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BETTER APPROXIMATION BY STANCU BETA OPERATORS IN COMPACT INTERVAL

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Abstract. The present paper deals with the study of Stancu-Beta operators which preserve the constant as well as linear functions but not the quadratic ones. We apply the King's approach to propose the modified form of these operators, so as they preserve the quadratic functions, which results in better approximation for the modified operators in the compact interval $(0,1)$ for these operators.

Keywords. Positive linear operators, Korovkin-type approximation theorem, Stancu-Beta operators.

1. INTRODUCTION

Many well-known approximating operators reproduce constant as well as linear functions for example the classical Bernstein polynomials, the Szász-Mirakjan operators, the Baskakov operators, and so on. In [1] the authors have studied the rate of convergence for the well known Stancu Beta operators, which also reproduce constant and linear functions. But the operators do not preserve quadratic functions. In this case a natural question arises: can we modify these operators such that the quadratic functions are preserved? In this paper we mainly focus on this problem and find affirmative answers. Actually, the basic reason of this idea is to make convergence faster to the function being approximated. King [3] was the first, who considered the Bernstein polynomials and obtained the faster convergence by modifying the well known Bernstein polynomials. Here we study the convergence behavior of Stancu Beta operators.

D. D. Stancu [5] introduced Beta operators L_n of second kind in order to approximate the Lebesgue integrable functions on the interval $(0,\infty)$ as

$$(L_n f)(x) = \tfrac{1}{B(nx,n+1)} \int_0^\infty \tfrac{t^{nx-1}}{(1+t)^{nx+n+1}} f(t)\mathrm{d}t. \tag{1}$$

Then, Abel and Gupta in [1], obtained the rate of convergence by means of decomposition technique. The moments of the operators (1) are given in the following lemmas:

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LEMMA 1. [5] *For all* $e_i(x) = x^i, i \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, $x > 0$ *with* $n > i$, *we have* $(L_n e_0)(x) = 1$, $(L_n e_1)(x) = e_1(x)$ *and* $(L_n e_2)(x) = x^2 + \frac{x(1+x)}{n-1}$. *Also we have the recurrence relation*

$$(L_n e_{i+1})(x) = \tfrac{nx+i}{n-i}(L_n e_i)(x), \text{ for all } n > i.$$

Proof. By the relationship (1) of the Beta operators of second kind, it is obvious that $(L_n e_0)(x) = 1$ and $(L_n e_1)(x) = e_1(x)$ (see [1], Proposition 2). Next

$$\begin{aligned}(L_n e_{i+1})(x) &= \tfrac{1}{B(nx,n+1)} B(nx+i+1, n-i) = (L_n e_i)(x) \cdot \tfrac{B(nx+i+1,n-i)}{B(nx+i,n-i+1)} \\ &= (L_n e_i)(x) \cdot \tfrac{nx+i}{n-i}.\end{aligned}$$

Since $B(\alpha, \beta)$ is only defined for $\alpha > 0$ and $\beta > 0$, it follows that the above recurrence is valid only for $n - i > 0$. The value of $(L_n e_2)(x)$ follows from the recurrence relation.
This completes the proof of Lemma 1. □

By simple computation and using Lemma 1, we have the following result:

LEMMA 2. *For fixed* $x \in (0, \infty)$, *if define the function* φ_x *by* $\varphi_x(t) = t - x$. *Then*

(*i*) $(L_n \varphi_x^0)(x) = 1$,
(*ii*) $(L_n \varphi_x^1)(x) = 0$,
(*iii*) $(L_n \varphi_x^2)(x) = \frac{x(1+x)}{n-1}$.

2. DIRECT RESULTS

In this section we compute the rates of convergence of the operators $(L_n f)(x)$. We define by $C_B(0, \infty)$, the space of all bounded and continuous functions on $(0, \infty)$ endowed with the norm $||f|| = \sup_{x \in (0,\infty)} |f(t)|$. For $f \in C_B(0, \infty)$ the first and second order modulus of continuity of f are denoted by $\omega(f, \delta)$ and $\omega_2(f, \delta)$ respectively and defined as

$$\omega(f, \delta) = \sup_{x-\delta \le t \le x+\delta,\, t \in (0,\infty)} |f(t) - f(x)|,$$

$$\omega_2(f, \sqrt{\delta}) = \sup_{x-\delta \le t \le x+\delta,\, t \in (0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|.$$

The Peetre's K-functional is defined as

$$K_2(f, \delta) = \inf\{||f - g|| + \delta ||g''|| : g \in W_\infty^2\}, \delta > 0,$$

where $W_\infty^2 = \{g \in C_B(0, \infty) : g', g'' \in C_B(0, \infty)\}$. Using [2], there exists a positive constant C such that

$$K_2(f, \delta) \le C\omega_2(f, \sqrt{\delta}).$$

Theorem 3. *For the Stancu-Beta operators we can write that, for every $f \in C_B(0,\infty), x > 0$ and $n > 1$,*

$$|(L_n f)(x) - f(x)| \leq 2\omega(f, \alpha_x), \tag{2}$$

where $\alpha_x = \sqrt{\frac{x(1+x)}{n-1}}$.

Proof. For every $\alpha > 0$ and $n \in N$, using linearity and monotonicity of L_n we easily get that

$$|(L_n f)(x) - f(x)| \leq \omega(f, \alpha)\left[1 + \tfrac{1}{\alpha}\sqrt{(L_n \varphi_x^2)(x)}\right].$$

Applying Lemma 2 and choosing $\alpha = \alpha_x$, the proof is completed. □

Theorem 4. *Let $f \in C_B(0,\infty)$, then for every $x \in (0,\infty)$ and for $C > 0$, we have*

$$|(L_n f)(x) - f(x)| \leq C\omega_2\left(f, \sqrt{\tfrac{x(1+x)}{n-1}}\right).$$

Proof. Let $g \in W_\infty^2$. By Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\mathrm{d}u,$$

and Lemma 2, we have

$$(L_n f)(x) - g(x) = \left(L_n \int_x^t (t-u)g''(u)\mathrm{d}u\right)(x),$$

we know that

$$\left|\int_x^t (t-u)g''(u)\mathrm{d}u\right| \leq (t-u)^2 ||g''||,$$

therefore

$$|(L_n f)(x) - f(x)| \leq \left(L_n (t-u)^2(x)\right)||g''|| = \tfrac{x(1+x)}{n-1}||g''||,$$

by Lemma 1, we have

$$|(L_n f)(x)| \leq \tfrac{1}{B(nx, n+1)} \int_0^\infty \tfrac{t^{nx-1}}{(1+t)^{nx+n+1}}|f(t)|\mathrm{d}t \leq ||f||.$$

Hence

$$\begin{aligned}|(L_n f)(x) - f(x)| &\leq |(L_n(f-g))(x) - (f-g)(x)| + |(L_n f)(x) - f(x)| \\ &\leq 2||f-g|| + \tfrac{x(1+x)}{n-1}||g''||,\end{aligned}$$

taking the infimum on the right side over all $g \in W_\infty^2$ and using K-functional, the required result is obtained. □

3. CONSTRUCTION OF OPERATORS

Let $r_n(x)$ be sequence of real valued continuous functions defined on $(0,\infty)$ with $0<r_n(x)<\infty)$ and defined by

$$r_n(x)=\tfrac{-1+\sqrt{1+4n(n-1)x^2}}{2n}.$$

Then we can define the modified form of operators (1) as

$$(L_n^*f)(x)=\tfrac{1}{B(nr_n(x),n+1)}\int_0^\infty \tfrac{t^{nr_n(x)-1}}{(1+t)^{nr_n(x)+n+1}}f(t)\mathrm{d}t. \tag{3}$$

For these modified operators, we have the following lemmas:

Lemma 5. *For each $x>0$, we have*

(i) $(L_n^*1)(x)=1$,

(ii) $(L_n^*t)(x)=\tfrac{-1+\sqrt{1+4n(n-1)x^2}}{2n}$,

(iii) $(L_n^*t^2)(x)=x^2$.

Lemma 6. *For fixed $x\in(0,\infty)$, define the function φ_x by $\varphi_x(t)=t-x$. The central moments for the operators L_n^* are given by*

(i) $(L_n^*\varphi_x^0)(x)=1$,

(ii) $(L_n^*\varphi_x^1)(x)=-x+\tfrac{-1+\sqrt{1+4n(n-1)x^2}}{2n}$,

(iii) $(L_n^*\varphi_x^2)(x)=2x\left[x+\tfrac{1}{2n}-\tfrac{\sqrt{1+4n(n-1)x^2}}{2n}\right]$.

Theorem 7. *For every $f\in C_B(0,\infty)$, $x>0$ and $n>1$, we have*

$$|(L_n^*f)(x)-f(x)|\le 2\omega(f,\delta_x),$$

where

$$\delta_x=2x\left[x+\tfrac{1}{2n}-\tfrac{1}{2n}\sqrt{1+4n(n-1)x^2}\right].$$

The proof of this can be carried out on the same lines as of Theorem 3.

Remark 8. Now considering the above remark the similar claim is valid for the operators L_n^* on the interval $(0,1)$. In order to get a better estimation we must show that $\delta_x<\alpha_x$ for appropriate x's. Indeed, for $0<x<1$, we have $x^2<1$. Also since

$$x^2\left[(2n-1)^2-4n(n-1)\right]<1,$$

or

$$1+4n(n-1)x^2>(2n-1)^2x^2,$$

which gives

$$\sqrt{1+4n(n-1)x^2}>(2n-1)x,$$

then we obtain

$$-\tfrac{1}{2n}+\tfrac{1}{2n}\sqrt{1+4n(n-1)x^2}>-\tfrac{1}{2n}+\tfrac{2n-1}{2n}x,$$

thus we have $x - r_n(x) < \frac{1+x}{2n}$,
i.e.

$$2x(x - r_n(x)) < \frac{x(1+x)}{n} < \frac{x(1+x)}{n-1},$$

for $x \in (0,1)$ and $n > 1$. This guarantees that $\delta_x < \alpha_x$ for $x \in (0,1)$ and $n > 1$, which corrects our claim.

A function $f \in C_B(0,\infty)$ belongs to $\mathrm{Lip}_M(\alpha)$, if the following inequality holds,

$$|f(y) - f(x)| \leq M|y - x|^\alpha (x, y \in (0,\infty)). \tag{4}$$

□

THEOREM 9. *For every $f \in \mathrm{Lip}_M(\alpha), x > 0$ and $n > 1$, we have*

$$|(L_n^* f)(x) - f(x)| \leq M \left\{ 2x \left(x + \frac{1-\sqrt{1+4n(n-1)x^2}}{2n} \right) \right\}^{\alpha/2}.$$

Proof. Since $f \in \mathrm{Lip}_M(\alpha)$ and $x > 0$, using (4) and then applying the Holder's inequality with $p = \alpha, q = \frac{2}{2-\alpha}$, we get

$$\begin{aligned} |(L_n^* f)(x) - f(x)| &\leq \left(L_n^* |f(y) - f(x)|\right)(x) \\ &\leq M(L_n^* |y - x|^\alpha)(x) \leq M\{(L_n^* \varphi_x^2)(x)\}^{\alpha/2}, \end{aligned}$$

using Lemma 6, we get the required result. □

Consider the class

$$C_\gamma(0,\infty) = \{f \in C(0,\infty) : |f(x)| \leq M(1+x)^\gamma \text{ for some } M > 0, \gamma > 0\},$$

with the norm

$$||f||_\gamma = \sup_{x\in(0,\infty)} \frac{|f(x)|}{(1+x)^\gamma}.$$

Let $C_\gamma^{(m)}(0,\infty), m = 0,1,2,...$, where $f \in C_\gamma(0,\infty)$. Following [4], we consider the m-th order generalization of the positive linear operators L_n^* as

$$(L_{n,m}^* f)(x) = \frac{1}{B(nr_n(x), n+1)} \sum_{i=0}^{m} \int_0^\infty \frac{t^{nr_n(x)-1}}{(1+t)^{nr_n(x)+n+1}} f^{(i)}(t) \frac{(t-x)^i}{i!} \mathrm{d}t. \tag{5}$$

Here $(L_{n,0}^* f)(x) = (L_n^* f)(x)$, and

$$(L_{n,m}^* f)(x) = \sum_{i=0}^{m} \int_0^\infty V_n(x,t) f^{(i)}(t) \frac{(t-x)^i}{i!} \mathrm{d}t, \tag{6}$$

where

$$V_n(x,t) = \frac{1}{B(nr_n(x), n+1)} \frac{t^{nr_n(x)-1}}{(1+t)^{nr_n(x)+n+1}}.$$

THEOREM 10. *For all $f \in C_\gamma^{(m)}(0,\infty), \gamma > 0$, such that $f^{(m)} \in \mathrm{Lip}_M(\alpha)$, and for every $x > 0$ we have*

$$|(L_{n,m}^* f)(x) - f(x)| \leq \frac{M}{(m-1)!} \frac{\alpha}{\alpha+m} B(\alpha, m) |(L_n^* |t - x|^{m+\alpha})(x)|,$$

where $m = 1,2,...$ and $B(\alpha, m)$ is the beta function.

Proof. By (6) and Lemma 5 we have

$$f(x)-(L_{n,m}^{*}f)(x)=\int_0^\infty V_n(x,t)\{f(x)-\sum_{i=0}^{m} f^{(i)}(t)\tfrac{(t-x)^i}{i!}\}\mathrm{d}t. \tag{7}$$

By Taylor's formula

$$f(x)-\sum_{i=0}^{m} f^{(i)}(t)\tfrac{(x-t)^i}{i!}=$$
$$=\tfrac{(x-t)^m}{(m-1)!}\int_0^1(1-s)^{m-1}\left\{f^{(m)}(t+s(x-t))-f^{(m)}(t)\right\}\mathrm{d}s.$$

Since $f^{(m)}\in \mathrm{Lip}_m(\alpha)$,

$$|f^{(m)}(t+s(x-t))-f^{(m)}(t)|\le Ms^\alpha|t-x|^\alpha. \tag{8}$$

From above equation and by Beta integral, we get

$$|f(x)-\sum_{i=0}^{m} f^{(m)}(t)\tfrac{(x-t)^i}{i!}|\le \tfrac{M}{(m-1)!}\tfrac{\alpha}{\alpha+m}B(\alpha,m)|t-x|^{\alpha+m}. \tag{9}$$

Hence the proof. □

Finally, for the uniform convergence of the operators $L_{n,m}^{*}$ we obtain the following result:

THEOREM 11. *For every* $f\in C_\gamma^{(m)}(0,\infty),\gamma>0,m=1,2,...,$ *such that* $f^{(m)}\in \mathrm{Lip}_M(\alpha)$, *we have*

$$\lim_{n\to\infty}(L_{n,m}^{*}f)(x)=f(x)$$

uniformly with respect to $x\in(0,\infty)$.

Proof. Define the function g by $g(t)=|t-x|^{m+\alpha}$. Then by Theorem 4, we have

$$\lim_{n\to\infty}(L_{n,m}^{*}f)(x)=f(x)=0$$

uniformly with respect to $x\in(0,\infty)$. Thus the proof is followed by previous theorem.

4. CONCLUDING REMARKS

By applying this approach the modified Stancu Beta operators $(L_n^{*}f)(x)$ preserve the test function $e_2(x)$ but the modified operators do not preserve the test function $e_1(x)$. Also we can have better approximation on the compact interval $(0,1)$, while the modified operators are defined on the interval $(0,\infty)$.

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