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DOUBLE INEQUALITIES FOR QUADRATURE FORMULA OF GAUSS TYPE

MARIUS HELJIU

Abstract. Double inequalities for the remainder term of the Gauss quadrature formula are given. These inequalities are sharp. It also will consider particular cases for n = 1, 2.

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1. INTRODUCTION

In this work we will consider Gauss's quadrature rule(see [2])

$$\int_a^b f(x)\mathrm{d}x = \sum_{i=1}^{n} C_i f(x_i) + \ R[f] \tag{1.1}$$

where the remainder term for $f : [a, b] \to \mathbb{R}$, $f \in C^{2n}[a, b]$, has the representation

$$R[f] = \int_a^b \varphi(x) f^{(2n)}(x)\mathrm{d}x. \tag{1.2}$$

The nodes x_i, $i = \overline{1, n}$ from quadrature formula (1.1) are given by the relation

$$x_i = \tfrac{a+b}{2} + \tfrac{b-a}{2}\xi_i \tag{1.3}$$

where ξ_i is replaced by the roots $\xi_1, \xi_2, ..., \xi_n$ of Legendre polynomial

$$X_n(\xi) = \tfrac{1}{2^n n!}\tfrac{d^n(1-\xi^2)^n}{d\xi^n}. \tag{1.4}$$

and the coefficients C_i, $i = \overline{1, n}$, from Gauss's formula shall be determined by putting the conditions that the quadrature formula must be accurate for any polynomial of degree $2n - 1$.

[†]Department of Mathematics, University of Petroşani, Romania, e-mail: mheljiu@upet.ro.

The function $\varphi(x)$, from remaining term expression, being symmetric to the line $x = \frac{a+b}{2}$, is sufficient to provide the equation $y = \varphi(x)$ in the intervals $[a, x_1], [x_1, x_2], [x_2, x_3]$... to middle of interval (a,b). So the function $\varphi(x)$ coincide on all the the intervals $[a, x_1], [x_1, x_2], [x_2, x_3]$..., with the relations:

$$
\begin{aligned}
&\varphi_1(x) = \int_a^x \tfrac{(x-s)^{2n-1}}{(2n-1)!}\mathrm{d}s \\
&\varphi_2(x) = \int_a^x \tfrac{(x-s)^{2n-1}}{(2n-1)!}\mathrm{d}s - C_1\tfrac{(x-x_1)^{2n-1}}{(2n-1)!} \\
&\varphi_3(x) = \int_a^x \tfrac{(x-s)^{2n-1}}{(2n-1)!}\mathrm{d}s - C_1\tfrac{(x-x_1)^{2n-1}}{(2n-1)!} - C_2\tfrac{(x-x_2)^{2n-1}}{(2n-1)!} \\
&\dots
\end{aligned}
\tag{1.5}
$$

In the next paragraph we will establish double integral inequalities for the remainder term of the Gauss quadrature formula. We will also establish conditions under which the results are sharp. At the end of the paper we will analyze the particular cases of Gauss formula with one respectively two nodes.

2. MAIN RESULTS

In this section we prove the following theorems:

THEOREM 2.1. *If $f \in C^{2n}[a, b]$ then*

$$
\begin{aligned}
&M(\gamma - S_{2n-1})(b-a) - \tfrac{\gamma(n!)^4}{[(2n)!]^3}\tfrac{(b-a)^{2n+1}}{(2n+1)} \leq \\
&\leq \sum_{i=1}^{n} C_i f(x_i) - \int_a^b f(x)\mathrm{d}x \leq \\
&\leq M(\Gamma - S_{2n-1})(b-a) - \tfrac{\Gamma(n!)^4}{[(2n)!]^3}\tfrac{(b-a)^{2n+1}}{(2n+1)}
\end{aligned}
\tag{2.1}
$$

where $\gamma, \Gamma \in \mathbb{R}$, $\gamma \leq f^{(2n)}(x) \leq \Gamma$, for all $x \in [a, b]$ and $S_{2n-1} = \frac{f^{(2n-1)}(b) - f^{(2n-1)}(a)}{b-a}$. Moreover,

$$
\gamma = \min_{x\in[a,b]} f^{(2n)}(x), \quad \Gamma = \max_{x\in[a,b]} f^{(2n)}(x)
\tag{2.2}
$$

the inequalities (5) *are sharp.*

Proof. Using (1.1) and (1.2), we obtain

$$
\int_a^b \varphi(x) f^{(2n)}(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x - \sum_{i=1}^{n} C_i f(x_i).
\tag{2.3}
$$

But it is well-known from [3] pp 283 that

$$
\int_a^b \varphi(x)\mathrm{d}x = \tfrac{(n!)^4}{[(2n)!]^3}\tfrac{(b-a)^{(2n+1)}}{(2n+1)}.
\tag{2.4}
$$

From (2.3) and (2.4) it follows

$$\int_a^b [f^{(2n)}(x)-\gamma]\varphi(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x - \sum_{i=1}^{n} C_i f(x_i) - \tfrac{\gamma(n!)^4}{[(2n)!]^3}\tfrac{(b-a)^{(2n+1)}}{(2n+1)} \tag{2.5}$$

and

$$\int_a^b [\Gamma - f^{(2n)}(x)]\varphi(x)\mathrm{d}x = -\int_a^b f(x)\mathrm{d}x + \sum_{i=1}^{n} C_i f(x_i) + \tfrac{\Gamma(n!)^4}{[(2n)!]^3}\tfrac{(b-a)^{(2n+1)}}{(2n+1)} \tag{2.6}$$

On the other hand, we have

$$\int_a^b [f^{(2n)}(x)-\gamma]\varphi(x)\mathrm{d}x \le \max_{x\in[a,b]} |\varphi(x)| \int_a^b |f^{(2n)}(x)-\gamma|\mathrm{d}x. \tag{2.7}$$

By calculating the right member of the inequality(2.7), we get

$$\begin{aligned}\int_a^b |f^{(2n)}(x)-\gamma|\mathrm{d}x &= \int_a^b (f^{(2n)}(x)-\gamma)\mathrm{d}x \\ &= f^{(2n-1)}(b) - f^{(2n-1)}(a) - \gamma(b-a) \\ &= (S_{2n-1}-\gamma)(b-a).\end{aligned} \tag{2.8}$$

From (2.7), (2.8) and noting

$$\max_{x\in[a,b]} |\varphi(x)| = M, \tag{2.9}$$

we obtain

$$\int_a^b [f^{(2n)}(x)-\gamma]\varphi(x)\mathrm{d}x \le M(S_{2n-1}-\gamma)(b-a). \tag{2.10}$$

From (2.5), by using previous relation, result

$$\int_a^b f(x)\mathrm{d}x - \sum_{i=1}^{n} C_i f(x_i) \le M(S_{2n-1}-\gamma)(b-a) + \tfrac{\gamma(n!)^4}{[(2n)!]^3}\tfrac{(b-a)^{(2n+1)}}{(2n+1)} \tag{2.11}$$

In the same way we have

$$\int_a^b [\Gamma - f^{(2n)}(x)]\varphi(x)\mathrm{d}x \le \max_{x\in[a,b]} |\varphi(x)| \int_a^b |\Gamma - f^{(2n)}(x)|\mathrm{d}x \tag{2.12}$$

and

$$\begin{aligned}\int_a^b |\Gamma - f^{(2n)}(x)|\mathrm{d}x &= \int_a^b (\Gamma - f^{(2n)}(x))\mathrm{d}x \\ &= \Gamma(b-a) - f^{(2n-1)}(b) + f^{(2n-1)}(a) \\ &= (\Gamma - S_{2n-1})(b-a)\end{aligned} \tag{2.13}$$

Using (2.6), (2.9), (2.12) and (2.13) we obtain the inequality

$$
\begin{aligned}
(2.14) \qquad & -\int_a^b f(x)\mathrm{d}x + \sum_{i=1}^{n} C_i f(x_i) \\
& \le M(\Gamma - S_{2n-1})(b-a) - \tfrac{\Gamma(n!)^4}{[(2n)!]^3}\tfrac{(b-a)^{(2n+1)}}{(2n+1)}.
\end{aligned}
$$

The inequalities (2.1) follow from the inequalities (2.11) and (2.14).

To prove the second part of the theorem we consider the function $f(x) = (x-a)^{2n}$. I have $f^{(2n)}(x) = (2n)!, \gamma = \Gamma = (2n)!$ and $S_{2n-1} = (2n)!$. It is easy to show that all the three members of the double inequality (2.1) are equal. This completes the proof. □

The following theorem offer us $R[f]$ and analogous inequality with the one given by Theorem 2.1.

THEOREM 2.2. *Under the assumptions of Theorem* 2.1, *we have*

$$
\begin{aligned}
(2.15) \qquad & M(\gamma - S_{2n-1})(b-a) + \tfrac{\gamma(n!)^4}{[(2n)!]^3}\tfrac{(b-a)^{2n+1}}{(2n+1)} \le \\
& \le \int_a^b f(x)\mathrm{d}x - \sum_{i=1}^{n} C_i f(x_i) \le \\
& \le M(\Gamma - S_{2n-1})(b-a) + \tfrac{\Gamma(n!)^4}{[(2n)!]^3}\tfrac{(b-a)^{2n+1}}{(2n+1)}
\end{aligned}
$$

Moreover,

$$
\gamma = \min_{x\in[a,b]} f^{(2n)}(x), \quad \Gamma = \max_{x\in[a,b]} f^{(2n)}(x)
$$

the inequalities (18) *are sharp.*

Proof. By using the relations (2.3), (2.5), (2.7) and (2.8) it follows

$$
\begin{aligned}
(2.16) \qquad & -\int_a^b f(x)\mathrm{d}x + \sum_{i=1}^{n} C_i f(x_i) \\
& \le -M(\gamma - S_{2n-1})(b-a) - \tfrac{\gamma(n!)^4}{[(2n)!]^3}\tfrac{(b-a)^{2n+1}}{(2n+1)}
\end{aligned}
$$

Analogous by using the relations (2.3), (2.6), (2.12) and (2.13) we obtain

$$
\begin{aligned}
(2.17) \qquad & \int_a^b f(x)\mathrm{d}x - \sum_{i=1}^{n} C_i f(x_i) \\
& \le M(\Gamma - S_{2n-1})(b-a) + \tfrac{\Gamma(n!)^4}{[(2n)!]^3}\tfrac{(b-a)^{(2n+1)}}{(2n+1)}
\end{aligned}
$$

From the relations (2.16) and (2.17) result the inequalities (2.15). To prove that the double inequalities (2.15) are exact we follow the steps of the proof from Theorem 2.1. □

Theorem 2.3 gives us the inequalities which do not depend on S_3.

THEOREM 2.3. *Under the assumptions of Theorem* 1, *we have*

$$\begin{aligned}(2.18)\qquad & \tfrac{1}{2}M(\gamma-\Gamma)(b-a)-\tfrac{\gamma+\Gamma}{2}\tfrac{(n!)^4}{[(2n)!]^3}\tfrac{(b-a)^{2n+1}}{2n+1}\\ &\leq \sum_{i=1}^{n} C_i f(x_i)-\int_a^b f(x)\mathrm{d}x\\ &\leq \tfrac{1}{2}M(\Gamma-\gamma)(b-a)-\tfrac{\gamma+\Gamma}{2}\tfrac{(n!)^4}{[(2n)!]^3}\tfrac{(b-a)^{2n+1}}{2n+1}.\end{aligned}$$

If

$$\gamma=\min_{x\in[a,b]} f^{(2n)}(x), \quad \Gamma=\max_{x\in[a,b]} f^{(2n)}(x)$$

then the inequalities (21) *are sharp.*

Proof. Multiplying the inequality (2.15) with (-1) we obtain

$$\begin{aligned}(2.19)\qquad & -M(\Gamma-S_{2n-1})(b-a)-\tfrac{\Gamma(n!)^4}{[(2n)!]^3}\tfrac{(b-a)^{2n+1}}{(2n+1)}\leq\\ &\leq -\int_a^b f(x)\mathrm{d}x+\sum_{i=1}^{n} C_i f(x_i)\leq\\ &\leq -M(\gamma-S_{2n-1})(b-a)-\tfrac{\gamma(n!)^4}{[(2n)!]^3}\tfrac{(b-a)^{(2n+1)}}{(2n+1)}\end{aligned}$$

From the inequalities (2.1) and (2.19) we have

$$\begin{aligned}(2.20)\qquad & M(\gamma-\Gamma)(b-a)-(\gamma+\Gamma)\tfrac{(n!)^4}{[(2n)!]^3}\tfrac{(b-a)^{2n+1}}{2n+1}\\ &\leq 2\left[\sum_{i=1}^{n} C_i f(x_i)-\int_a^b f(x)\mathrm{d}x\right]\\ &\leq M(\Gamma-\gamma)(b-a)-(\gamma+\Gamma)\tfrac{(n!)^4}{[(2n)!]^3}\tfrac{(b-a)^{2n+1}}{2n+1}.\end{aligned}$$

Multiplying (2.20) with $\frac{1}{2}$ results the double inequality (2.18). Considering the function $f(x)=(x-a)^{2n}$ we will show that the inequality is sharp. This completes the proof. □

3. PARTICULAR CASES

Using the results of the before paragraph, we will present the following double inequalities for the rest of Gauss quadrature formulas in the cases n=1, n=2.

The Gauss's formula with a single node (n = 1), also called the mid-point, has the form

$$(3.1)\qquad \int_a^b f(x)\mathrm{d}x=(b-a)\left[f\left(\tfrac{a+b}{2}\right)\right]+R[f],$$

where

$$(3.2)\qquad R[f]=\int_a^b \varphi(x)f''(x)\mathrm{d}x,$$

The function $\varphi(x)$ is given by the relation

$$\varphi(x) = \begin{cases} \frac{(x-a)^2}{2}, & \text{if} \quad x \in \left[a, \frac{a+b}{2}\right] \\ \frac{(b-x)^2}{2}, & \text{if} \quad x \in \left[\frac{a+b}{2}, b\right]. \end{cases} \tag{3.3}$$

Applying Theorem 2.1 for n=1, we get a result established by Ujevic [3].

THEOREM 3.1. *Let $f : [a,b] \to \mathbb{R}$, $f \in C^2(a,b)$, then we have the inequality*

$$\begin{aligned} &\tfrac{2\gamma-3S_1}{24}(b-a)^3 \\ &\le (b-a) f\left(\tfrac{a+b}{2}\right) - \int_a^b f(x)\mathrm{d}x \\ &\le \tfrac{2\Gamma-3S_1}{24}(b-a)^3 \end{aligned} \tag{3.4}$$

where $\gamma,\ \Gamma \in \mathbb{R}$, $\gamma \le f^{(2)}(x) \le \Gamma$, $\forall\ x \in [a,b]$ and

$$S_1 = \tfrac{f'(b)-f'(a)}{b-a}.$$

If $\gamma = \min\limits_{x\in[a,b]} f^{(2)}(x)$, $\Gamma = \max\limits_{x\in[a,b]} f^{(2)}(x)$ then inequalities (25) *are sharp.*

Similar with Theorem 2.2 and 2.3, the Gauss's formula with a single node are given in the next theorems :

THEOREM 3.2. *Under the assumptions of Theorem* 3.1 *we have*

$$\begin{aligned} &\tfrac{4\gamma-3S_1}{24}(b-a)^3 \le \\ &\le \int_a^b f(x)\mathrm{d}x - (b-a) f\left(\tfrac{a+b}{2}\right) \\ &\le \tfrac{4\Gamma-3S_1}{24}(b-a)^3. \end{aligned} \tag{3.5}$$

If $\gamma = \min\limits_{x\in[a,b]} f^{(2)}(x)$ and $\Gamma = \max\limits_{x\in[a,b]} f^{(2)}(x)$ then inequalities (26) *are sharp.*

THEOREM 3.3. *Under the assumptions of Theorem* 3.1 *we have*

$$\begin{aligned} &\tfrac{(b-a)^3}{24}(\gamma - 2\Gamma) \le \\ &\le (b-a) f\left(\tfrac{a+b}{2}\right) - \int_a^b f(x)\mathrm{d}x \le \\ &\le \tfrac{(b-a)^3}{24}(\Gamma - 2\gamma). \end{aligned} \tag{3.6}$$

If $\gamma = \min\limits_{x\in[a,b]} f^{(2)}(x)$ and $\Gamma = \max\limits_{x\in[a,b]} f^{(2)}(x)$ then inequalities (27) *are sharp.*

Similar results, we obtain in the case of Gauss formula with two nodes. The results was established by the author in the paper [1].

Let $f : [a,b] \to \mathbb{R}$, $f \in C^4[a,b]$ and $x_1, x_2 \in [a,b]$ so that $x_1 = \frac{a+b}{2} - \frac{b-a}{2} \cdot \xi$, $x_2 = \frac{a+b}{2} + \frac{b-a}{2} \cdot \xi$, where $\xi = \frac{1}{\sqrt{3}} = 0,57735027\ldots$.

Gauss's quadrature formula with two nodes has the following form

$$\int_a^b f(x)\mathrm{d}x = \tfrac{b-a}{2}[f(x_1) + f(x_2)] + R[f]. \tag{3.7}$$

The error $R[f]$ from the formula (3.7) is given by

$$R[f] = \int_a^b \varphi(x) f^{(4)}(x)\mathrm{d}x, \tag{3.8}$$

where the function φ has the form

$$\varphi(x) = \begin{cases} \frac{(x-a)^4}{4!}, & \text{if} \quad x \in [a, x_1] \\ \frac{(x-a)^4}{4!} - \frac{b-a}{2}\frac{(x-x_1)^3}{3!}, & \text{if} \quad x \in]x_1, x_2[\\ \frac{(b-x)^4}{4!}, & \text{if} \quad x \in [x_2, b] \end{cases} \tag{3.9}$$

In this case we obtain the following theorems:

Theorem 3.4. *If $f \in C^4[a,b]$ then*

$$\begin{aligned} &\tfrac{1}{17280}(b-a)^5(41\gamma - 45S_3 + 180\xi^3 S_3 - 180\xi^3\gamma) \\ &\le \tfrac{b-a}{2}[f(x_1) + f(x_2)] - \int_a^b f(x)\mathrm{d}x \\ &\le \tfrac{1}{17280}(b-a)^5(41\Gamma - 45S_3 + 180\xi^3 S_3 - 180\xi^3\Gamma) \end{aligned} \tag{3.10}$$

where $\gamma, \Gamma \in \mathbb{R}$, $\gamma \le f^{(4)}(x) \le \Gamma$, for all $x \in [a,b]$ and $S_3 = \frac{f'''(b)-f'''(a)}{b-a}$. Moreover,

$$\gamma = \min_{x\in[a,b]} f^{(4)}(x), \quad \Gamma = \max_{x\in[a,b]} f^{(4)}(x) \tag{3.11}$$

the inequalities (31) *are sharp.*

Theorem 3.5. *Under the assumptions of Theorem* 3.4 *we have*

$$\begin{aligned} &\tfrac{1}{17280}(b-a)^5(49\gamma - 45S_3 + 180\xi^3 S_3 - 180\xi^3\gamma) \le \\ &\le \int_a^b f(x)\mathrm{d}x - \tfrac{b-a}{2}[f(x_1) + f(x_2)] \le \\ &\le \tfrac{1}{17280}(b-a)^5(49\Gamma - 45S_3 + 180\xi^3 S_3 - 180\xi^3\Gamma) \end{aligned} \tag{3.12}$$

If

$$\gamma = \min_{x\in[a,b]} f^{(4)}(x), \quad \Gamma = \max_{x\in[a,b]} f^{(4)}(x)$$

the inequalities (33) *are sharp.*

THEOREM 3.6. *Under the assumptions of Theorem* 3.4 *we have*

$$\begin{aligned}(3.13)\qquad & \tfrac{1}{34560}(b-a)^5(41\gamma-49\Gamma-180\xi^3\gamma+180\xi^3\Gamma)\leq\\ &\leq \tfrac{b-a}{2}[f(x_1)+f(x_2)]-\int_a^b f(x)\mathrm{d}x\leq\\ &\leq \tfrac{1}{34560}(b-a)^5(41\Gamma-49\gamma+180\xi^3\gamma-180\xi^3\Gamma)\end{aligned}$$

If

$$\gamma=\min_{x\in[a,b]} f^{(4)}(x),\qquad \Gamma=\max_{x\in[a,b]} f^{(4)}(x)$$

the inequalities (34) *are sharp.*

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