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NECESSARY AND SUFFICIENT CONDITIONS FOR OSCILLATION OF THE SOLUTIONS OF EVEN ORDER DIFFERENTIAL EQUATIONS‡

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Abstract. In this paper, we establish several necessary and sufficient conditions for oscillation of the solutions of the following even order differential equation

$$x^{(n)}(t) + q(t)x^{\gamma}(t) = 0, \quad n \text{ is even},$$

where $q(t) \in C([t_0, \infty), \mathbb{R}^+)$ and γ is the quotient of odd positive integers.

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1. INTRODUCTION

Considering the n-order differential equation

$$x^{(n)}(t) + q(t)x^{\gamma}(t) = 0, \qquad n \text{ is even}, \tag{1}$$

where $q(t) \in C([t_0, \infty), \mathbb{R}^+)$ and γ is the quotient of odd positive integers.

In the recent past, the asymptotic and oscillatory properties of the solutions of n-order differential equations have been researched by many authors (see [1–3, 7–9]).

A solution of Eq.(1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise, the solution is said to be nonoscillatory.

We say that Eq.(1) is strictly superlinear if $\gamma > 1$, strictly sublinear if $0 < \gamma < 1$, and linear if $\gamma = 1$.

In particular, if $n = 2$, then Eq.(1) reduced to

$$x''(t) + q(t)x^{\gamma}(t) = 0, \tag{2}$$

Eq.(2) is the well-known Emden-Fowler equation (see [10–12]).

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Many remarkable results have been established for the oscillation of solutions of the second and higher order functional differential equations. For example, the following well-known Theorems A-C are presented in [4–6].

THEOREM A (see [4, 6]). *If* $\gamma > 0$, *then Eq.*(2) *has a bounded nonoscillatory solution if and only if*

$$\int_{t_0}^{\infty} sq(s)\mathrm{d}s < \infty.$$

THEOREM B (see [4, 5]). *If* $\gamma > 1$, *then all solutions of Eq.*(2) *are oscillatory if and only if*

$$\int_{t_0}^{\infty} sq(s)\mathrm{d}s = \infty.$$

THEOREM C (see [6]). *If* $0 < \gamma < 1$, *then Eq.*(2) *is oscillatory if and only if*

$$\int_{t_0}^{\infty} s^{\gamma}q(s)\mathrm{d}s = \infty.$$

For Eq.(1) with $\gamma = 1$, the following Theorem D is presented in [7].

THEOREM D. *If* $\gamma = 1$, *then every bounded solution of Eq.*(2) *oscillates if and only if*

$$\int_{t_0}^{\infty} s^{n-1}q(s)\mathrm{d}s = \infty.$$

Due to some obstacles of theoretical and technical character in handling with higher order nonlinear differential equation, and there are a few results which presented the necessary and sufficient conditions for the oscillatory behavior when $\gamma \neq 1$. So there are a lot of problems worth to be considered further for the Eq.(1).

The main aim of this paper is to prove the following Theorem 1.1:

THEOREM 1.1. *If* $\gamma \neq 1$ *is the quotient of odd positive integers and* n *is even, then the following statements are true*:

(a) *If*

$$\int_{t_0}^{\infty} s^{n-1}q(s)\mathrm{d}s < \infty, \tag{3}$$

then Eq.(1) *has a bounded nonoscillatory solution*;

(b) *If* $\gamma > 1$, *then every solution of Eq.*(1) *oscillates if and only if*

$$\int_{t_0}^{\infty} s^{n-1}q(s)\mathrm{d}s = \infty; \tag{4}$$

(c) *If* $0 < \gamma < 1$, *then every solution of Eq.*(1) *oscillates if and only if*

$$\int_{t_0}^{\infty} s^{(n-1)\gamma}q(s)\mathrm{d}s = \infty. \tag{5}$$

We clearly see that Theorems A-C are the special case of our Theorem 1.1.

2. PROOF OF THEOREM 1.1

In order to prove Theorem 1.1 we need the following Lemma 2.1.

LEMMA 2.1 (see [1-2, 7]). *Let $x(t)$ be a positive and n-times differentiable function on $[t_0,\infty)$, and $x^{(n)}(t)$ be nonpositive and not identically zero on any subinterval $[t_1,\infty)$. Then there exist $T\geq t_0$ and integer $k\in\{0,1,...,n-1\}$, such that $n+k$ is odd and*

(i) $x^{(i)}(t)\geq 0$ *for* $t\geq T, i=0,1,...,k-1$;

(ii) $(-1)^{i+k}x^{(i)}(t)>0$ *for* $i=k,k+1,...,n$;

(iii) $(t-T)|x^{(k-i)}(t)|\leq(1+i)|x^{(k-i-1)}(t)|$ *for* $t\geq T, i=0,1,...,k-1, k=1,...,n-1$.

Proof of the Theorem 1.1.

(a) Assume that (3) holds, we first prove that Eq.(1) has a nonoscillatory solution.

Observing that if $x(t)$ satisfies the equation

$$x(t)=1-\tfrac{1}{(n-1)!}\int_t^\infty(s-t)^{n-1}q(s)x^\gamma(s)\mathrm{d}s, \tag{6}$$

then $x(t)$ is a solution of Eq.(1). Therefore it suffices to show that Eq.(6) has bounded nonoscillatory solution. To this end, choose sufficient large $t\geq T$ such that

$$\max\left\{\int_t^\infty s^{n-1}q(s)\mathrm{d}s, 2\gamma\int_t^\infty s^{n-1}q(s)\mathrm{d}s\right\}<\tfrac{1}{2}(n-1)!. \tag{7}$$

Next, we consider the functional set

$$M=\{x\in C([T,\infty),\mathbb{R}):\tfrac{1}{2}\leq x(t)\leq 1\}$$

and define the operator $S:M\to C([T,\infty),\mathbb{R})$ as follows:

$$Sx(t)=1-\tfrac{1}{(n-1)!}\int_t^\infty(s-t)^{n-1}q(s)x^\gamma(s)\mathrm{d}s. \tag{8}$$

We clearly see that $x(t)^\gamma\leq 1$ and

$$(Sx)(t)\geq 1-\tfrac{1}{(n-1)!}\int_t^\infty(s-t)^{n-1}q(s)\mathrm{d}s\geq\tfrac{1}{2}\quad for\ t\geq T.$$

Therefore, $(Sx)(t)\leq 1$ and $S:M\to M$. Now, we claim that S is a contraction on M. In fact, let $f(x)=x^\gamma$, then for $x_1,x_2\in(\frac{1}{2},1)$ one has

$$|x_1^\gamma-x_2^\gamma|=|f'(\xi)||x_1-x_2|,\quad\text{where } \xi\in(\min\{x_1,x_2\},\max\{x_1,x_2\}),$$

where

$$|f'(\xi)|=|\gamma\xi^{\gamma-1}|\quad\leq\quad\begin{cases}\gamma, & if\ \gamma\geq 1,\\ 2\gamma, & if\ 0<\gamma<1.\end{cases}$$

Therefore

$$|x_1^\gamma - x_2^\gamma| \le 2\gamma|x_1 - x_2|, \quad for\ x_1, x_2 \in (\tfrac{1}{2}, 1).$$

Let $x, w \in M$, then for $n \ge N$ one has

$$\begin{aligned} |(Sx)(t) - (Sw)(t)| &\le \tfrac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}q(s)|x^\gamma(s) - w^\gamma(s)|\mathrm{d}s \\ &\le \tfrac{2\gamma}{(n-1)!}\int_t^\infty (s-t)^{n-1}q(s)|x(s) - w(s)|\mathrm{d}s \\ &\le \tfrac{2\gamma}{(n-1)!}||x(s) - w(s)||\int_t^\infty (s-t)^{n-1}q(s)\mathrm{d}s \le \tfrac{1}{2}||x-w||. \end{aligned}$$

Hence

$$||Sx - Sw|| \le \tfrac{1}{2}||x - w|| \tag{9}$$

and S is a contraction on M. The (unique) fixed point of T is the desired bounded, nonoscillatory solution of Eq.(1).

(b) Sufficiency. Assume that $\gamma > 1$ and $\int_{t_0}^\infty s^{n-1}q(s)\mathrm{d}s = \infty$, we prove that every solution of Eq.(1) oscillates. Otherwise, Eq.(1) has a nonoscillatory solution $x(t)$. Without loss of generality, we assume that $x(t) > 0$ for $t \ge t_0$. Then Lemma 2.1 implies that there exist odd integer $k \in \{1, ..., n-1\}$ and $T_k \ge t_0$ such that

$$x^{(i)}(t) > 0, \quad for \quad t \ge T_k, 0 \le i \le k, \tag{10}$$

$$(-1)^{i+k}x^{(i)}(t) > 0, \quad for \quad t \ge T_k, k \le i \le n. \tag{11}$$

The proof is divided into two cases.

Case 1 $k = 1$. That is

$$x'(t) > 0, x''(t) < 0, x^{(3)}(t) > 0, ..., x^n(t) < 0. \tag{12}$$

From (10) and (11) together with the Taylor expansion we get

$$x'(t) = \sum_{j=0}^{n-2} \tfrac{(-1)^j}{j!}x^{(1+j)}(\tau)(\tau - t)^j + \tfrac{(-1)^{n-1}}{(n-2)!}\int_t^\tau (s-t)^{n-2}x^{(n)}(s)\mathrm{d}s. \tag{13}$$

Using (12) we have

$$x'(t) > \int_t^\tau \tfrac{(s-t)^{n-2}}{(n-2)!}q(s)x^\gamma(s)\mathrm{d}s, \tag{14}$$

which implies

$$(15)\qquad x'(t) > \int_t^\infty \frac{(s-t)^{n-2}}{(n-2)!} q(s) x^\gamma(s) \mathrm{d}s > \int_t^\infty \frac{(s-t)^{n-2}}{(n-2)!} q(s) \mathrm{d}s x^\gamma(t).$$

From inequality

$$\int_T^t (u-s)^{n-k-1} \mathrm{d}s = -\frac{(u-s)^{u-k}}{n-k}|_T^t = \frac{1}{n-k}[(u-T)^{n-k} - (u-t)^{n-k}]$$
$$\geq \frac{1}{n-k}(t-T)(u-T)^{n-k-1}$$

we obtain

$$\int_T^t \frac{x'(s)}{x^\gamma(s)} \mathrm{d}s > \int_T^t \mathrm{d}s \int_s^\infty \frac{(u-s)^{n-2}}{(n-2)!} q(u) \mathrm{d}u$$
$$= \int_T^t q(u) \mathrm{d}u \int_T^u \frac{(u-s)^{n-2}}{(n-2)!} \mathrm{d}s + \int_t^\infty q(u) \mathrm{d}u \int_T^t \frac{(u-s)^{n-2}}{(n-2)!} \mathrm{d}s$$
$$\geq \int_T^t \frac{(u-T)^{n-1}}{(n-1)!} q(u) \mathrm{d}u + (t-T) \int_t^\infty \frac{(u-T)^{n-2}}{(n-1)!} q(u) \mathrm{d}u.$$

Therefore

$$(16)\qquad \int_T^t \frac{(u-T)^{n-1}}{(n-1)!} q(u) \mathrm{d}u < \int_T^t \frac{x'(s)}{x^\gamma(s)} \mathrm{d}s$$

or

$$(17)\qquad \int_T^t \frac{(u-T)^{n-1}}{(n-1)!} q(u) \mathrm{d}u < \frac{x^{1-\gamma}(t)}{\gamma-1} < \infty,$$

which contradicts with

$$\int_T^\infty u^{n-1} q(u) \mathrm{d}u = \infty.$$

Case 2 $k > 1$. It follows from (iii) of Lemma 2.1 that for $t \geq T_k$,

$$(18)\qquad x(t) \geq \frac{(t-T_k)^{k-1}}{k!} x^{(k-1)}(t).$$

For sufficient large t, we have

$$x^\gamma(t) \geq \frac{(t-T_k)^{(k-1)\gamma}}{(k!)^\gamma} (x^{(k-1)}(t))^\gamma > \frac{(t-T_k)^{k-1}}{(k!)^\gamma} (x^{(k-1)}(t))^\gamma, \gamma > 1.$$

Let $z(t) = x^{(k-1)}(t)$, then

$$z(t) > 0, z'(t) > 0, z''(t) < 0, ...$$

and so

$$z^{(n-k+1)}(t)+q(t)\tfrac{(t-T_k)^{k-1}}{(k!)^\gamma}z^\gamma(t)<0. \tag{19}$$

Making use of the same method as in the proof of case 1, we get

$$\int_{t_0}^{\infty} s^{n-k}q(s)\tfrac{(s-T_k)^{k-1}}{(k!)^\gamma}\mathrm{d}s<\infty$$

or

$$\int_{t_0}^{\infty} s^{n-1}q(s)ds<\infty, \tag{20}$$

which also contradicts with

$$\int\limits_{t_0}^{\infty} s^{n-1}q(s)\mathrm{d}s=\infty.$$

Conversely, we prove that (4) holds if every solution of Eq.(1) oscillates and $\gamma>1$. Otherwise (3) holds, then from Theorem 1.1(a) we get the contradiction that Eq.(1) has a nonoscillatory solution.

(c) Sufficiency. For $0<\gamma<1$, there are two cases as follows.

Case 1 $k=1$. That is

$$x(t)>0, x'(t)>0, x''(t)<0, ..., x^n(t)<0.$$

Making use of the same method as Case 1 in Theorem 1.1(b), we have

$$x'(t)>\int\limits_{t}^{\infty}\tfrac{(s-t)^{n-2}}{(n-2)!}q(s)x^\gamma(s)\mathrm{d}s. \tag{21}$$

Integrating (21) from T to t yields

$$\begin{aligned}x(t)&>x(t)-x(T)\\&>\int\limits_{T}^{t}\tfrac{(u-T)^{n-1}}{(n-1)!}q(u)x^\gamma(u)\mathrm{d}u+(t-T)\int\limits_{t}^{\infty}\tfrac{(u-T)^{n-2}}{(n-1)!}q(u)x^\gamma(u)\mathrm{d}u\\&>(t-T)\int\limits_{t}^{\infty}\tfrac{(u-T)^{n-2}}{(n-1)!}q(u)x^\gamma(u)\mathrm{d}u\end{aligned}$$

or

$$\tfrac{x(t)}{t-T}>\int\limits_{t}^{\infty}\tfrac{(u-T)^{n-2}}{(n-1)!}q(u)x^\gamma(u)\mathrm{d}u. \tag{22}$$

Let

$$z(t)=\int_t^{\infty}\frac{(u-T)^{n-2}}{(n-1)!}q(u)x^{\gamma}(u)\mathrm{d}u, \tag{23}$$

then $z'(t)<0, 0<z(t)<\frac{x(t)}{t-T}$ and

$$z'(t)=-\frac{(t-T)^{n-2}}{(n-1)!}q(t)x^{\gamma}(t)\le-\frac{(t-T)^{n-2+\gamma}}{(n-1)!}q(t)z^{\gamma}(t), \tag{24}$$

$$\frac{z'(t)}{z^{\gamma}(t)}\le-\frac{(t-T)^{n-2+\gamma}}{(n-1)!}q(t) \tag{25}$$

for $T_2>T$. Then we get

$$\int_{T_2}^{t}\frac{z'(u)}{z^{\gamma}(u)}\mathrm{d}u\le-\int_{T_2}^{t}\frac{(u-T)^{n-2+\gamma}}{(n-1)!}q(u)\mathrm{d}u, \tag{26}$$

$$\frac{1}{1-\gamma}[z^{1-\gamma}(t)-z^{1-\gamma}(T_2)]\le-\frac{1}{(n-1)!}\int_{T_2}^{t}(u-T)^{n-2+\gamma}q(u)\mathrm{d}u. \tag{27}$$

Therefore

$$\int_{T_2}^{t}(u-T)^{n-2+\gamma}q(u)\mathrm{d}u<+\infty. \tag{28}$$

Inequality (28) and $(n-1)\gamma<n-2+\gamma$ leads to

$$\int_{T_2}^{t}(u-T)^{(n-1)\gamma}q(u)\mathrm{d}u<+\infty, \tag{29}$$

which contradicts with the assumption.

Case 2 $k>1$. That is

$$x(t)>0, x'(t)>0, ..., x^{(k-1)}(t)>0, x^{(k)}(t)>0, x^{(k+1)}(t)<0, ..., x^{(n)}(t)<0.$$

Lemma 2.1 implies

$$x(t)\ge\frac{(t-T_k)^{k-1}}{k!}x^{(k-1)}(t)$$

or

$$x^{\gamma}(t)\ge\frac{(t-T_k)^{(k-1)\gamma}}{(k!)^{\gamma}}[x^{(k-1)}(t)]^{\gamma}. \tag{30}$$

Let $z(t)=x^{(k-1)}(t)$, then $z(t)>0, z'(t)>0, z''(t)<0, ..., z^{(n-k+1)}<0$ and

$$z^{(n-k+1)}(t)+q(t)\frac{(t-T_k)^{(k-1)\gamma}}{(k!)^{\gamma}}z^{\gamma}(t)<0, \tag{31}$$

where $n-k+1$ is also even. Making use of the same method as in Case 1, we conclude that

$$\int_{t_0}^{\infty} s^{(n-k)\gamma} q(s) \tfrac{(s-T_k)^{(k-1)\gamma}}{(k!)^\gamma} \mathrm{d}s < +\infty \tag{32}$$

or

$$\int_{t_0}^{\infty} s^{(n-1)\gamma} q(s) \mathrm{d}s < +\infty, \tag{33}$$

which also contradicts with the assumption.

Necessity. For $0 < \gamma < 1$ and (5) holds, we prove that Eq.(1) has a nonoscillatory solution. Otherwise, from (33) we know that there exists $t \geq T$ such that

$$\int_{t}^{\infty} s^{(n-1)\gamma} q(s) \mathrm{d}s \leq \tfrac{1}{2}. \tag{34}$$

Let M be a set defined by

$$M = \{x \in C([T,\infty), R) : \tfrac{1}{2(n-1)!}(t-T)^{n-1} \leq x(t) \leq \tfrac{1}{(n-1)!}(t-T)^{n-1}, t \geq T\}$$

and the mapping T on M defined by

$$Sx(t) = \int_T^t \mathrm{d}s_1 \int_T^{s_1} \mathrm{d}s_2 \cdots \int_T^{s_{n-2}} [\tfrac{1}{2} + \int_{s_{n-1}}^{\infty} q(u)x^\gamma(u)\mathrm{d}u]\mathrm{d}s_{n-1}. \tag{35}$$

Then $(Sx)(t) \geq \frac{1}{2(n-1)!}(t-T)^{n-1}$ for $x(t) \in M$ and $t \geq T$. Moreover, from the definition of the operator S we get $(Sx)(t) \leq \frac{1}{(n-1)!}(t-T)^{n-1}$. Therefore, $TM \subseteq M$.

Next, we define the function $u_n : [T,\infty) \to \mathbb{R}$ as follows

$$u_n = (Su_{n-1})(t), \quad n \in \mathbb{N} \tag{36}$$

and

$$u_0(t) = \tfrac{1}{2(n-1)!}(t-T)^{n-1}, \quad t \geq T.$$

A straightforward verification leads to

$$\tfrac{1}{2(n-1)!}(t-T)^{n-1} \leq u_{n-1}(t) \leq u_n(t) \leq \tfrac{1}{(n-1)!}(t-T)^{n-1}, \quad t \geq T.$$

Therefore, there exists the limit $\lim\limits_{n\to\infty} u_n(t) = u(t)$ for $t \geq T$. It follows from the Lebesgue convergence theorem that $u \in M$ and $u(t) = (Su)(t)$. It is easy to see that $u(t)$ is the solution of the Eq.(1).

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