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ON THE SEMILOCAL CONVERGENCE OF DERIVATIVE FREE METHODS FOR SOLVING NONLINEAR EQUATIONS[‡]

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Abstract. We introduce a *Derivative Free Method* (DFM) for solving nonlinear equations in a Banach space setting. We provide a semilocal convergence analysis for DFM using recurrence relations. Numerical examples validating our theoretical results are also provided in this study to show that DFM is faster than other derivative free methods [9] using similar information.

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1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of an equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet–differentiable operator defined on a non–empty, open subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

A large number of problems in applied mathematics and engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, we assume that a time–invariant system is driven by the equation $\dot{x} = Q(x)$, for some suitable operator Q, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations

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can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative. In fact, starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

A classic iterative process for solving nonlinear equations is Chebyshev's method (see [5], [9], [14], [17]):

$$\begin{cases} x_0 \in \mathcal{D}, \\ y_k = x_k - F'(x_k)^{-1}\ F(x_k), \\ x_{k+1} = y_k - \frac{1}{2}\ F'(x_k)^{-1} F''(x_k)(y_k - x_k)^2, \quad k \geq 0. \end{cases}$$

This one-point iterative process depends explicitly on the two first derivatives of F (namely, $x_{n+1} = \psi(x_n, F(x_n), F'(x_n), F''(x_n))$). Ezquerro and Hernández introduced in [14] some modifications of Chebyshev's method that avoid the computation of the second derivative of F and reduce the number of evaluations of the first derivative of F. Actually, these authors have obtained a modification of the Chebyshev iterative process which only need to evaluate the first derivative of F, (namely, $x_{n+1} = \overline{\psi}(x_n, F'(x_n))$, but with third-order of convergence [14]. In this paper we recall this method as the *Chebyshev–Newton–type method* (CNTM) and it is written as follows:

$$\begin{cases} x_0 \in \mathcal{D}, \\ y_k = x_k - F'(x_k)^{-1}\ F(x_k), \\ z_k = x_k + a\ (y_k - x_k) \\ x_{k+1} = x_k - \frac{1}{a^2}\ F'(x_k)^{-1}\ ((a^2 + a - 1)\ F(x_k) + F(z_k)), \quad k \geq 0, \end{cases}$$

where $F'(x)$ $(x \in \mathcal{D})$ is the Fréchet–derivative of F.

There is an interest in constructing families of iterative processes free of derivatives. To obtain a new family in [9] we considered an approximation of the first derivative of F from a divided difference of first order, that is, $F'(x_n) \approx [x_{n-1}, x_n, F]$, where, $[x, y; F]$ is a divided difference of order one for the operator F at the points x, $y \in \mathcal{D}$. Then, we introduce the *Chebyshev–Secant–type method* (CSTM)

$$\begin{cases} x_{-1},\ x_0 \in \mathcal{D}, \\ y_k = x_k - B_k^{-1}\ F(x_k), \quad B_k = [x_{k-1}, x_k; F], \\ z_k = x_k + a\ (y_k - x_k), \\ x_{k+1} = x_k - B_k^{-1}\ (b\ F(x_k) + c\ F(z_k)), \quad k \geq 0, \end{cases}$$

where a, b, c are non–negative parameters to be chosen so that sequence $\{x_k\}$ converges to $x^\star$. Note that CSTM is reduced to the *secant method* (SM) if

$a = 0$, $b = c = 1/2$, and $y_k = x_{k+1}$. Moreover, if $x_{k-1} = x_k$, and F is differentiable on $\mathcal{D}$, then, $F'(x_k) = [x_k, x_k; F]$, and CSTM reduces to *Newton's method* (NM).

We provided a semilocal convergence analysis for CSTM using recurrence sequences, and also illustrated its effectiveness through numerical examples. Bosarge and Falb [10], Dennis [13], Potra [23], Argyros [1]–[5], Hernández *et al.* [15] and others [16], [22], [26], have provided sufficient convergence conditions for the SM based on Lipschitz-type conditions on divided difference operator (see, also relevant works in [8]–[13], [18], [21], [24], [27]).

In this paper, we continue the study of inverse free iterative processes. We introduce the *derivative free method* (DFM):

$$\begin{cases} x_{-1},\ x_0 \in \mathcal{D}, \\ y_k = x_k - A_k^{-1}\ F(x_k), \quad A_k = [2x_k - x_{k-1}, x_{k-1}; F], \\ z_k = x_k + a\ (y_k - x_k), \\ x_{k+1} = x_k - A_k^{-1}\ (b\ F(x_k) + c\ F(z_k)), \quad k \geq 0. \end{cases}$$

Note that DFM reduces to the *Kurchatov-type method* (KTM)

$$x_{k+1} = x_k - A_k^{-1} F(x_k),$$

if $a = 0, b = c = 0.5$, and $y_k = x_{k+1}$ [20], [25].

In this special case the quadratic convergence of KTM was first established in [20], [25] and then in [6], [7] under different sets of sufficient conditions. We provide a semilocal convergence analysis for DFM. Then, we give numerical examples to show that DFM is faster than CSTM. In particular, two numerical examples are also provided. Firstly, we consider a scalar equation where the main study of the paper is applied. Secondly, we discretize a nonlinear integral equation and approximate a numerical solution using DFM.

2. SEMILOCAL CONVERGENCE ANALYSIS OF DFM

We shall show the semilocal convergence of DFM under the following conditions

($\mathcal{C}_1$) $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ is a Fréchet–differentiable operator, and there exists divided difference denoted by $[x, y; F]$ satisfying

$$[x, y; F](x - y) = F(x) - F(y) \quad \text{for all} \quad x, y \in \mathcal{D};$$

($\mathcal{C}_2$) There exist x_{-1} and x_0 in $\mathcal{D}$ and $\beta > 0$ such that

$$A_0^{-1} = [2x_0 - x_{-1}, x_{-1}; F]^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$$

exists and

$$0 < \parallel A_0^{-1} \parallel \leq \beta;$$

($\mathcal{C}_3$) There exists $d > 0$ such that

$$\parallel x_0 - x_{-1} \parallel \leq d;$$

($\mathcal{C}_4$) There exists $\eta > 0$ such that

$$0 < \| A_0^{-1} F(x_0) \| \leq \eta;$$

($\mathcal{C}_5$) There exists constant $M > 0$, such that for all x, y, u, v in $\mathcal{D}$

$$\| [x, y; F] - [u, v; F] \| \leq \tfrac{M}{2} (\| x - u \| + \| y - v \|);$$

($\mathcal{C}_6$) For $a \in [0, 1]$, $b \in [0, 1]$ and $c > 0$ given in DFM, we suppose

$$(1 - a)\, c = 1 - b;$$

($\mathcal{C}_7$)

$$\alpha = 2 \left(1 + d_0 + a\, c\, \gamma\, (a + 2\, d_0) \right) \gamma < 1,$$

where,

$$\gamma = \tfrac{\beta\, M\, \eta}{2}, \quad d_0 = \tfrac{d}{\eta};$$

($\mathcal{C}_8$)

$$\overline{U}(x_0, R = r\, \eta) = \{x \in \mathcal{X} \, : \| x - x_0 \| \leq R\} \subseteq \mathcal{D},$$

for some $r > 1$ to be precised later in Theorem 5;

($\mathcal{C}_9$)

$$x, y \in \mathcal{D} \Rightarrow 2\, y - x \in \mathcal{D}.$$

Delicate condition (C_9) is certainly satisfied, if $\mathcal{D} = \mathcal{X}$. It is also satisfied, if (C_9) is replaced by

($\mathcal{C}_8$)′

$$\overline{U}(x_0, 3\, R) \subseteq \mathcal{D}.$$

Indeed, if $x, y \in \overline{U}(x_0, R)$, then

$$\|2\, y - x - x_0\| \leq \|y - x_0\| + \|y - x\| \leq 2\, \|y - x_0\| + \|x - x_0\| \leq 3\, R.$$

That is, $2y - x \in \overline{U}(x_0, 3R)$.

We note by ($\mathcal{C}$) the set of conditions ($\mathcal{C}_1$)–($\mathcal{C}_9$).

Definition 1. *Let γ and d_0 as defined in $(\mathcal{C}_7)$. It is convenient to define for $\mu_0 = w_0 = 1$, $q_{-1} = d_0$, and $n \geq 0$, the following sequences*

$$p_n = a\, c\, \gamma\, \mu_n (a\, w_n + 2\, q_{n-1})\, w_n,$$

$$q_n = p_n + w_n,$$

$$\mu_{n+1} = \tfrac{\mu_n}{1 - 2\, \gamma\, \mu_n\, (q_{n-1} + q_n)},$$

$$c_n = \tfrac{M}{2}\, ((q_n + 2\, q_{n-1})\, q_n + a\, c\, (a\, w_n + 2\, q_{n-1})\, w_n),$$

and

$$w_{n+1} = \gamma\, \mu_{n+1}\, ((q_n + 2\, q_{n-1})\, q_n + a\, c\, (a\, w_n + 2\, q_{n-1})\, w_n).$$

Note that

$$w_{n+1} = \beta\, \eta\, \mu_{n+1}\, c_n.$$

We need an Ostrowski–type approximations for DFM. The proof is given in [5], [9].

Lemma 2. *Assume sequence* $\{x_k\}$ *generated by* DFM *is well defined,* $(1-a)\,c = 1-b$ *holds for* $a \in [0,1]$*,* $b \in [0,1]$*, and* $c > 0$*.*

Then, the following items hold for all $k \geq 0$*:*

$$\begin{aligned}F(z_k) &= (1-a)F(x_k)+a\int_0^1 (F'(x_k+a(y_k-x_k))-F'(x_k))(y_k-x_k)\mathrm{d}t+\\&\quad + a(F'(x_k)-A_k)(y_k-x_k),\end{aligned} \tag{2.1}$$

$$\begin{aligned}x_{k+1}-y_k &= -acA_k^{-1}\Big(\int_0^1 (F'(x_k+a(y_k-x_k))-F'(x_k))(y_k-x_k)\mathrm{d}t+\\&\quad + (F'(x_k)-A_k)(y_k-x_k)\Big),\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}F(x_{k+1}) = &\int_0^1 (F'(x_k+t(x_{k+1}-x_k))-F'(x_k))(x_{k+1}-x_k)\mathrm{d}t+\\&+(F'(x_k)-A_k)(x_{k+1}-x_k)-ac\Big(\int_0^1 (F'(x_k+at(y_k-x_k))-\\&-F'(x_k))(y_k-x_k)\mathrm{d}t+(F'(x_k)-A_k)(y_k-x_k)\Big).\end{aligned} \tag{2.3}$$

The following relates DFM with scalar sequences introduced in Definition 1.

Lemma 3. *Under the* $(\mathcal{C})$ *conditions, we assume:*

$$x_n \in \mathcal{D} \quad \text{and} \quad 2\,\gamma\,\mu_n\,(q_{n-1}+q_n) < 1 \quad (n \geq 0).$$

Then, the following items hold for all $n \geq 0$*:*

$(\mathbf{I_n})$ $\parallel A_n^{-1} \parallel \leq \mu_n\,\beta$,
$(\mathbf{II_n})$ $\parallel y_n - x_n \parallel = \parallel A_n^{-1}\,F(x_n) \parallel \leq w_n\,\eta$,
$(\mathbf{III_n})$ $\parallel x_{n+1}-y_n \parallel \leq p_n\,\eta$,
$(\mathbf{IV_n})$ $\parallel x_{n+1}-x_n \parallel \leq q_n\,\eta$.

Proof. We use induction.

We have $\parallel y_0 - x_0 \parallel \leq \eta$, and $\parallel z_0 - x_0 \parallel \leq a\,\eta$, so that $x_0,\, z_0 \in \mathcal{D}$.

Items $(\mathbf{I_0})$ and $(\mathbf{II_0})$ hold by $(\mathcal{C}_2)$ and $(\mathcal{C}_4)$, respectively. To prove $(\mathbf{III_0})$, we use Lemma 2 for $n=0$ to obtain by $(\mathcal{C}_2)$–$(\mathcal{C}_5)$

$$\begin{aligned}\parallel x_1 - y_0 \parallel &\leq a\,c \parallel A_0^{-1} \parallel \tfrac{M}{2}\,(a \parallel y_0 - x_0 \parallel + 2 \parallel x_0 - x_{-1} \parallel) \parallel y_0 - x_0 \parallel\\&\leq \tfrac{a\,c\,\beta\,M}{2}\,(a\,\eta + 2\,d)\,\eta\\&= a\,c\,\gamma\,(a+2d_0)\,\eta = p_0\,\eta.\end{aligned}$$

Moreover,

$$\parallel x_1 - x_0 \parallel \leq \parallel x_1 - y_0 \parallel + \parallel y_0 - x_0 \parallel \leq p_0\,\eta + \eta = (1+p_0)\,\eta = q_0\,\eta,$$

which implies $(\mathbf{IV_0})$. Note also that $z_1 \in \mathcal{D}$. Following an inductive argument, assume $x_k \in \mathcal{D}$, and $2\,\gamma\,\mu_k\,(q_{k-1}+q_k) < 1$. Then, we have

$$\begin{aligned}
&\| A_k^{-1} \| \, \| A_{k+1} - A_k \| \leq \\
&\leq \| A_k^{-1} \| \, \tfrac{M}{2} [\| 2\,(x_{k+1}-x_k) - (x_k - x_{k-1}) \| + \| x_k - x_{k-1} \|] \\
&\leq 2 \, \| A_k^{-1} \| \, \tfrac{M}{2} \, (\| x_k - x_{k-1} \| + \| x_{k+1} - x_k \|) \\
&\leq 2 \, \tfrac{\beta\,M}{2} \, \mu_k\,(q_{k-1}+q_k)\,\eta = 2\,\gamma\,\mu_k\,(q_{k-1}+q_k) < 1.
\end{aligned}$$

It follows from the Banach lemma on invertible operators [1], [5], [19] that A_{k+1}^{-1} exists, and

$$\begin{aligned}
\| A_{k+1}^{-1} \| &\leq \frac{\|A_k^{-1}\|}{1-2\,\|A_k^{-1}\| \, \frac{M}{2} \, (\| x_k - x_{k-1} \| + \| x_{k+1} - x_k \|)} \\
&\leq \frac{\beta\,\mu_k}{1-2\,\gamma\,\mu_k\,(q_{k-1}+q_k)} = \mu_{k+1}\,\beta,
\end{aligned}$$

which shows $(\mathbf{I_{k+1}})$. Using Lemma 2, $(\mathcal{C}_5)$, and the induction hypotheses, we get

(2.4)
$$\begin{aligned}
\| F(x_{k+1}) \| &\leq \tfrac{M}{2} \, \| x_{k+1} - x_k \|^2 + M \, \| x_{k+1} - x_k \| \, \| x_k - x_{k-1} \| + \\
&\quad + ac \left(\tfrac{a\,M}{2} \, \| y_k - x_k \|^2 + M \, \| x_k - x_{k-1} \| \, \| y_k - x_k \| \right) \\
&\leq \tfrac{M}{2} \, q_k^2\,\eta^2 + M\,q_k\,\eta\,q_{k-1}\,\eta + a\,c\,\left(\tfrac{a\,M}{2}\,w_k^2\,\eta^2 + M\,q_{k-1}\,\eta\,w_k\,\eta\right) \\
&= c_k\,\eta^2.
\end{aligned}$$

Then, we get

$$\| y_{k+1} - x_{k+1} \| \leq \| A_{k+1}^{-1} \| \, \| F(x_{k+1}) \| \leq \mu_{k+1}\,\beta\,c_k\,\eta^2 = w_{k+1}\,\eta.$$

Moreover, by Lemma 2, we have

$$\begin{aligned}
&\| x_{k+2} - y_{k+1} \| \leq \\
&\leq a\,c \, \| A_{k+1}^{-1} \| \, \tfrac{M}{2} \left(a \, \| y_{k+1} - x_{k+1} \| + 2 \, \| x_{k+1} - x_k \| \right) \| y_{k+1} - x_{k+1} \| \\
&\leq a\,c\,\mu_{k+1} \, \tfrac{\beta\,M}{2} \, (a\,w_{k+1} + 2\,q_k)\,w_{k+1}\,\eta^2 = p_{k+1}\,\eta,
\end{aligned}$$

and consequently,

$$\| x_{k+2} - x_{k+1} \| \leq \| x_{k+2} - y_{k+1} \| + \| y_{k+1} - x_{k+1} \| \leq (p_{k+1} + w_{k+1})\,\eta = q_{k+1}\,\eta.$$

This completes the proof of Lemma 3. □

We shall establish the convergence of sequence $\{x_n\}$ generated by DFM. This can be achieved by showing that $\{q_n\}$ is a Cauchy sequence, if the following conditions hold for $n \geq 0$:

$(\mathcal{A}_1)$ $x_n \in \mathcal{D}$,

and

$(\mathcal{A}_2)$ $2\,\gamma\,\mu_n\,(q_{n-1}+q_n) < 1$.

In the next result, we show the Cauchy property for sequence $\{q_n\}$.

LEMMA 4. *Assume* $(\mathcal{C}_8)$. *Note that* $\alpha \in [0, 1)$ *implies* $2\,\gamma\,(q_{-1} + q_0) < 1$. *Then, scalar sequence:*

(a) $\{\mu_n\}$ *is increasing.*
(b) $\{q_n\}$ *is decreasing and* $\lim\limits_{n \longrightarrow \infty} q_n = 0$.

Proof. (a) We show using induction that all scalar sequences involved are positive. By Definition 1, and $(\mathcal{C}_8)$, we have for $j = 0$: μ_j, p_j, q_j, w_j, c_j, and $1 - 2\,\gamma\,\mu_j\,(q_{j-1} + q_j)$ are positive. Assume μ_k, p_k, q_k, w_k, c_k, and $1 - 2\,\gamma\,\mu_k\,(q_{k-1} + q_k)$ are positive for all $k \le n$. Since $c_k > 0$, it follows from the definition of the scalar sequences that w_{k+1}, μ_{k+1}, p_{k+1}, c_{k+1} have the same sign. Assume the common sign to be negative. Then

$$\begin{aligned}
&q_{k-1} + q_k + q_{k+1} < q_{k-1} + q_k \\
&\Longrightarrow 1 - 2\,\gamma\,\mu_k\,(q_{k-1} + q_k + q_{k+1}) > 1 - 2\,\gamma\,\mu_k\,(q_{k-1} + q_k) \\
&\Longrightarrow \tfrac{1 - 2\,\gamma\,\mu_k\,(q_{k-1} + q_k + q_{k+1})}{1 - 2\,\gamma\,\mu_k\,(q_{k-1} + q_k)} > 1.
\end{aligned}$$

But it follows from the definition of sequence $\{\mu_k\}$ that

$$\begin{aligned}
1 - 2\,\gamma\,\mu_{k+1}\,(q_k + q_{k+1}) &= \tfrac{1 - 2\,\gamma\,\mu_k\,(q_{k-1} + 2\,q_k + q_{k+1})}{1 - 2\,\gamma\,\mu_k\,(q_{k-1} + q_k)} \\
\Longrightarrow 1 - 2\,\gamma\,\mu_{k+1}\,q_{k+1} &= \tfrac{1 - 2\,\gamma\,\mu_k\,(q_{k-1} + 2\,q_k + q_{k+1})}{1 - 2\,\gamma\,\mu_k\,(q_{k-1} + q_k)} + 2\,\gamma\,\mu_{k+1}\,q_k \\
&= \tfrac{1 - 2\,\gamma\,\mu_k\,(q_{k-1} + q_k + q_{k+1})}{1 - 2\,\gamma\,\mu_k\,(q_{k-1} + q_k)} > 1,
\end{aligned}$$

which is a contradiction, since we get $2\,\gamma\,\mu_{k+1}\,q_{k+1} < 0$, but μ_{k+1}, q_{k+1} have the same sign, and $\gamma > 0$. The induction is then completed.

By the definition of sequence $\{\mu_n\}$ and $\mu_0 = 1$, we have

$$\begin{aligned}
&1 - 2\,\gamma\,\mu_k\,(q_{k-1} + q_k) = \tfrac{\mu_k}{\mu_{k+1}} \\
&\Longrightarrow q_{k-1} + q_k = \tfrac{1}{2\,\gamma}\,(\tfrac{1}{\mu_k} - \tfrac{1}{\mu_{k+1}}) \\
&\Longrightarrow \sum_{i=0}^{k-1} (q_{i-1} + q_i) = \tfrac{1}{2\,\gamma}\,(\tfrac{1}{\mu_0} - \tfrac{1}{\mu_k}) = \tfrac{1}{2\,\gamma}\,(1 - \tfrac{1}{\mu_k}) \\
&\Longrightarrow \mu_k = \frac{1}{1 - 2\,\gamma \displaystyle\sum_{i=0}^{k-1} (q_{i-1} + q_i)}.
\end{aligned}$$

But $1 - 2\,\gamma \displaystyle\sum_{i=0}^{k-1} (q_{i-1} + q_i)$ decreases. Therefore, sequence $\{\mu_k\}$ increases, and consequently $\mu_k \ge \mu_0 = 1$.

(b) We have that sequence $\mu_k > 1$ is increasing, so that $0 \le \frac{1}{\mu_k} \le 1$. Since $\{\frac{1}{\mu_k}\}$ is monotonic on a compact set, it converges to $\frac{1}{\mu_\infty}$. Then, we have

$$\lim_{k \longrightarrow \infty} (q_{k-1} + q_k) = \tfrac{1}{2\,\gamma} \lim_{k \longrightarrow \infty} (\tfrac{1}{\mu_k} - \tfrac{1}{\mu_{k+1}}) = \tfrac{1}{2\,\gamma}\,(\tfrac{1}{\mu_\infty} - \tfrac{1}{\mu_\infty}) = 0.$$

This completes the proof of Lemma 4. □

We can show the main semilocal convergence theorem for DFM.

THEOREM 5. *Let* $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ *be a Fréchet–differentiable operator defined on a non–empty open, convex domain* $\mathcal{D}$ *of a Banach space* $\mathcal{X}$*, with values in a Banach space* $\mathcal{Y}$*. Assume that the* $(\mathcal{C})$ *conditions hold. Then, sequence* $\{x_n\}$ $(n \geq -1)$*, generated by* DFM*, is well defined, remains in* $\overline{U}(x_0, r\,\eta)$ *for all* $n \geq 0$*, and converges to a solution* $x^\star \in \overline{U}(x_0, r\,\eta)$ *of equation* $F(x) = 0$*, where,*

$$r = \sum_{n=0}^{\infty} q_n. \tag{2.5}$$

Moreover, the following estimate holds

$$\| x_n - x^\star \| \leq \sum_{k=n+1}^{\infty} q_k\,\eta < r\,\eta.$$

Furthermore, $x^\star$ *is the unique solution of* $F(x) = 0$ *in* $U(x_0, r_0) \cap \mathcal{D}$*, provided that* $r_0 \geq r\,\eta$*, where,*

$$r_0 = \tfrac{2}{\beta\,M} - 2\,d - r\,\eta. \tag{2.6}$$

Proof. According to Lemmas 3, and 4, sequence $\{x_n\}$ is of Cauchy ($\{q_n\}$ is of Cauchy) in a Banach space $\mathcal{X}$, and it converges to some $x^\star \in \overline{U}(x_0, r\,\eta)$ (since, $\overline{U}(x_0, r\,\eta)$ is a closed set). The sequence $\{\mu_n\}$ is bounded above. Indeed, we have

$$\mu_n = \frac{1}{1-2\,\gamma\displaystyle\sum_{i=0}^{n-1}(q_{i-1}+q_i)} \leq \frac{1}{1-2\,\gamma\displaystyle\sum_{i=0}^{\infty}(q_{i-1}+q_i)},$$

and $\lim\limits_{n\longrightarrow\infty} q_n = 0$, which imply $\lim\limits_{n\longrightarrow\infty} c_n = 0$. By letting $n \longrightarrow \infty$ in (2.4), we get $F(x^\star) = 0$.

We also have

$$\| x_{n+1} - x_0 \| \leq \sum_{i=0}^{n} \| x_{i+1} - x_i \| \leq \sum_{i=0}^{n} q_i\,\eta < r\,\eta, \tag{2.7}$$

which imply $x_n \in \overline{U}(x_0, r\,\eta)$. Consequently, we obtain $x^\star \in \overline{U}(x_0, r\,\eta)$.

Finally, we shall show the uniqueness of the solution $x^\star$ in $U(x_0, r_0)$. Let $y^\star$ be a solution of equation $F(x) = 0$ in $U(x_0, r_0)$. Define linear operator

$$\mathcal{L} = \int_0^1 F'(x_t^\star)\mathrm{d}t, \quad \text{where} \quad x_t^\star = x^\star + t\,(y^\star - x^\star).$$

We shall show $\mathcal{L}^{-1}$ exists. Using $(\mathcal{C}_2)$ and $(\mathcal{C}_7)$, we get

$$\begin{aligned}
\| A_0^{-1} \| \| A_0 - \mathcal{L} \| &\leq \tfrac{\beta M}{2} \int_0^1 (\| 2\, x_0 - x_{-1} - x_t^\star \| + \| x_{-1} - x_t^\star \| \,\mathrm{d}t \\
&\leq 2\, \tfrac{\beta M}{2} \int_0^1 (\| x_0 - x_{-1} \| + 2 \ \| x_0 - x_t^\star \|)\mathrm{d}t \qquad (2.8)\\
&\leq \tfrac{\beta M}{2} \,(2\, d + \| x_0 - x^\star \| + \| y^\star - x_0 \|) \\
&< \tfrac{\beta M}{2}\,(2\, d + r\, \eta + r_0) = 1.
\end{aligned}$$

It follows from (2.8), and the Banach lemma on invertible operators, that $\mathcal{L}$ is invertible.

Finally, in view of the equality

$$0 = F(y^\star) - F(x^\star) = \mathcal{L}\,(y^\star - x^\star),$$

we obtain

$$x^\star = y^\star.$$

This completes the proof of Theorem 5. □

Remark 6. (a) It follows from the proof of Lemma 4 that

$$\mu_k = \frac{1}{1 - 2\,\gamma \displaystyle\sum_{i=0}^{k-1} (q_{i-1} + q_i)},$$

so that

$$\sum_{i=0}^{k-1} (q_{i-1} + q_i) = \tfrac{1}{2\,\gamma}\,(1 - \tfrac{1}{\mu_k}). \qquad (2.9)$$

By (2.9), the following relation between μ_∞ and r holds:

$$r = .5\,\Big(- q_{-1} + \tfrac{1}{2\,\gamma}\,(1 - \tfrac{1}{\mu_\infty})\Big).$$

Set

$$\overline{r}_n = .5\,\Big(-q_{-1} + \tfrac{1}{2\,\gamma}\,(1 - \tfrac{1}{\mu_n})\Big), \quad \overline{r} = .5\,(-q_{-1} + \tfrac{1}{2\,\gamma}) \quad \text{and} \quad \overline{r}_0 = \tfrac{2}{\beta M} - 2\,d - \overline{r}\,\eta.$$

Then, we have

$$\overline{r} > r \quad \text{and} \quad \overline{r}_0 < r_0.$$

In view of the proof of Theorem 5, $\overline{r}$ can replace r. However, this approach is less accurate but it avoids the computation of μ_∞.

(b) Condition $(\mathcal{C}_5)$ implies that for $x = y$ and $u = v$

$$\| F'(x_0)^{-1}\ (F'(x) - F'(u)) \| \leq M \ \| x - u \| \quad \text{for all} \quad x, u \in \mathcal{D}.$$

Then the conclusions of [14, Theorem 4.4] can be obtained from Theorem 5 for

$$b = \tfrac{a^2+a-1}{a^2}, \qquad c = \tfrac{1}{a^2}. \quad \square$$

3. NUMERICAL EXAMPLES

To illustrate the theoretical results introduced previously, we present some numerical examples. In these examples we show some situations where the results provided in the paper can be applied.

EXAMPLE 7. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$ be equipped with the max–norm. Choose:

$$x_{-1} = (.999, .999)^T, \quad x_0 = (1,1)^T, \quad \mathcal{D} = U(x_0, 1-\kappa), \quad \kappa \in [0,1).$$

Define function F on D by

$$F(x) = (\theta_1^3 - \kappa,\, \theta_2^3 - \kappa)^T, \qquad x = (\theta_1, \theta_2)^T. \tag{3.1}$$

The Fréchet–derivative of operator F is given by

$$F'(x) = \begin{bmatrix} 3\,\theta_1^2 & 0 \\ 0 & 3\,\theta_2^2 \end{bmatrix}, \tag{3.2}$$

and the divided difference of F is defined by

$$[y, x; F] = \int_0^1 F'(x + t(y-x))\mathrm{d}t.$$

By the $(\mathcal{C})$ conditions, Definition 1, and Remark 6 (a), we have:

$$M = 6\ (2-\kappa), \quad \eta = (1-\kappa)\ \beta.$$

Let $\kappa = .75$. Then, using Maple 13, we get for $a = b = .5$, and $c = 1$:

$$\beta = .333333222, \qquad M = 7.5,$$
$$q_{-1} = d = .001, \qquad \eta = .083333306,$$
$$\gamma = .104166597, \quad d_0 = .012000004, \quad \alpha = .216518950,$$

$p_0 = .027291649$	$q_0 = p_0 + w_0 = 1.027291649$	
$\mu_1 = 1.276355057$	$\overline{r}_1 = .513645824$	$\overline{r}_1\,\eta = .042803804$
$w_1 = .178421441$	$p_1 = .02542729$	$q_1 = p_1 + w_1 = .203848731$
$\mu_2 = 1.897556100$	$\overline{r}_2 = 1.129216014$	$\overline{r}_2\,\eta = .094101303$
$w_2 = .128802051$	$p_2 = .006009641$	$q_2 = p_2 + w_2 = .134811692$
$\mu_3 = 2.190871176$	$\overline{r}_3 = 1.298546226$	$\overline{r}_3\,\eta = .108212149$
$w_3 = .023629489$	$p_3 = .000758844$	$q_3 = p_3 + w_3 = .024388333$
$\mu_4 = 2.362542637$	$\overline{r}_4 = 1.378146238$	$\overline{r}_4\,\eta = .114845482$
$w_4 = .00258294$	$p_4 = 1.59131E-05$	$q_4 = p_4 + w_4 = .002598853$
$\mu_5 = 2.394346713$	$\overline{r}_5 = 1.391639832$	$\overline{r}_5\,\eta = .115969947$
$w_5 = 4.9428E-05$	$p_5 = 3.21907E-08$	$q_5 = p_5 + w_5 = 4.94602E-05$
$\mu_6 = 2.397513917$	$\overline{r}_6 = 1.392963989$	$\overline{r}_6\,\eta = .116080294$

$w_6 = 9.70475E{-}08$	$p_6 = 1.19934E{-}12$	$q_6 = p_6{+}w_6 = 9.70487E{-}08$
$\mu_7 = 2.397573264$	$\overline{r}_7 = 1.392988767$	$\overline{r}_7\,\eta = .116082359$
$w_7 = 3.59932E{-}12$	$p_7 = 8.72398E{-}20$	$q_7 = p_7{+}w_7 = 3.59932E{-}12$
$\mu_8 = 2.397573381$	$\overline{r}_8 = 1.392988816$	$\overline{r}_8\,\eta = .116082363$
$w_8 = 2.61721E{-}19$	$p_8 = 2.35266E{-}31$	$q_8 = p_8{+}w_8 = 2.61721E{-}19$
$\mu_9 = 2.397573381$	$\overline{r}_9 = 1.392988816$	$\overline{r}_9\,\eta = .116082363$

We can stop the process, since $\overline{r}_9 = \overline{r}_8$. Then, we set $r \simeq \overline{r}_9 = 1.392988816$. Consequently

$$\overline{r}_0 = .681917904$$

and

$$\mathcal{D}_0 = U(x_0, .681917904) \cap \mathcal{D} = \mathcal{D}.$$

The hypotheses of Theorem 5 are satisfied. Hence, equation $F(x) = 0$ has a solution

$$x^\star = (\sqrt[3]{.75}, \sqrt[3]{.75})^T = (.908560296, .908560296)^T,$$

which is unique in $\mathcal{D}_0$ and can be obtained as the limit of $\{x_k\}$ starting at x_0.

We can make a comparison between CSTM and DFM. Table 1 shows the comparison results for CSTM and DFM for this example. From Table 1, we can conclude that DFM is faster than CSTM.

Table 1. The comparison results for CSTM and DFM

	CSTM		DFM	
n	$\|y_n - x_n\|$	$\|x_{n+1} - x_n\|$	$\|y_n - x_n\|$	$\|x_{n+1} - x_n\|$
0	0.170170113	0.001	0.169999943	0.001
1	0.026874237	0.177126154	0.032667629	0.177020256
2	0.004721205	0.029570382	0.001370181	0.033232850
3	0.000115413	0.004813812	2.19776E-06	0.001371179
4	3.68019E-07	0.000115768	5.70322E-12	2.19776E-06
5	2.71547E-11	3.68046E-07	1.11022E-16	5.7031E-12

□

Example 8. In this example we present an application of the previous analysis to the Chandrasekhar equation [1], [5], [12], [19]:

$$x(s) = 1 + \tfrac{s}{4}\, x(s) \int_0^1 \tfrac{x(t)}{s+t} \mathrm{d}t, \quad s \in [0,1]. \tag{3.3}$$

We determine where a solution is located, along with its region of uniqueness. Later, the solution is approximated by an iterative method of DFM.

Note that solving (3.3) is equivalent to solve $F(x) = 0$, where $F : C[0,1] \to C[0,1]$ and

$$[F(x)](s) = x(s) - 1 - \tfrac{s}{4}\, x(s) \int_0^1 \tfrac{x(t)}{s+t}\, \mathrm{d}t, \quad s \in [0,1]. \tag{3.4}$$

To obtain the existence of a unique solution of $F(x)=0$, where F is given in (3.4), we need evaluate d, β, η, M from operator (3.4) and the starting points x_{-1} and x_0. In addition, from (3.4), we have

$$[F'(x)y](s) = y(s) - \tfrac{s}{4}\, x(s) \int_0^1 \tfrac{y(t)}{s+t}\, \mathrm{d}t - \tfrac{s}{4}\, y(s) \int_0^1 \tfrac{x(t)}{s+t}\, \mathrm{d}t, \quad s \in [0,1],$$

$$\begin{aligned}[2\,y-x, x; F]z(s) &= \int_0^1 F'(x+2\,\tau(y-x))z(s)\,\mathrm{d}\tau \\ &= z(s) - \tfrac{1}{4}\int_0^1 \tfrac{s}{s+t}(y(s)z(t)+z(s)y(t))\,\mathrm{d}t.\end{aligned}$$

On the other hand, from (3.3), we infer that $x(0)=1$, so that reasonable choices of initial approximations seem to be $x_{-1}(s)=.99$ and $x_0(s)=1$, for all $s\in[0,1]$, and $d=\|x_0-x_{-1}\|=.01$. In consequence,

$$\|I-A_0\| = \tfrac{1}{2}\max_{s\in[0,1]}\left|\int_0^1 \tfrac{s}{s+t}\mathrm{d}t\right| = \tfrac{1}{2}\max_{s\in[0,1]} s\ln(1+\tfrac{1}{s}) = \tfrac{\ln 2}{2} < 1.$$

Hence, by the Banach lemma, there exists A_0^{-1} and

$$\|A_0^{-1}\| \le \tfrac{1}{1-\|I-A_0\|} \le \tfrac{2}{2-\ln 2} = 1.17718382 = \beta.$$

Moreover,

$$\|A_0^{-1}F(x_0)\| \le \beta * \tfrac{1}{4}\max_{s\in[0,1]} s\ln(1+\tfrac{1}{s}) = \beta * \tfrac{\ln 2}{4} = 0.08859191 = \eta.$$

Furthermore,

$$\|[x,y;F]-[u,v;F]\| \le \tfrac{\ln 2}{4}\left(\|x-u\|+\|y-v\|\right) \text{ and } M = \tfrac{\ln 2}{2} = 0.150514998.$$

If we now choose $a=b=1/2$, $c=1$, and using Maple 13, then

$$\gamma = .007848527, \quad q_{-1}=d=.01, \quad d_0 = .112877124, \quad \alpha = .017513597 < 1,$$

$p_0 = .002848051$	$q_0 = 1.002848051$	$\mu_1 = 1.017825791$
$\overline{r}_1 = .501424025$	$\overline{r}_1\,\eta = .044422112$	
$w_1 = .012741379$	$p_1 = .000102398$	$q_1 = .012843777$
$\mu_2 = 1.034615081$	$\overline{r}_2 = 1.009269939$	$\overline{r}_2\,\eta = .089413152$
$w_2 = .000314609$	$p_2 = 3.30127E-08$	$q_2 = .000314642$
$\mu_3 = 1.034836224$	$\overline{r}_3 = 1.015849148$	$\overline{r}_3\,\eta = .089996016$
$w_3 = 9.94684E-08$	$p_3 = 2.54212E-13$	$q_3 = 9.94687E-08$
$\mu_4 = 1.034841515$	$\overline{r}_4 = 1.016006519$	$\overline{r}_4\,\eta = 0.090009958$
$w_4 = 7.6268E-13$	$p_4 = 6.16157E-22$	$q_4 = 7.6268E-13$
$\mu_5 = 1.034841516$	$\overline{r}_5 = 1.016006568$	$\overline{r}_5\,\eta = 0.090009963$
$w_5 = 1.84847E-21$	$p_5 = 1.14503E-35$	$q_5 = 1.84847E-21$
$\mu_6 = 1.034841516$	$\overline{r}_6 = 1.016006568$	$\overline{r}_6\,\eta = 0.090009963$

We stop the process, since $\overline{r}_6 = \overline{r}_5$. Then, we set $r \simeq \overline{r}_6 = 1.016006568$. Consequently

$$\overline{r}_0 = 11.17770242.$$

The conditions of Theorem 5 are satisfied. In consequence, equation (3.3) has a solution $x^\star$ in $\{\varphi \in C[0,1]; \|\varphi - 1\| \leq .090009963\}$.

To obtain a numerical solution of (3.3), we first discretize the problem and approach the integral by a Gauss-Legendre numerical quadrature with eight nodes,

$$\int_0^1 f(t)\,\mathrm{d}t \approx \sum_{j=1}^{8} w_j f(t_j),$$

where

t_1= 0.019855072	t_2= 0.101666761	t_3= 0.237233795	t_4= 0.408282679
t_5= 0.591717321	t_6= 0.762766205	t_7= 0.898333239	t_8= 0.980144928
w_1= 0.050614268	w_2= 0.111190517	w_3= 0.156853323	w_4= 0.181341892
w_5= 0.181341892	w_6= 0.156853323	w_7= 0.111190517	w_8= 0.050614268

If we denote $x_i = x(t_i)$, $i = 1, 2, \ldots, 8$, equation (3.3) is transformed into the following nonlinear system:

$$x_i = 1 + \tfrac{x_i}{4} \sum_{j=1}^{8} a_{ij} x_j, \quad i = 1, 2, \ldots, 8,$$

where $a_{ij} = \frac{t_i w_j}{t_i + t_j}$.

Denote now $\overline{x} = (x_1, x_2, \ldots, x_8)^T$, $\overline{1} = (1, 1, \ldots, 1)^T$, $A = (a_{ij})$ and write the last nonlinear system in the matrix form:

$$\overline{x} = \overline{1} + \tfrac{1}{4}\,\overline{x} \odot (A\overline{x}), \tag{3.5}$$

where $\odot$ represents the inner product. If we choose $\overline{x}_0 = (1, 1, \ldots, 1)^T$ and $\overline{x}_{-1} = (.99, .99, \ldots, .99)^T$, after eight iterations by applying method DFM with $a = b = 1/2$ and $c = 1$, and using the stopping criterion $\|\overline{x}_{n+1} - \overline{x}_n\| < 10^{-20}$, we obtain the numerical solution $\overline{x}^\star = (x_1^\star, x_2, \ldots, x_8^\star)^T$ given in Table 2.

j	$x_j^\star$	j	$x_j^\star$	j	$x_j^\star$	j	$x_j^\star$
1	1.0220...	3	1.1291...	5	1.2102...	7	1.2510...
2	1.0747...	4	1.1751...	6	1.2350...	8	1.2595...

Table 2. Numerical solution $\overline{x}^\star = (x_1^\star, x_2^\star, \ldots, x_8^\star)^T$ of system (3.5)

□

CONCLUSION

We provided a semilocal convergence analysis of DFM for approximating a locally unique solution of an equation in a Banach space, which shows that

DFM is faster than CSTM given in [9]. These advantages are obtained under the same computational cost as in [9]. DFM is also a useful derivative free alternative to the usage of *Newton's method* (NW). The latter method requires Fréchet-derivative operator at each step but its computation may be too expensive or impossible, especially if the analytic representation of Fréchet-derivative operator involved is unavailable [5]–[7], [14]–[17], [19], [20], [22], [25].

REFERENCES

[1] I.K. Argyros, *Polynomial operator equations in abstract spaces and applications*, St. Lucie/CRC/Lewis Publ. Mathematics series, 1998, Boca Raton, Florida, U.S.A.

[2] I.K. Argyros, *On the Newton–Kantorovich hypothesis for solving equations*, J. Comput. Appl. Math., **169**, pp. 315–332, 2004.

[3] I.K. Argyros, *A unifying local–semilocal convergence analysis and applications for two–point Newton–like methods in Banach space*, J. Math. Anal. Appl., **298**, pp. 374–397, 2004.

[4] I.K. Argyros, *New sufficient convergence conditions for the Secant method*, Chechoslovak Math. J., **55**, pp. 175–187, 2005.

[5] I.K. Argyros, *Convergence and applications of Newton–type iterations*, Springer–Verlag Publ., New–York, 2008.

[6] I.K. Argyros, *On a two-point Newton-like method of convergent order two*, Int. J. Comput. Math., **88, 2**, pp. 219–234, 2005.

[7] I.K. Argyros, *A Kantorovich-type analysis for a fast method for solving nonlinear equations*, J. Math. Anal. Appl. **332**, pp. 97–108, 2007.

[8] I.K. Argyros and S. Hilout, *On the weakening of the convergence of Newton's method using recurrent functions*, J. Complexity, **25**, pp. 530–543, 2009.

[9] I.K. Argyros, J. Ezquerro, J.M. Gutiérrez, M. Hernández and S. Hilout, *On the semilocal convergence of efficient Chebyshev-Secant-type methods*, J. Comput. Appl. Math., **235**, pp. 3195–3206, 2011.

[10] W.E. Bosarge and P.L. Falb, *A multipoint method of third order*, J. Optimiz. Th. Appl., **4**, pp. 156–166, 1969.

[11] E. Catinas, *On some iterative methods for solving nonlinear equations*, Rev. Anal. Numér. Théor. Approx., **23, 1**, pp. 47–53, 1994.

[12] S. Chandrasekhar, *Radiative transfer, Dover Publ.*, New–York, 1960.

[13] J.E. Dennis, *Toward a unified convergence theory for Newton–like methods, in Nonlinear Functional Analysis and Applications L.B. Rall*, Ed., Academic Press, New York, pp. 425–472, 1971.

[14] J.A. Ezquerro and M.A. Hernández, *An optimization of Chebyshev's method*, J. Complexity, **25**, pp. 343–361, 2009.

[15] M.A., Hernández, M.J. Rubio and J.A., Ezquerro, *Solving a special case of conservative problems by Secant–like method*, Appl. Math. Cmput., **169**, pp. 926–942, 2005.

[16] M.A. Hernández, M.J. Rubio and J.A. Ezquerro, *Secant–like methods for solving nonlinear integral equations of the Hammerstein type*, J. Comput. Appl. Math., **115**, pp. 245–254, 2000.

[17] M. Grau and M. Noguera, *A variant of Cauchy's method with accelerated fifth-order convergence*. Appl. Math. Lett., **17**, pp. 509–517, 2004.

[18] Z. Huang, *A note of Kantorovich theorem for Newton iteration*, J. Comput. Appl. Math., **47**, pp. 211–217, 1993.

[19] L.V. Kantorovich and G.P. Akilov, *Functional Analysis*, Pergamon Press, Oxford, 1982.

[20] V.A. Kurchatov, *On a method of linear interpolation for the solution of functional equations*, Soviet Math. Dokl., **12, 3**, pp. 835–838, 1971.

[21] P. Laasonen, *Ein überquadratisch konvergenter iterativer algorithmus*, Ann. Acad. Sci. Fenn. Ser I, **450**, pp. 1–10, 1969.

[22] J.M. Ortega and W.C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, 1970.

[23] F.A. Potra, *Sharp error bounds for a class of Newton–like methods, Libertas Mathematica*, **5**, pp. 71–84, 1985.

[24] J.W. Schmidt, *Untere Fehlerschranken fur Regula–Falsi Verhafren*, Period. Hungar., **9**, pp. 241–247, 1978.

[25] S.M. Shakhno, *About the difference method with quadratic convergence for solving nonlinear operator equations*, PAMM, **4, 1**, pp. 650–651, 2004.

[26] T. Yamamoto, *A convergence theorem for Newton–like methods in Banach spaces*, Numer. Math., **51**, pp. 545–557, 1987.

[27] M.A. Wolfe, *Extended iterative methods for the solution of operator equations*, Numer. Math., **31**, pp. 153–174, 1978.

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