \max_{j \neq k} |z_j - \eta_k|^{-1}, \quad \sigma_k = \sum_{j \neq k} \tfrac{|W_j|}{|z_j - \sigma_k|}, \quad k \in I_n.$$

If $\sqrt{1+\gamma_k} > \sqrt{\gamma_k} + \sqrt{\sigma_k}$, then there is exactly one zero of P in the disk centered in η_k and of radius

$$|W_k| \cdot \left(1 - \tfrac{2(1-2\sigma_k-\gamma_k)}{1-\sigma_k-2\gamma_k+\sqrt{(1-\sigma_k-2\gamma_k)^2+4\sigma_k(1-2\sigma_k-\gamma_k)^2}}\right).$$

If

$$\sqrt{1+\gamma_k} > \sqrt{\gamma_k} + \sqrt{\sigma_k} \text{ and } \gamma_k + 2\sigma_k < 1, \tag{8}$$

then there is exactly one zero of P, in disk centered in η_k and of radius

$$|W_k| \tfrac{\gamma_k + \sigma_k}{1-\sigma_k}.$$

Proof. See [12] and [5, 8, 13]. □

THEOREM 2. *If the* i-factor $c(n)$ *appearing in* (6) *is*

$$c(n) < \phi(n) = \tfrac{1}{n+2\sqrt{n-1}} \text{ and } c(n) < \psi(n) = \tfrac{1}{2n}, \tag{9}$$

then both inequalities (8) *hold and the minimal radius of the inclusion disk given in Theorem* 1 *is not greater than $|W_k|$.*

Proof. See [31, T 1.5.]. □

THEOREM 3. *Let* $c(a,b,n) = \frac{1}{an+b}$*,* $a \geq 2$*,* $b > (2-a)n$*, and let us assume that* $w < c(a,b,n)d$*, holds. Then for* $n \in \mathbf{d}_3$*, the disks*

$$D_1 = D\left(z_1 - W_1; \tfrac{n}{(a-1)n+b}\,|W_1|\right), \ldots, D_n = D\left(z_n - W_n; \tfrac{n}{(a-1)n+b}\,|W_n|\right)$$

are mutually disjoint and each of then contain unique zero of P*.*

Proof. See [31, T 1.6]. □

COROLLARY 4. *Under the conditions of Theorem* 3*, each of disks* D_k^* *defined by*

$$\begin{aligned} D_k^* &= D\left(z_k - W_k; \tfrac{n}{(a-1)n+b}\,|W_k|\right) \\ &= D\left(z_k; \tfrac{1}{1-nc(a,b,n)}\,|W_k|\right),\ k \in I_n \end{aligned} \tag{10}$$

contains exactly one zero of P*.*

Proof. See [31, Corollary 1.1]. □

Let $g : (0,1) \to \mathbb{R}$

$$g(t) = \begin{cases} 1 + 2t, & 0 < t \leq \frac{1}{2}, \\ \frac{1}{1-t}, & \frac{1}{2} < t < 1. \end{cases} \tag{11}$$

LEMMA 5. *Let*

$$s_m(t) = t^m + \sum_{k=0}^{m} t^k\ ,\ t \in (0,1),\ m = 1, 2, \ldots\ .$$

Then $s_m(t) < g(t)$*.*

Proof. Proof is elementary. □

THEOREM 6. *Let the iterative method* (5) *have the iterative correction of the form* (4) *and let* $z_1^{(0)}$*,* $z_2^{(0)}$*,* $\ldots z_n^{(0)}$ *be distinct initial approximations of zeros for the polynomial* P*. If there exists a real number* β *such that the following two inequalities:*

(1) $\left|C_k^{(m+1)}\right| \leq \beta \left|C_k^{(m)}\right|$, for $m = 0, 1, \ldots,$

(2) $\left|z_k^{(0)} - z_j^{(0)}\right| > g(\beta)\left(\left|C_k^{(0)}\right| + \left|C_j^{(0)}\right|\right)$, for $k \neq j,\ k, j \in I_n$*,*

are valid, then the iterative method (5) *is convergent.*

Proof. See [31, T 3.1]. □

All the simultaneous inclusion methods convergence theorems try to determine the values of a and b such that the initial condition (6) assures the convergence for the method. We try to find the optimum values for a and b, namely an *optimum i-factor* for every simultaneous method with corrections, [31, pp. 80-81], [30].

Let us consider the nonlinear general problem with restrictions for simultaneous inclusion methods with the disks

$$D_k = D\big(\widehat{z}_k; |C_k|\big), \quad k \in I_n, \tag{12}$$

$$\left\{\begin{array}{ll}
\max c(a,b,n) & \text{maximization,}\\
a>1,\ b\geq 0,\ n\in \mathbf{d}_5 & \text{defining } c,\\
c(a,b,n)\leq \phi(n) & \text{asymptotic condition to } \phi(n),\\
0<\lambda(a,b,n)<\frac{1}{2} & \Rightarrow |C_k| \leq \frac{\lambda(a,b,n)}{c(a,b,n)}\,|W_k|\,,\\
0<\delta(a,b,n)<1 & \Rightarrow \lim\limits_{m\to\infty}\left|W_k^{(m)}\right| = 0,\\
0<\beta(a,b,n)<1 & \Rightarrow \lim\limits_{m\to\infty}\left|C_k^{(m)}\right| = 0,\\
\theta(a,b,n)\leq 1 & \Rightarrow \mathit{if}\ w\leq c(a,b,n)d \Rightarrow \widehat{w}\leq c(a,b,n)\widehat{d},\\
\eta(a,b,n)<0 & \Rightarrow\ D_k\cap D_j = \emptyset,
\end{array}\right. \tag{13}$$

for $k, j \in I_n$. The function $\phi(n)$ is given by (9). Functions δ, β, θ and η are defined by

$$\left\{\begin{array}{lll}
\lambda & = & \lambda(c,n)\ ,\\
\Pi(\lambda,n) & = & \left(1+\frac{\lambda}{1-2\lambda}\right)^{n-1}\ ,\\
\delta(\lambda,c,n) & = & \mu_\delta(\lambda,c,n)\cdot \Pi(\lambda,n)\ ,\\
\beta(\lambda,c,n) & = & \mu_\beta(\lambda,c,n)\cdot \delta(\lambda,c,n)\ ,\\
\mu_\theta(\lambda) & = & \frac{1}{1-2\lambda}\ ,\\
\theta(\lambda,c,n) & = & \mu_\theta(\lambda)\cdot \beta(\lambda,c,n)\ ,\\
\eta(\lambda,c,n) & = & 2\lambda - \frac{1}{g\left(\beta(\lambda,c,n)\right)}\ .
\end{array}\right. \tag{14}$$

The functions λ, μ_δ, μ_β, will be defined for every inclusion method from the lemmas and convergence theorems. The functions λ depend on c and n while c depends on a b and n. The function g is given in (11).

Let us consider the nonlinear general problem with restrictions for simultaneous inclusion methods with the disks D_k^* given by (10):

$$\left\{\begin{array}{ll}
\max c(a,b,n) & \text{maximization,}\\
a\geq 2,\ b\geq 0,\ n\in \mathbf{d}_5 & \text{defining } c,\\
c(a,b,n)\leq \psi(n) & \text{asymptotic condition to } \psi(n),\\
0<\lambda(a,b,n)<\frac{1}{2}, & \Rightarrow |C_k| \leq \frac{\lambda(a,b,n)}{c(a,b,n)}\,|W_k|\,,\\
0<\delta(a,b,n)<1 & \Rightarrow \lim\limits_{m\to\infty}\left|W_k^{(m)}\right| = 0,\\
0<\beta(a,b,n)<1 & \Rightarrow \lim\limits_{m\to\infty}\left|C_k^{(m)}\right| = 0,\\
\theta(a,b,n)\leq 1 & \Rightarrow \mathit{if}\ w\leq c(a,b,n)d \Rightarrow \widehat{w}\leq c(a,b,n)\widehat{d},
\end{array}\right. \tag{15}$$

for $k, j \in I_n$, where functions are defined by (14).

PROPOSITION 7. *If the functions $f_1, f_2 : I \subset \mathbb{R} \to \mathbb{R}_+$ are monotonically increasing (monotonically decreasing), then the function $f : I \subset \mathbb{R} \to \mathbb{R}_+$, $f(t) = f_1(t)\cdot f_2(t)$ is monotonically increasing (monotonically decreasing).*

Proof. If the functions f_1, f_2 are positive and increasing (strictly increasing), then $f_1'(t),\ f_2'(t) \geq 0,\ (>0)$, and $f'(t) = f_1'(t)\cdot f_2(t) + f_1(t)\cdot f_2'(t) \geq 0,\ (>0)$. Then it follows that the function $f(t)$ is increasing (strictly increasing). □

Remark 8.

(1) If the functions $\mu_\beta(a,b,n)$ and $\mu_\theta(a,b,n)$, are monotonically and ≥ 1, then we have $\delta(a,b,n) \leq \beta(a,b,n) \leq \theta(a,b,n)$.

(2) If the function $\Pi = \Pi(a,b,n) \nearrow$ it is necessary for the functions $\delta = \delta(a,b,n) \nearrow$, $\beta = \beta(a,b,n) \nearrow$ and $\theta = \theta(a,b,n) \nearrow$. These involve the followings:
  - If $\Pi \nearrow$, then is necessary that : $\mu_\delta \nearrow$ for as $\delta = \mu_\delta \Pi \nearrow$.
  - If $\delta \nearrow$, then is necessary that: $\mu_\beta \nearrow$ for as $\beta = \mu_\beta \delta \nearrow$ or if $\mu_\beta \searrow$, then $\mu_{\beta\delta} = \mu_\beta \mu_\delta \nearrow$ for as $\beta = \mu_{\beta\delta}\Pi \nearrow$.
  - If $\beta \nearrow$, then is necessary that: $\mu_\theta \nearrow$ for as $\theta = \mu_\theta \beta \nearrow$ or if $\mu_\theta \searrow$, then:
    - $\mu_{\beta\delta} = \mu_\beta\mu_\delta \nearrow$ for as $\theta = \mu_{\beta\delta}\delta \nearrow$ or if $\mu_{\beta\delta} \searrow$, then:
    - $\mu_{\theta\beta\delta} = \mu_\theta\mu_\beta\mu_\delta \nearrow$ for as $\theta = \mu_{\theta\beta\delta}\Pi \nearrow$.

The following notations where used: $\mu_\delta = \mu_\delta(a,b,n)$, $\mu_\beta = \mu_\beta(a,b,n)$, $\mu_\theta = \mu_\theta(a,b,n)$.

If $\Pi = \Pi(a,b,n) \searrow$ then we have conditions similar to the case $\nearrow$. □

Lemma 9. *The function $\Pi(\lambda, n)$ is strictly increasing if $\lambda(n) < \Lambda(n)$ and strictly decreasing if $\lambda(n) > \Lambda(n)$, where*

$$\Lambda(n) = \frac{1 - \sqrt[n-1]{e}}{1 - 2\sqrt[n-1]{e}}\ . \tag{16}$$

Proof. If we derive the function $\Pi(\lambda, n)$ in respect to the variable n we obtain

$$\frac{\partial}{\partial n}\Pi(\lambda,n) = \Big[(n-1)\lambda'(n) + \big(1-2\lambda(n)\big)\big(1-\lambda(n)\big)\ln\Big(\frac{1-\lambda(n)}{1-2\lambda(n)}\Big)\Big] \times \Big(\frac{1-\lambda(n)}{1-2\lambda(n)}\Big)^n .$$

Since $\lambda(n) \in (0, \frac{1}{2})$, under the constraints of nonlinear problems with restrictions (13) and (15), we have that

$$\Big(\frac{1-\lambda(n)}{1-2\lambda(n)}\Big)^n = \Big(1 + \frac{\lambda(n)}{1-2\lambda(n)}\Big)^n > 0.$$

The solution of differential equation

$$\lambda'(n) = -\frac{1}{n-1}\big(1-2\lambda(n)\big)\big(1-\lambda(n)\big)\ln\Big(\frac{1-\lambda(n)}{1-2\lambda(n)}\Big),$$

is $\Lambda(n)$, given by (16). Then:

- if $\lambda(n) < \Lambda(n)$, for $n \in \mathbf{d}_5$, then $\Pi(\lambda, n)$ is increasing,
- if $\lambda(n) > \Lambda(n)$, for $n \in \mathbf{d}_5$, then $\Pi(\lambda, n)$ is decreasing.

□

3. THE DURAND–KERNER'S METHOD

Durand Kerner method, [3, 6], or Weierstrass-Docev method, [2, 4], [15] is defined by

$$z_k^{(m+1)} = z_k^{(m)} - W_k^{(m)} \text{ for } k \in I_n \text{ and } m = 0, 1, \dots \ . \tag{17}$$

This method is part of the correction methods of form $C_k = W_k = \frac{P(z_k)}{F_k(z)}$ where

$$F_k(z) = F_k(z_1, z_2, \dots, z_n) = \prod_{j \neq k} (z_k - z_j) \ , \ k \in I_n.$$

3.1. Durand–Kerner's method convergence. Proof of Durand–Kerner's method convergence requires the next lemma and theorem.

LEMMA 10. *Let z_1, z_2, ..., z_n be distinct approximations and let initial conditions*

$$w \leq c(a, b, n)d,$$

$$0 < c(a, b, n) < \tfrac{1}{2}, \tag{18}$$

$$\theta(a, b, n) = \tfrac{\delta(a,b,n)}{1-2c(a,b,n)} \leq 1, \tag{19}$$

where

$$\delta(a, b, n) = \tfrac{(n-1)c(a,b,n)}{1-c(a,b,n)} \left(1 + \tfrac{c(a,b,n)}{1-2c(a,b,n)}\right)^{n-1} . \tag{20}$$

hold. Then:

(1) $\left|\widehat{W}_k\right| \leq \delta(a, b, n) \left|W_k\right|$,
(2) $\widehat{w} \leq c(a, b, n)\widehat{d}$.

Proof. See [31, Lemma 3.3.]. □

THEOREM 11. *Let the i-factor*

$$c_P(n) = \tfrac{1}{1.76325n+0.8689425}, \tag{21}$$

then the Durnad-Kerner method is convergent if the initial condition is satisfied, namely if

$$w^{(0)} < c_P(n)d^{(0)}.$$

Proof. See [31, T 3.3.]. □

Petković in [31, pp. 80] stated that the *i–factor* is "*almost optimum*".

3.2. The analytically optimum i-factor for Durand–Kerner's method with inclusion disks D_k. These inclusion disks are $D_k = D(\widehat{z}_k, |C_k|)$, where for Durand-Kerner method the corrections C_k are Weierstrass factors W_k.

The solution for the Lambert equation, $x\mathrm{e}^x = 1$, [32, LambertW-Function], [19], is denoted by $W(1)$. The constant $1/W(1)$, is called *omega constant* [32, OmegaConstant], [32, A030178] prints 59 decimals for the constant $W(1)$,

$$W(1) \approx 0.56714329040978387299996866221\ldots.$$

Next $1/W(1)$ will be denoted by ω,

$$\omega \approx 1.76322283435189671022520177695\ldots.$$

Let the constant

$$\tau \approx 0.88049674007368891\ldots, \tag{22}$$

be the real solution of

$$b^3 + (11\omega - 3)b^2 + (35\omega^2 - 30\omega + 2)b + 25\omega^3 - 75\omega^2 + 18\omega = 0. \tag{23}$$

For Durand–Kerner's method we have the function $c(a, b, n)$, given by (7), and the functions:

$$\begin{cases} \lambda(c) = c, \\ \mu_\delta(\lambda, n) = \frac{(n-1)\lambda}{1-\lambda}, \\ \mu_\beta = 1, \end{cases} \tag{24}$$

resulting from Lemma 10 relations (19) and (20), and the functions δ, β, μ_θ, θ and η are defined in (14).

Proposition 12. *If $b > h(a, 5)$, then the function $\lambda(a, b, n) = c(a, b, n) < \Lambda(n)$, where*

$$h(a, n) = \tfrac{1}{\Lambda(n)} - an$$

and Λ is given by (16)*.*

Proof. The function

$$h(a, n) = \frac{1 - \sqrt[n-1]{e}}{1 - 2\sqrt[n-1]{e}} - an, \tag{25}$$

is decreasing, because

$$h'(a, n) = \frac{\partial}{\partial n} h(a, n) = \frac{1}{\left(2(n-1)\sinh\left(\frac{1}{2(n-1)}\right)\right)^2} - a < 0,$$

for $a > 1$ and $n \in \mathbf{d}_5$. Then, if $b > h(a, 5)$ resulting that $\lambda(a, b, n) < \Lambda(n)$. $\square$

Lemma 13. *Let the* i-factor

$$c(n) = c(\omega, \tau, n) = \frac{1}{\omega n + \tau}, \tag{26}$$

then:

(1) $0 < \lambda(\omega, \tau, n) < 0.10313$,
(2) $0.707 < \delta(\omega, \tau, n) < 1$,
(3) $0.944 < \theta(\omega, \tau, n) < 1$

(4) $\eta(\omega,\tau,n) < 0$,

for $n \in \mathbf{d}_5$.

Proof. The derivative function $\lambda(a,b,n) = c(a,b,n)$ with respect to n is

$$-\tfrac{a}{(an+b)^2} < 0,$$

which implies that the functions c and λ are strictly decreasing. For

$$\lambda(\omega,\tau,n) = c(\omega,\tau,n)$$

we have

$$\lim_{n\to\infty} \lambda(\omega,\tau,n) = 0 \quad \text{and} \quad \lambda(5) = \tfrac{1}{5\omega+\tau} \approx 0.10312881573707708\ldots < \tfrac{1}{2},$$

then we have (1) of lemma.

If the function $\lambda(a,b,n) \to 0$, then the function $\Pi(\lambda(a,b,n),n)$ converges and we have

$$\lim_{n\to\infty} \Pi\big(\lambda(a,b,n),n\big) = \lim_{n\to\infty} \Pi(a,b,n) = \mathrm{e}^{\frac{1}{a}},$$

for $a > 1$. In these conditions we have that

$$\lim_{n\to\infty} \delta(a,b,n) = \tfrac{1}{a}\mathrm{e}^{\frac{1}{a}} \quad \text{and} \quad \lim_{n\to\infty} \theta(a,b,n) = \tfrac{1}{a}\mathrm{e}^{\frac{1}{a}}.$$

Since $\mu_\theta(a,b,n) = (an+b)/(an+b-2) > 1$ it follows that $\theta(a,b,n) > \beta(a,b,n) = \delta(a,b,n)$. Imposing the inequality $\theta(a,b,n) \le 1$ it follows that the equation

$$\frac{\mathrm{e}^{\frac{1}{a}}}{a} = 1.$$

This equation is a Lambert type equation, [19], whose solution is the constant ω.

The function $\mu_\delta(\omega,b,n) = (n-1)/(an+b-1)$ is increasing, because

$$\tfrac{\partial}{\partial n}\mu_\delta(\omega,b,n) = \tfrac{\omega+b-1}{(\omega n+b-1)^2} > 0,$$

for $b > 1-\omega \approx -0.7632228\ldots$ and $n \in \mathbf{d}_5$.

Under Lemma 9 and Proposition 12, for function $\Pi(\omega,b,n)$ to be increasing it is necessary that $b > h(\omega,5) \approx -3.295\ldots$, where h is given by (25). Then, it follows that $\delta(\omega,b,n) \nearrow 1$, for $b \ge 0$ when $n \to \infty$.

The derivative for the function $\mu_\theta(\omega,b,n)$ is

$$\tfrac{\partial}{\partial n}\mu_\theta(\omega,b,n) = \tfrac{-2\omega}{(\omega n+b-2)^2} < 0.$$

Therefore we can not say that the function $\theta(\omega,b,n) = \mu_\theta(\omega,b,n)\delta(\omega,b,n)$ is increasing on Proposition 7. But the function $\theta(a,b,n)$ can be expressed as

$$\theta(a,b,n) = \mu_\theta(a,b,n)\delta(a,b,n) = \mu_\theta(a,b,n)\mu_\delta(a,b,n)\Pi(a,b,n). \tag{27}$$

Using the notation $\mu_{\theta\delta}(a,b,n) = \mu_\theta(a,b,n)\mu_\delta(a,b,n)$, it follows that

$$\mu_{\theta\delta}(a,b,n) = (n-1)\tfrac{an+b}{(an+b-1)(an+b-2)} \tag{28}$$

and

$$\mu'_{\theta\delta}(a,b,n) = \frac{\partial}{\partial n}\mu_{\theta\delta}(a,b,n) = \frac{P_3(a,b,n)}{(an+b-1)^2(an+b-2)^2}. \tag{29}$$

Thus we have the polynomial

$$P_3(a,b,n) = b^3 + \alpha_2(a,n)b^2 + \alpha_1(a,n)b + \alpha_0(a,n), \tag{30}$$

with coefficients

$$\begin{array}{rcl} \alpha_2(a,n) & = & (2n+1)a-3, \\ \alpha_1(a,n) & = & n(n+2)a^2-6na+2, \\ \alpha_0(a,n) & = & a(n^2a^2-3n^2a+4n-2), \end{array} \tag{31}$$

where

$$\begin{array}{lll} \alpha_2(a,n) & \geq 0 & for\ a \geq \frac{3}{11} \text{ and } n \in \mathbf{d}_5, \\ \alpha_1(a,n) & \geq 0 & for\ a \notin \left(\frac{3}{7}-\sqrt{\frac{31}{7}}, \frac{3}{7}+\sqrt{\frac{31}{7}}\right) \approx (0.073, 0.748) \text{ and } n \in \mathbf{d}_5, \\ \alpha_0(a,n) & \leq 0 & for\ a \in \left(\frac{3}{2}-\frac{3\sqrt{17}}{10}, \frac{3}{2}+\frac{3\sqrt{17}}{10}\right) \approx (0.263, 2.737) \text{ and } n \in \mathbf{d}_5. \end{array}$$

Then, according to Cauchy's theorem, [29, pp. 3], the equation $P_3(a,b,n) = 0$ has a unique positive real root, in relation to b. The greatest root results for $n = 5$. For $a = \omega$ and $n = 5$ we have the equation (23). For $a = \omega$ and $n = 6$ we have the solution $\approx 0.92989\ldots$, for $a = \omega$ and $n = 7$ we have the solution $\approx 0.96706\ldots$ and so on.

Then it follows that $\mu_{\theta\delta}(\omega,b,n)$ is strictly increasing if $\mu'_{\theta\delta}(\omega,b,5) > 0$. We have this inequality if $b > \tau \approx 0.8804967400681223\ldots$ is true. The function $\theta(\omega,b,n)$ is the product of two positive and increasing functions, $\Pi(\omega,b,n)$ and $\mu_{\theta\delta}(\omega,b,n)$. If $b > \tau$, then, according to Proposition 7, the function $\theta(\omega,b,n)$ is strictly increasing.

We compute the function values δ and θ for $n = 5$,

$$\begin{array}{rcl} \delta(\omega,\tau,5) & \approx & 0.7497422461897223\ldots, \\ \theta(\omega,\tau,5) & \approx & 0.9445662420442020\ldots. \end{array}$$

Because functions δ and θ are strictly increasing resulting that claims the second and the third from the lemma are true.

For Durand–Kerner's method we have $\delta(a,b,n) = \beta(a,b,n)$, when taking into account (2) of the lemma and the definition of function g, given by (11), we have

$$\eta(\omega,\tau,n) = 2\lambda(\omega,\tau,n) + \beta(\omega,\tau,n) - 1 = 2\lambda(\omega,\tau,n) + \delta(\omega,\tau,n) - 1.$$

To demonstrate that $\eta(\omega,\tau,n) < 0$, for $n \in \mathbf{d}_5$, but this relation is equivalent to $\theta(\omega,\tau,n) < 1$, for $n \in \mathbf{d}_5$, relation that has already been demonstrated. $\square$

THEOREM 14. *The function $c(n)$ given by* (26) *is the optimum* i-factor *for Durand–Kerner's method with inclusion disks D_k, given by* (12).

Proof. Since ω was computed from the limit condition

$$\tfrac{1}{\omega}\mathrm{e}^{\frac{1}{\omega}} = 1,$$

and τ from the extreme condition, namely τ is the real solution for the equation, it implies that ω and τ are the best values for the given conditions. We can then state that $c(n)$ given by (26) is the optimum *i-factor* for Durand–Kerner's method. $\square$

THEOREM 15. *If the initial distinct approximations $z_1^{(0)}, z_2^{(0)}, \ldots, z_n^{(0)}$ satisfy the initial condition*

$$w^{(0)} < c(n)d^{(0)}\ , \tag{32}$$

for $n \in \mathbf{d}_5$, where c given by (26)*, then the Durand-Kerner method with inclusion disks D_k, given by* (12)*, is convergent.*

Proof. The conclusions in the Lemma 13 assure that the conclusions of Lemma 10 are satisfied, which, in turn assure the convergence of the Durand-Kerner method if the initial condition is verified (32). $\square$

REMARK 16. If $n \in \mathbf{d}_3$, case considered by Petković et al., $a = \omega$ and $b = \tau_3$, where τ_3 is the solution for the equation $\mu'_{\theta\delta}(\omega, b, 3) = 0$, and the function $\mu'_{\theta\delta}$ is given by (29). So we have

$$b^3 + (7\omega - 3)b^2 + (15\omega^2 - 18\omega + 2)b + 9\omega^3 - 27\omega^2 + 10\omega = 0,$$

and the optimum *i-factor*

$$c(n) = \tfrac{1}{\omega n + \tau_3}, \quad \text{with} \quad \tau_3 = 0.7071447767242046\ldots,$$

for $n \in \mathbf{d}_3$. $\square$

3.3. The analytically optimum i-factor for Durand–Kerner's method with inclusion disks D_k^*. The inclusion disks D_k^* are given by (10).

LEMMA 17. *Let the* i-factor *be*

$$c(n) = c(2, \tau_*, n) = \tfrac{1}{2n + \tau_*}, \quad \tau_* \approx 0.67211423631036255\ldots, \tag{33}$$

where τ_* is a root of equation

$$b^3 + 19b^2 + 82b - 64 = 0, \tag{34}$$

then:

(1) $0 < \lambda(2, \tau_*, n) < 0.094$,
(2) $0.639 < \delta(2, \tau_*, n) \leq \frac{\sqrt{e}}{2} \approx 0.824$,
(3) $0.787 < \theta(2, \tau_*, n) \leq \frac{\sqrt{e}}{2} \approx 0.824$,

for $n \in \mathbf{d}_5$.

Proof. According to corollary 4 if we consider the inclusion disks D_k^* given by (10), then $a \geq 2$ and $b > (2-a)n$. Let $a = 2$, the lowest value of a, then it follows that $b > 0$.

The function $\lambda(2, b, n) \searrow 0$ when $n \to \infty$, because the derivative function λ in relation to the variable n is $-2/(2n+b)^2 < 0$. Then, $\Pi(2, b, n) \to \sqrt{\mathrm{e}}$ when $n \to \infty$. According to Proposition 7, the function $\delta(2, b, n)$ is increasing if the functions $\mu_\delta(2, b, n)$ and $\Pi(2, b, n)$ are increasing. The derivative of the function $\mu_\delta(2, b, n)$ is

$$\tfrac{\partial}{\partial n}\mu_\delta(2, b, n) = \tfrac{b+1}{(b+2n-1)^2}\ ,$$

and is positive if $b > -1$. The function $\Pi(2, b, n)$ is increasing if $b > h(2, 5) \approx -4.479$, where h is given by (25). Therefore it follows that $\delta(2, b, n)$ is increasing for $b \geq 0$ and $n \in \mathbf{d}_5$. In these circumstances we have

$$\lim_{n\to\infty} \delta(2, b, n) = \lim_{n\to\infty} \theta(2, b, n) = \tfrac{\sqrt{\mathrm{e}}}{2} \approx 0.824 < 1.$$

The derivative of the function $\mu_\theta(2, b, n)$ is

$$\tfrac{\partial}{\partial n}\mu_\theta(2, b, n) = \tfrac{-4}{(2n+b-2)^2} < 0.$$

We can not say that the function $\theta(2, b, n) = \mu_\theta(2, b, n)\delta(2, b, n)$ is increasing in the Proposition 7. But the function $\theta(a, b, n)$ can be expressed as (27). We denote by $\mu_{\theta\delta}(a, b, n) = \mu_\theta(a, b, n)\mu_\delta(a, b, n)$ and consider the polynomial (30) with coefficients (31). If $0.748 < a < 2.737$, then, according to Cauchy's theorem, [29, pp. 3], the equation $P_3(a, b, n) = 0$, in relation to the variable b, has a unique real positive solution. The largest positive real root results for $n = 5$. For $a = 2$ and $n = 5$ resulting the equation (34), with real positive solution $\tau_* \approx 0.67211423631036255\ldots$. For $a = 2$ and $n = 6$ we have the solution $0.71912\ldots$, for $a = 2$ and $n = 7$ we have the solution $0.75419\ldots$ and so on.

Then it follows that the function $\mu_{\theta\delta}(2, b, n)$ is increasing if $b > \tau_*$. The function $\theta(2, b, n)$ is the product of two positive and increasing functions, namely $\Pi(2, b, n)$ and $\mu_{\theta\delta}(2, b, n)$, if $b > \tau_*$, then, according to Proposition 7, the function $\theta(2, b, n)$ is increasing.

Monotonous function values θ, δ and λ for $n = 5$ are:

$$\begin{aligned}
\theta(2, \tau_*, 5) &\approx 0.7874985412073337\ldots,\\
\delta(2, \tau_*, 5) &\approx 0.6399179355710173\ldots,\\
\lambda(2, \tau_*, 5) &\approx 0.09370214540972971.
\end{aligned}$$

□

THEOREM 18. *The function $c(n)$ given by* (33) *is the optimum* i-factor *for Durand–Kerner's method with the inclusion disks D_k^* given by* (10).

Proof. Since $a = 2$ and $b = \tau_*$ are the best values under the circumstances, then we say that $c(n)$ given by (33) is optimum *i-factor* for Durand–Kerner's method with the inclusion disks D_k^*, given by (10). □

THEOREM 19. *If the initial distinct approximations* $z_1^{(0)}, z_2^{(0)}, \dots, z_n^{(0)}$ *satisfy the initial condition*

$$w^{(0)} < c(n)d^{(0)}, \tag{35}$$

for $n \in \mathbf{d}_5$*, where* c *is given by* (33)*, then Durand–Kerner's method with inclusion disks* D_k^**, given by* (10)*, is convergent.*

Proof. The conclusions in the Lemma 17 assure that the conclusions of Lemma 10 are satisfied, which, in turn assure the convergence of the Durand-Kerner method if the initial condition is verified (35). □

4. THE TANABE'S METHOD

The Tanabe's method, [10], is given by the formula:

$$z_k^{(m+1)} = z_k^{(m)} - W_k^{(m)} \left(1 - \sum_{j \neq k} \frac{W_j^{(m)}}{z_k^{(m)} - z_j^{(m)}} \right), \ k \in I_n \ m = 0, 1, \dots. \tag{36}$$

If we denote by

$$t_k = \sum_{j \neq k} \frac{W_j}{z_k - z_j}, \ k \in I_n,$$

then for reasonably small values of t_k, we can state that $1/(1 + t_k) = 1 - t_k + O(t_k^2)$. It is well known that Tanabe's method results form Börsch-Supan's method based on this observation. Tanabe's method is one with corrections, where

$$C_k(z_1, z_2, \dots, z_n) = \frac{P(z_k)}{F_k(z_1, z_2, \dots, z_n)}, \ k \in I_n,$$

but

$$F_k(z_1, z_2, \dots, z_n) = \frac{1}{1 - \sum_{j \neq k} \frac{W_k^{(m)}}{z_k^{(m)} - z_j^{(m)}}} \cdot \prod_{j \neq k} (z_k - z_j).$$

4.1. Tanabe's method convergence. In order to prove the convergence we state the following theorem and 4 lemmas.

LEMMA 20. *Let* $z_1, z_2, \dots, z_n \in \mathbb{C}$ *be distinct numbers and the* i-factor $c(a, b, n)$ *that satisfies the conditions*

$$0 < c(a, b, n) < \psi(n) = \frac{1}{1 + \sqrt{2n - 1}} \tag{37}$$

and

$$w \leq c(a, b, n)d. \tag{38}$$

Then:

$$(1)\qquad \frac{\lambda(a,b,n)}{c(a,b,n)} \geq \left|1-\sum_{j\neq k}\frac{W_j}{z_k-z_j}\right| \geq 2-\frac{\lambda(a,b,n)}{c(a,b,n)},$$

$$(2)\qquad |\widehat{z}_k - z_k| \leq \frac{\lambda(a,b,n)}{c(a,b,n)}\,|W_k| \leq \lambda(a,b,n)d,$$

$$(3)\qquad |\widehat{z}_k - z_j| \geq \big(1-\lambda(a,b,n)\big)d,$$

$$(4)\qquad |\widehat{z}_k - \widehat{z}_j| \geq \big(1-2\lambda(a,b,n)\big)d,$$

$$(5)\qquad \left|1+\sum\frac{W_j}{\widehat{z}_k-z_j}\right| \leq \frac{(n-1)\big(\lambda(a,b,n)+(n-1)c(a,b,n)\big)c(a,b,n)^2}{\big(2c(a,b,n)-\lambda(a,b,n)\big)\big(1-\lambda(a,b,n)\big)},$$

$$(6)\qquad \left|\prod_{j\neq k}\frac{\widehat{z}_k-z_j}{\widehat{z}_k-\widehat{z}_j}\right| \leq \left(1+\frac{\lambda(a,b,n)}{1-2\lambda(a,b,n)}\right)^{n-1},$$

where

$$\lambda(a,b,n) = \big(1+(n-1)c(a,b,n)\big)c(a,b,n). \tag{39}$$

Proof. See [31, Lemma 3.8.]. □

LEMMA 21. *Let z_1, z_2, ..., z_n be distinct approximations for the roots ζ_1, ζ_2, ..., ζ_n of the polynomial P and let us assume the conditions* (37) *and* (38) *from Lemma* 20 *to be true. Let us also consider true the following inequality*

$$\theta(a,b,n) = \frac{\delta(a,b,n)}{1-2\lambda(a,b,n)} \leq 1, \tag{40}$$

where

(41)
$$\delta(a,b,n) = \frac{(n-1)c(a,b,n)\lambda(a,b,n)\big(\lambda(a,b,n)+(n-1)c(a,b,n)\big)}{\big(2c(a,b,n)-\lambda(a,b,n)\big)\big(1-\lambda(a,b,n)\big)} \times \left(1+\frac{\lambda(a,b,n)}{1-2\lambda(a,b,n)}\right)^{n-1}.$$

Then:

$$(1)\qquad \left|\widehat{W_k}\right| \leq \delta(a,b,n)\,|W_k|\,,\ \ k\in I_n$$

$$(2)\qquad \widehat{w} \leq c(a,b,n)w.$$

Proof. See [31, Lemma 3.9.]. □

LEMMA 22. *Let us consider all the conditions form Lemma* 20 *and* 21 *and the next two ones to be true*

$$\beta(a,b,n) = \frac{\lambda(a,b,n)\delta(a,b,n)}{2c(a,b,n)-\lambda(a,b,n)} < 1 \tag{42}$$

and

$$\eta(a,b,n) = 2\lambda(a,b,n) - \frac{1}{g\big(\beta(a,b,n)\big)} < 0\ . \tag{43}$$

If the initial distinct approximations $z_1^{(0)}, z_2^{(0)}, \ldots, z_n^{(0)}$ *satisfy the initial condition*

$$w^{(0)} \leq c(a,b,n)d^{(0)}, \tag{44}$$

then Tanabe's method given by (36) *is convergent.*

Proof. See [31, T 3.6.]. □

LEMMA 23. *The* i-factor *function*

$$c_P(n) = \tfrac{1}{3n}. \tag{45}$$

satisfies then conditions (37)*,* (40)*,* (42) *and* (43)*.*

Proof. See [31, Lemma 3.10.]. □

THEOREM 24. *The Tanabe's method, given by* (36)*, is convergent if the initial condition* (38) *is true for the initial distinct approximations* $z_1^{(0)}$, $z_2^{(0)}$, $\ldots z_n^{(0)}$*, where the* i-factor *is given by* (45)*.*

4.2. The optimum analytic i-factor. For Tanabe's method we have the function $c(a,b,n)$, given by (7) and the functions:

$$\begin{cases} \lambda(c,n) & = & c+(n-1)c^2 \\ \mu_\delta(\lambda,c,n) & = & \frac{(n-1)^2c^2\lambda+(n-1)c\lambda^2}{(2c-\lambda)(1-\lambda)}, \\ \mu_\beta(\lambda,c) & = & \frac{\lambda}{2c-\lambda}, \end{cases} \tag{46}$$

resulting from the Lemmas 20, 21 and 22, respectively from the relations (39), (41) and (42) and the functions δ, β, μ_θ and θ defined in (14). Let the constant ω_{T} be the solution for the Lambert type equation, [19],

$$\tfrac{x+1}{x^2}\exp\left(\tfrac{x+1}{x^2}\right) = \tfrac{(x-1)^2}{x+1}\ . \tag{47}$$

The approximative value ω_{T} is $2.7480500253477966212\ldots$.

PROPOSITION 25. *If* $b > h_{\mathrm{T}}(a,5)$*, then* $\lambda(a,b,n) < \Lambda(n)$*, where* $\lambda(a,b,n) = c(a,b,n)+(n-1)c(a,b,n)^2$*,*

$$h_{\mathrm{T}}(a,n) = \tfrac{-2(a-1)\sqrt[n-1]{e}+2an-1}{2(\sqrt[n-1]{e}-1)} + \tfrac{\sqrt{4(2n-1)\sqrt[n-1]{e^2}-4(3n-2)\sqrt[n-1]{e}+4n-3}}{2(\sqrt[n-1]{e}-1)} \tag{48}$$

and Λ *is given by* (16)*.*

Proof. The equation $\lambda(a,b,n) = \Lambda(n)$ has solutions:

$$\tfrac{-2(a-1)\sqrt[n-1]{e}+2an-1}{2(\sqrt[n-1]{e}-1)} \pm \tfrac{\sqrt{4(2n-1)\sqrt[n-1]{e^2}-4(3n-2)\sqrt[n-1]{e}+4n-3}}{2(\sqrt[n-1]{e}-1)}.$$

The function $h_{\mathrm{T}}(a,n)$, given by (48), for $a > 1$ and $n \in \mathbf{d}_5$. Then, if $b > h_{\mathrm{T}}(a,5)$ resulting that $\lambda(a,b,n) < \Lambda(n)$. □

LEMMA 26. *If* $a = \omega_{\mathrm{T}}$ *and* $b = 0$*, then for the* i-factor $c(n) = c(\omega_{\mathrm{T}},0,n)$ *the following inequalities must be true for* $n \in \mathbf{d}_5$*,*

(1) $0 < c(\omega_{\mathrm{T}}, 0, n) < \psi(n)$,
(2) $0 < \lambda(\omega_{\mathrm{T}}, 0, n) < 0.094$,
(3) $0.349 < \delta(\omega_{\mathrm{T}}, 0, n) < 0.4664$,
(4) $0.636 < \beta(\omega_{\mathrm{T}}, 0, n) < 1$,
(5) $0.783 < \theta(\omega_{\mathrm{T}}, 0, n) < 1$,

where $\psi(n)$ *is given by* (37).

Proof. The function $c(\omega_{\mathrm{T}}, 0, n)$ is clearly greater than 0 and verifies the inequality $c(\omega_{\mathrm{T}}, 0, n) < \psi(n)$, for $n \in \mathbf{d}_5$.

For the function

$$\lambda(a, b, n) = \lambda\big(c(a, b, n), n\big) = \tfrac{1}{an+b} + \tfrac{n-1}{(an+b)^2},$$

we have

$$\lim_{n\to\infty} \lambda(a, b, n) = 0, \quad \text{for} \quad a > 0.$$

The function $\Pi(a, b, n)$ is

$$\Pi(a, b, n) = \Big(1 + \tfrac{\lambda(a,b,n)}{1-2\lambda(a,b,n)}\Big)^{n-1}.$$

Since the limit the function λ is 0, then we move to limit the function $\Pi(a, b, n)$ and we have

$$L_{\Pi}(a) = \lim_{n\to\infty} \Pi(a, b, n) = \mathrm{e}^{\frac{a+1}{a^2}}.$$

The function $\mu_\delta(a, b, n)$ is

$$\mu_\delta(a, b, n) = \tfrac{(n-1)[(a+1)n+b-1][an^2+(b+1)n-1]}{(an+b)[(a-1)n+b+1][a^2n^2+(2ab-a-1)n+b^2-b+1]}, \tag{49}$$

and its limit is

$$L_{\mu_\delta}(a) = \lim_{n\to\infty} \mu_\delta(a, b, n) = \tfrac{a+1}{a^2(a-1)}.$$

Then the function $\delta(a, b, n) = \mu_\delta(a, b, n)\Pi(a, b, n)$, has the limit

$$L_\delta(a) = \lim_{n\to\infty} \delta(a, b, n) = \tfrac{a+1}{a^2(a-1)}\mathrm{e}^{\frac{a+1}{a^2}}.$$

The function $\mu_\beta(a, b, n)$ is

$$\mu_\beta(a, b, n) = \tfrac{(a+1)n+b-1}{(a-1)n+b+1},$$

and its limit is

$$L_{\mu_\beta}(a) = \lim_{n\to\infty} \mu_\beta(a, b, n) = \tfrac{a+1}{a-1}.$$

Then the function $\beta(a, b, n) = \mu_\beta(a, b, n)\delta(a, b, n)$, has the limit

$$L_\beta(a) = \lim_{n\to\infty} \beta(a, b, n) = \tfrac{(a+1)^2}{a^2(a-1)^2}\mathrm{e}^{\frac{a+1}{a^2}}.$$

The function $\mu_\theta(a, b, n)$ is

$$\mu_\theta(a, b, n) = \tfrac{(an+b)^2}{a^2n^2-2(a+1-ab)n+b^2-2b+2},$$

and its limit is 1 when $n \to \infty$. Then it follows that the function

$$\theta(a,b,n) = \mu_\theta(a,b,n)\beta(a,b,n),$$

has the limit

$$L_\theta(a) = \tfrac{(a+1)^2}{a^2(a-1)^2}\mathrm{e}^{\frac{a+1}{a^2}}.$$

Impose conditions $L_\delta(a) \leq 1$ and $L_\beta(a) = L_\theta(a) \leq 1$. Lambert type equations, $L_\delta(a) = 1$ has solution $\approx 2.236180389652745005$ and $L_\theta(a) = 1$ has solution $\omega_{\mathrm{T}} \approx 2.7480500253477966212\ldots$. Let $a = \omega_{\mathrm{T}}$, sine for any $a > \omega_{\mathrm{T}}$ we have $L_\delta(a) < 1$.

To prove that the function $\Pi(\omega_{\mathrm{T}}, b, n)$ is monotonically increasing to its limit. According to Lemma 9 and Proposition 25, the function $\Pi(\omega_{\mathrm{T}}, b, n)$ is increasing for $b > h_{\mathrm{T}}(\omega_{\mathrm{T}}, 5) \approx -5.529\ldots$. It follows that for $b \geq 0$ the function $\Pi(\omega_{\mathrm{T}}, 0, n)$ is monotonically increasing to its limit. Let further $b = 0$.

To prove that the function $\delta(\omega_{\mathrm{T}}, 0, n)$ is monotonically increasing to its limit, we derivative the function $\mu_\delta(\omega_{\mathrm{T}}, 0, n)$, where $\mu_\delta(a,b,n)$ is given by (49), so

$$\mu'_\delta(\omega_{\mathrm{T}}, 0, n) = \tfrac{\alpha_6 n^6 + \alpha_5 n^5 + \alpha_4 n^4 + \alpha_3 n^3 + \alpha_2 n^2 + \alpha_1 n + \alpha_0}{\omega_{\mathrm{T}}^2 n^2 [(\omega_{\mathrm{T}}-1)n+1]^2 [\omega_{\mathrm{T}}^2 n^2 - (\omega_{\mathrm{T}}+1)n+1]^2},$$

where

$$\begin{aligned}
\alpha_6 &= \omega_{\mathrm{T}}^5 - 2\omega_{\mathrm{T}}^3 + 2\omega_{\mathrm{T}}^2 + \omega_{\mathrm{T}} \approx 133.06582372905672183,\\
\alpha_5 &= 2\omega_{\mathrm{T}}^4 + 4\omega_{\mathrm{T}}^3 - 10\omega_{\mathrm{T}}^2 - 4\omega_{\mathrm{T}} \approx 110.55940610241204683,\\
\alpha_4 &= -3\omega_{\mathrm{T}}^4 - 6\omega_{\mathrm{T}}^3 + 14\omega_{\mathrm{T}}^2 + 6\omega_{\mathrm{T}} + 1 \approx -172.39088809543209589,\\
\alpha_3 &= 4\omega_{\mathrm{T}}^3 - 6\omega_{\mathrm{T}}^2 - 4\omega_{\mathrm{T}} - 4 \approx 22.707791497616619358,\\
\alpha_2 &= \omega_{\mathrm{T}} + 6 \approx 8.7480500253477966212,\\
\alpha_1 &= -4,\\
\alpha_0 &= 1.
\end{aligned}$$

The largest real root of the nominator polynomial function $\mu'_\delta(\omega_{\mathrm{T}}, 0, n)$ is $\approx 0.63786\ldots$, hence for $n \in \mathbf{d}_5$, $\mu'_\delta(\omega_{\mathrm{T}}, 0, n) > 0$. Then, according to Proposition 7, the function $\delta(\omega_{\mathrm{T}}, 0, n) = \mu_\delta(\omega_{\mathrm{T}}, 0, n)\Pi(\omega_{\mathrm{T}}, 0, n)$ is monotonically increasing.

To prove that $\beta(\omega_{\mathrm{T}}, 0, n)$ is monotone increasing function to its limit, we derivative the function $\mu_\beta(\omega_{\mathrm{T}}, 0, n)$, is $2\omega_{\mathrm{T}}/[(\omega_{\mathrm{T}} - 1)n + 1]^2 > 0$. Then, according to Proposition 7, the function $\beta(\omega_{\mathrm{T}}, 0, n) = \mu_\beta(\omega_{\mathrm{T}}, 0, n)\delta(\omega_{\mathrm{T}}, 0, n)$ is monotonically increasing.

To prove that $\theta(\omega_{\mathrm{T}}, 0, n)$ is monotonically increasing function to its limit. The derivative function $\mu_\theta(\omega_{\mathrm{T}}, 0, n)$ is

$$-\tfrac{2\omega_{\mathrm{T}}^2 n[(\omega_{\mathrm{T}}+1)n-2]}{[\omega_{\mathrm{T}}^2 n^2 - 2(\omega_{\mathrm{T}}+1)+2]^2} < 0\ .$$

Then we can say that the function $\theta(\omega_{\mathrm{T}},0,n)=\mu_\theta(\omega_{\mathrm{T}},0,n)\beta(\omega_{\mathrm{T}},0,n)$ is monotonically increasing. Let the function

$$\begin{aligned}\mu_{\theta\beta\delta}(\omega_{\mathrm{T}},0,n)&=\mu_\theta(\omega_{\mathrm{T}},0,n)\mu_\beta(\omega_{\mathrm{T}},0,n)\mu_\delta(\omega_{\mathrm{T}},0,n)\\&=\tfrac{\omega_{\mathrm{T}}n(n-1)[(\omega_{\mathrm{T}}+1)n-1]^2(\omega_{\mathrm{T}}n^2+n-1)}{[(\omega_{\mathrm{T}}-1)n+1]^2[\omega_{\mathrm{T}}^2n^2-2(\omega_{\mathrm{T}}+1)n+2][\omega_{\mathrm{T}}^2n^2-(\omega_{\mathrm{T}}+1)n+1]},\end{aligned}$$

its derivative is

$$\mu'_{\theta\beta\delta}(\omega_{\mathrm{T}},0,n)=\tfrac{\omega_{\mathrm{T}}[(\omega_{\mathrm{T}}+1)n-1]P_8(\omega_{\mathrm{T}},n)}{[(\omega_{\mathrm{T}}-1)n+1]^3[\omega_{\mathrm{T}}^2n^2-2(\omega_{\mathrm{T}}+1)n+2]^2[\omega_{\mathrm{T}}^2n^2-(\omega_{\mathrm{T}}+1)n+1]^2}$$

where polynomial $P_8(\omega_{\mathrm{T}},n)=\alpha_8n^8+\alpha_7n^7+\alpha_6n^6+\alpha_5n^5+\alpha_4n^4+\alpha_3n^3+\alpha_2n^2+\alpha_1n+\alpha_0$ has the following coefficients:

$$\begin{aligned}\alpha_8&=\omega_{\mathrm{T}}^7-4\omega_{\mathrm{T}}^5+4\omega_{\mathrm{T}}^4+3\omega_{\mathrm{T}}^3\approx 847.0086516,\\
\alpha_7&=4\omega_{\mathrm{T}}^5-16\omega_{\mathrm{T}}^4-12\omega_{\mathrm{T}}^3-8\omega_{\mathrm{T}}^2-4\omega_{\mathrm{T}}\approx-606.0300766,\\
\alpha_6&=-3\omega_{\mathrm{T}}^6-8\omega_{\mathrm{T}}^5+18\omega_{\mathrm{T}}^4+28\omega_{\mathrm{T}}^3+43\omega_{\mathrm{T}}^2+18\omega_{\mathrm{T}}-2\approx-565.9812397,\\
\alpha_5&=10\omega_{\mathrm{T}}^5+8\omega_{\mathrm{T}}^4-18\omega_{\mathrm{T}}^3-72\omega_{\mathrm{T}}^2-24\omega_{\mathrm{T}}+12\approx 1052.2011279,\\
\alpha_4&=-14\omega_{\mathrm{T}}^4-15\omega_{\mathrm{T}}^3+38\omega_{\mathrm{T}}^2-4\omega_{\mathrm{T}}-30\approx-863.72570760,\\
\alpha_3&=14\omega_{\mathrm{T}}^3+8\omega_{\mathrm{T}}^2+36\omega_{\mathrm{T}}+40\approx 489.8813608,\\
\alpha_2&=-9\omega_{\mathrm{T}}^2-30\omega_{\mathrm{T}}-30\approx-180.4075112,\\
\alpha_1&=8\omega_{\mathrm{T}}+12\approx 33.9844002,\\
\alpha_0&=-2.\end{aligned}$$

The monomials $(\omega_{\mathrm{T}}+1)n-1$ and $(\omega_{\mathrm{T}}-1)n+1$ are equal to 0 in $0.2668054\ldots$ and $-0.572066\ldots$. The largest positive real root of polynomial $P_8(\omega_{\mathrm{T}},n)$ is $\approx 0.600633\ldots$, hence $\mu'_{\theta\beta\delta}(\omega_{\mathrm{T}},0,n)>0$ for $n\in\mathbf{d}_5$, i.e. $\mu_{\theta\beta\delta}(\omega_{\mathrm{T}},0,n)$ is increasing, then and the function $\theta(\omega_{\mathrm{T}},0,n)=\mu_{\theta\beta\delta}(\omega_{\mathrm{T}},0,n)\Pi(\omega_{\mathrm{T}},0,n)$ is increasing for $n\in\mathbf{d}_5$.

Monotonous function values λ, δ, β and θ for $n=5$ are:

$$\begin{aligned}\lambda(\omega_{\mathrm{T}},0,5)&\approx 0.093965939752349623079,\\
\delta(\omega_{\mathrm{T}},0,5)&\approx 0.349197650213356803400,\\
\beta(\omega_{\mathrm{T}},0,5)&\approx 0.636005603331306147860,\\
\theta(\omega_{\mathrm{T}},0,5)&\approx 0.783192428417692776398.\end{aligned}$$

For the function δ we have

$$\lim_{n\to\infty}\delta(\omega_{\mathrm{T}},0,n)=\tfrac{\omega_{\mathrm{T}}-1}{\omega_{\mathrm{T}}+1}\approx 0.466389192653742137777.$$

□

THEOREM 27. *The optimum* i-factor *for Tanabe's method is*

$$c(n)=\tfrac{1}{\omega_{\mathrm{T}}n}. \tag{50}$$

Proof. The constants $a=\omega_{\mathrm{T}}$ and $b=0$, are the best values in the given conditions, so the *i-factor* $c(n)$ given by (50), is optimal. □

THEOREM 28. *If the initial distinct approximation* $z_1^{(0)}, z_2^{(0)}, \ldots, z_n^{(0)}$ *satisfy the initial condition*

$$w^{(0)} < c(n)d^{(0)}, \tag{51}$$

for $n \in \mathbf{d}_5$*, where* c *given by* (50)*, then Tanabe's method with inclusion disks* D_k^**, given by* (10)*, is convergent.*

Proof. Since $\omega_{\text{T}} > 2$ and $b = 0$, then according to corollary 4 the disks D_k^*, given by (10), are mutually disjoint and each of them contain exactly one zero of polynomial P.

The conclusions on Lemma 26 assure the fulfillment of Lemmas 20, 21 and 22, fact, that in turn, assures the convergence for Tanabe's method if the initial condition (51) is verified. □

5. CONCLUSIONS

(1) For Durand–Kerner's method, given by (17), with the inclusion disks D_k, given by (12), the optimum *i-factor* is

$$c(n) = \frac{1}{\omega n + \tau}\ , \quad n \in \mathbf{d}_5,$$

where $\omega \approx 1.7632228343518967\ldots$, $\tau \approx 0.88049674007368891\ldots$.

(2) For Durand–Kerner's method, given by (17), with the inclusion disks D_k^*, given by (10), the optimum *i-factor* is

$$c(n) = \frac{1}{2n + \tau_*}, \quad n \in \mathbf{d}_5,$$

where $\tau_* \approx 0.67211423631036255\ldots$.

(3) For Tanabe's method, given by (36), with the inclusion disks D_k^*, given by (10), the optimum *i-factor* is

$$c(n) = \frac{1}{\omega_{\text{T}} n}, \quad n \in \mathbf{d}_5,$$

where $\omega_{\text{T}} \approx 2.7480500253477966212\ldots$.

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Received by the editors: April 11, 2011.