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SATURATION RESULTS FOR THE LAGRANGE MAX-PRODUCT INTERPOLATION OPERATOR BASED ON EQUIDISTANT KNOTS[‡]

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Abstract. In this paper we obtain the saturation order and a local inverse result in the approximation by the Lagrange max-product interpolation operator based on equidistant knots.

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1. INTRODUCTION

Based on the Open Problem 5.5.4, pp. 324-326 in [15], in a series of recent papers we have introduced and studied the so-called max-product operators attached to the Bernstein polynomials and to other linear Bernstein-type operators, like those of Favard-Szász-Mirakjan operators (truncated and nontruncated case), see [1], [3], Meyer-König and Zeller operators, see [4], Baskakov operators, see [6], [7] and Bleimann-Butzer-Hahn operators, see [5].

For example, in the recent paper [2], starting from the linear Bernstein operators $B_n(f)(x) = \sum_{k=0}^{n} b_{n,k}(x) f(k/n)$, where $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, written in the equivalent form

$$B_n(f)(x) = \frac{\sum\limits_{k=0}^{n} b_{n,k}(x) f(k/n)}{\sum\limits_{k=0}^{n} b_{n,k}(x)}$$

and then replacing the sum operator Σ by the maximum operator $\bigvee$, one obtains the nonlinear Bernstein operator of max-product kind

$$B_n^{(M)}(f)(x) = \frac{\bigvee\limits_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right)}{\bigvee\limits_{k=0}^{n} b_{n,k}(x)},$$

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where the notation $\bigvee_{k=0}^{n} b_{n,k}(x)$ means $\max\{b_{n,k}(x); k \in \{0, ..., n\}\}$ and similarly for the numerator.

For this max-product operator, nice approximation and shape preserving properties were found in the class of positive valued functions, in e.g. [2], [14].

In other two recent papers [11] and [12], this idea is applied to the Lagrange interpolation based on the Chebyshev nodes of second kind plus the endpoints, and to the Hermite-Fejér interpolation based on the Chebyshev nodes of first kind respectively, obtaining max-product interpolation operators which, in general, (for example, in the class of positive Lipschitz functions) approximates essentially better than the corresponding Lagrange and Hermite-Fejér interpolation polynomials.

Let $I = [a, b]$, $a < b$ and $f : [a, b] \to \mathbb{R}$. The max-product Lagrange interpolation operator on equidistant knots attached to the function f is given by (see [13])

$$L_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} l_{n,k}(x) f\left(x_{n,k}\right)}{\bigvee_{k=0}^{n} l_{n,k}(x)},\ x \in I,\ n \in \mathbb{N}, \tag{1.1}$$

where $x_{n,k} = a + (b - a)k/n$ for all $n \in \mathbb{N}$ and $k \in \{0, 1, ..., n\}$ and

$$l_{n,k}(x) = (-1)^{n-k} \left(\prod_{i=0}^{n} (x - x_{n,i}) \right) \cdot \frac{1}{x - x_{n,k}} \tag{1.2}$$

for all $x \in I$, $n \in \mathbb{N}$ and $k \in \{0, 1, ..., n\}$. Note that $L_n^{(M)}(f)$ is a well defined function. Indeed, using the fundamental Lagrange polynomials,

$$p_{n,k}(x) = \frac{(x - x_{n,0})(x - x_{n,1})...(x - x_{n,k-1})(x - x_{n,k+1})...(x - x_{n,n})}{(x_{n,k} - x_{n,0})(x_{n,k} - x_{n,1})...(x_{n,k} - x_{n,k-1})(x_{n,k} - x_{n,k+1})...(x_{n,k} - x_{n,n})},$$

we observe that we can rewrite $l_{n,k}(x)$, $x \in I$, in the form

$$l_{n,k}(x) = c_{n,k} \cdot p_{n,k}(x)$$

where

$$c_{n,k} = (x_{n,k} - x_{n,0})(x_{n,k} - x_{n,1})...(x_{n,k} - x_{n,k-1})(x_{n,k+1} - x_{n,k})...(x_{n,n} - x_{n,k}).$$

Then, since for any $x \in I$ we have $\sum_{i=0}^{n} p_{n,i}(x) = 1$ it follows the existence of $i(x) \in \{0, 1, ..., n\}$ such that $p_{n,i(x)}(x) > 0$ and noting that $c_{n,i(x)} > 0$ it easily results that $l_{n,i(x)}(x) > 0$ and this implies that $\bigvee_{k=0}^{n} l_{n,k}(x) > 0$ for all $x \in I$, which means that indeed $L_n^{(M)}(f)$ is a well defined function on $[a, b]$.

The max-product operator $L_n^{(M)}(f)(x)$ is continuous on $[a, b]$ and has the interpolation properties $L_n^{(M)}(f)(x_{n,j}) = f(x_{n,j})$ for all $j \in \{0, 1, ..., \}$.

Also, according to Corollary 3.2, (i), in [13], for positive valued functions, i.e. for $f : [a, b] \to \mathbb{R}_+$, it satisfies the Jackson-type estimate

$$|L_n^{(M)}(f)(x) - f(x)| \leq 2\omega_1 \left(f; \tfrac{b-a}{n}\right)_{[a,b]},\ \text{ for all } x \in [a, b],\ n \in \mathbb{N},$$

where $\omega_1\left(f;\frac{b-a}{n}\right)_{[a,b]}$ denotes the modulus of continuity of f on $[a,b]$. This estimate for the Lagrange max-product operator essentially improves for positive valued functions the order of approximation by the classical Lagrange interpolation polynomials on equidistant nodes, when as it is well-known, we can also have a very pronounced divergence phenomenon in $[a,b]$ (see e.g. Chapter 4 in the book [17], see also [16], [10]).

The goal of the present paper is to determine for $L_n^{(M)}$ the saturation order together with its special class of functions and to obtain a local inverse result.

The plan of the paper goes as follows. In Section 2 the saturation order together with its special class of functions are obtained. Section 3 contains a local inverse approximation result.

2. THE SATURATION ORDER

Firstly, we need three simple auxiliary results, Lemmas 2.1-2.3, where $l_{n,k}$ denote the fundamental Lagrange polynomials attached to the knots $x_{n,k} = k/n$, $k \in \{0,1,...,n\}$, $n \in \mathbb{N}$.

Lemma 2.1. *Let $n \in \mathbb{N}$, $j \in \{0,1,...,n-1\}$ and $x \in [j/n,(j+1)/n]$. We have*

$$\bigvee_{k=0}^{n} l_{n,k}(x) = l_{n,j}(x), \text{ for all } x \in \left[\frac{j}{n}, \frac{j+1/2}{n}\right]$$

and

$$\bigvee_{k=0}^{n} l_{n,k}(x) = l_{n,j+1}(x), \text{ for all } x \in \left[\frac{j+1/2}{n}, \frac{j+1}{n}\right].$$

Here $l_{n,k}$, $k \in \{0,1,...,n\}$ are given by (1.2).

Proof. Let us denote $J_n(x) = \{k \in \{0,1,...,n\} : l_{n,k}(x) > 0\}$. This implies that

$$\bigvee_{k=0}^{n} l_{n,k}(x) = \bigvee_{k\in J_n(x)} l_{n,k}(x).$$

We observe that $\{j,j+1\} \subseteq J_n(x)$. Indeed, for $a=0$ and $b=1$, by using (1.2) we have $sign(l_{n,j}(x)) = (-1)^{n-j}\cdot(-1)^{n-j} = 1$ and $sign(l_{n,j+1}(x)) = (-1)^{n-j-1}\cdot(-1)^{n-j-1} = 1$. Then, we denote $\Omega_n(x) = \prod_{i=0}^{n}(x-x_{n,i})$. The definitions of $l_{n,k}(x)$ and $J_n(x)$ imply $l_{n,k}(x) = \frac{|\Omega_n(x)|}{|x-x_{n,k}|}$ for all $k \in J_n(x)$. We thus obtain that $\bigvee_{k=0}^{n} l_{n,k}(x) = |\Omega_n(x)|\cdot \bigvee_{k\in J_n(x)} \frac{1}{|x-x_{n,k}|}$. Since $\{j,j+1\} \subseteq J_n(x)$ it is immediate that for $x \in \left[\frac{j}{n}, \frac{j+1/2}{n}\right]$ we have $\bigvee_{k\in J_n(x)} \frac{1}{|x-x_{n,k}|} = \frac{1}{|x-x_{n,j}|}$ and for $x \in \left[\frac{j+1/2}{n}, \frac{j+1}{n}\right]$ we have $\bigvee_{k\in J_n(x)} \frac{1}{|x-x_{n,k}|} = \frac{1}{|x-x_{n,j+1}|}$. From here we easily get the desired conclusion. $\square$

LEMMA 2.2. *For any function $f : [0,1] \to \mathbb{R}$, and for all $n \in \mathbb{N}$, $n \geq 1$, and $j \in \{0, 1, ..., n\}$, $j \leq n/2$, we have:*

(i) $L_n^{(M)}(f)(j/(n+1)) \geq f(j/n)$;
(ii) $L_{n+1}^{(M)}(f)(j/n) \geq f(j/(n+1))$.

Proof. (i) Firstly, by Lemma 2.1 we observe that for $x \in \left[\frac{(j-1)+1/2}{n}, \frac{j}{n}\right]$ we have $\bigvee_{k=0}^{n} l_{n,k}(x) = l_{n,j}(x)$. Now, if $j \leq n/2$ then it is easy to check that $x := j/(n+1) \in \left[\frac{(j-1)+1/2}{n}, \frac{j}{n}\right]$ which implies $\bigvee_{k=0}^{n} l_{n,k}(j/(n+1)) = l_{n,j}(j/(n+1))$. This implies that

$$\begin{aligned} L_n^{(M)}(f)(j/(n+1)) &= \frac{\bigvee_{k=0}^{n} l_{n,k}(j/(n+1)) f\left(\frac{k}{n}\right)}{l_{n,j}(j/(n+1))} \\ &\geq \frac{l_{n,j}(j/(n+1)) f\left(\frac{j}{n}\right)}{l_{n,j}(j/(n+1))} = f\left(\frac{j}{n}\right). \end{aligned}$$

(ii) Since $j \leq n/2$, one can easily prove that $j/n \in \left[\frac{j}{n+1}, \frac{j+1/2}{n+1}\right]$. Therefore, by Lemma 2.1 we obtain $\bigvee_{k=0}^{n+1} l_{n+1,k}(j/n) = l_{n+1,j}(j/n)$. This implies that

$$\begin{aligned} L_{n+1}^{(M)}(f)(j/n) &= \frac{\bigvee_{k=0}^{n+1} l_{n+1,k}(j/n) f\left(\frac{k}{n+1}\right)}{l_{n+1,j}(j/n)} \geq \frac{l_{n+1,j}(j/n) f\left(\frac{j}{n+1}\right)}{l_{n+1,j+1}(j/n)} \\ &= f\left(\frac{j}{n+1}\right). \end{aligned}$$

□

LEMMA 2.3. *For any function $f : [0,1] \to \mathbb{R}$, and for all $n \in \mathbb{N}$, $n \geq 1$, and $j \in \{0, 1, ..., n\}$, $j \geq n/2$, we have:*

(i) $L_n^{(M)}(f)((j+1)/(n+1)) \geq f(j/n)$;
(i) $L_{n+1}^{(M)}(f)((j/n) \geq f((j+1)/(n+1))$.

Proof. (i) Since $j \geq n/2$ by elementary calculus it is easy to prove that $(j+1)/(n+1) \in \left[\frac{j}{n}, \frac{j+1/2}{n}\right]$ and by Lemma 2.1 this implies that $\bigvee_{k=0}^{n} l_{n,k}((j+1)/(n+1)) = l_{n,j}((j+1)/(n+1))$. We obtain

$$\begin{aligned} L_n^{(M)}(f)((j+1)/(n+1)) &= \frac{\bigvee_{k=0}^{n} l_{n,k}((j+1)/(n+1)) f\left(\frac{k}{n}\right)}{l_{n,j}((j+1)/(n+1))} \\ &\geq \frac{l_{n,j}((j+1)/(n+1)) f\left(\frac{j}{n}\right)}{l_{n,j}((j+1)/(n+1))} = f\left(\frac{j}{n}\right). \end{aligned}$$

(ii) Since $j \geq n/2$, again it is easy to check that $j/n \in \left[\frac{j+1/2}{n+1}, \frac{j+1}{n+1}\right]$ and by Lemma 2.1 this implies that $\bigvee_{k=0}^{n+1} l_{n+1,k}(j/n) = l_{n+1,j+1}(j/n)$. We obtain

$$\begin{aligned} L_{n+1}^{(M)}(f)((j/n) &= \frac{\bigvee_{k=0}^{n+1} l_{n+1,k}(j/n) f\left(\frac{k}{n+1}\right)}{l_{n+1,j+1}(j/n)} \geq \frac{l_{n+1,j+1}(j/n) f\left(\frac{j+1}{n+1}\right)}{l_{n+1,j+1}(j/n)} \\ &= f\left(\frac{j+1}{n+1}\right). \end{aligned}$$

□

We are now in position to determine the saturation order and the associated special class of functions for the truncated max-product operator $L_n^{(M)}$.

THEOREM 2.4. *Denote* $C_+[a,b] = \{f : [a,b] \to \mathbb{R}_+; f$ *continuous on* $[a,b]\}$ *and* $\|f\| = \sup\{|f(x)|; x \in [a,b]\}$. *Then for the max-product* $L_n^{(M)}$ *operator, the saturation order in* $C_+[a,b]$ *is* $\frac{1}{n}$, *that is* $\|L_n^{(M)}(f) - f\| = o(1/n)$, *implies that* f *is a positive constant function on* $[a,b]$.

Proof. We begin with the particular case when $a = 0$ and $b = 1$. By hypothesis, there exists $a_n \in \mathbb{R}$, $n \in \mathbb{N}$ with the property $a_n \searrow 0$ as $n \to +\infty$, such that

$$\left|L_n^{(M)}(f)(x) - f(x)\right| \leq \frac{a_n}{n}, \text{ for all } x \in [0,1] \text{ and } n \in \mathbb{N}.$$

Let us choose arbitrary $\varepsilon > 0$. Since $a_n \searrow 0$ as $n \to +\infty$, it follows that there exists $n_0 \in \mathbb{N}$ such that $a_n < \varepsilon$ for all $n \in \mathbb{N}$, $n \geq n_0$. Noting the above relation we get

$$\left|L_n^{(M)}(f)(x) - f(x)\right| \leq \frac{\varepsilon}{n}, \text{ for all } x \in [0,1] \text{ and } n \in \mathbb{N}, n \geq n_0. \tag{2.1}$$

Then, from the uniform continuity of f it results the existence of $n_1 \in \mathbb{N}$ such that

$$|f(x) - f(y)| \leq \varepsilon \text{ for all } x, y \in [0,1] \text{ and } n \in \mathbb{N},\ |x - y| \leq 1/n, n \geq n_1. \tag{2.2}$$

We will obtain the desired conclusion in two steps: (A) we prove that f is constant on any interval $[a,b]$ with $0 < a < b < 1/2$; (B) we prove that f is constant on any interval $[a,b]$ with $1/2 < a < b < 1$. Indeed, if (A) holds then thanks to the continuity of f we easily obtain that f is constant on $[0,1/2]$. Similarly, if (B) holds then we obtain that f is constant on $[1/2,1]$. Then, from the continuity of f it easily follows that f is constant on $[0,1]$. So, we start by proving that (A) and (B) hold.

(A) Let us choose arbitrary $a, b \in \mathbb{R}$ such that $0 < a < b < 1/2$. Further one, let x_0 and y_0 be the points where f attaints its minimum and maximum respectively on the interval $[a,b]$. Without any loss of generality we may suppose that $x_0 \neq y_0$ (contrariwise it follows that f is constant on $[a,b]$ and there is nothing to prove). We have two subcases: A$_1$) $x_0 < y_0$ and A$_2$) $x_0 > y_0$.

Subcase A_1) Let $n \in \mathbb{N}$ be with $n > \max\{n_0, n_1, 2/(y_0 - x_0)\}$. By relation (2.1) it follows that

$$L_n^{(M)}(f)(j/(n+1)) - f(j/(n+1)) \leq \frac{\varepsilon}{n} \text{ for all } j \in \{0, 1, ..., n\}.$$

Moreover, combining the inequality in Lemma 2.2 (i) with the above inequality, we get

$$f(j/n) - f(j/(n+1)) \leq \frac{\varepsilon}{n} \text{ for all } j \in \{0, 1, ..., n\}, j \leq n/2. \tag{2.3}$$

Further one, let us choose $j_1 \in \{0, 1, ..., n-1\}$ such that $j_1/n \leq y_0 \leq (j_1+1)/n$ and $x_0 \leq j_1/n$. Note that there exists such an index j_1, because the previous inequalities are equivalent to $ny_0 - 1 \leq j_1 \leq ny_0$, $nx_0 \leq j_1 \leq ny_0$, while the condition $n > 2/(y_0 - x_0)$ is equivalent to the condition $ny_0 - nx_0 > 2$.

Also, from $j_1/n \leq y_0 \leq b < 1/2$ it easily follows that $j_1 \leq n/2$.

As a first consequence, from the relation (2.2) we obtain

$$|f(j_1/n) - f(y_0)| < \varepsilon. \tag{2.4}$$

Then, since $\lim\limits_{l\to\infty} \frac{j_1}{n+l} = 0$, by $x_0 > 0$ and $x_0 \leq j_1/n$ it follows that there exists $l_0 \in \mathbb{N}$ such that $\frac{j_1}{n+l_0+1} \leq x_0 \leq \frac{j_1}{n+l_0}$.

It is worth noting here that indeed, the above l_0 cannot be equal to 0, because if we would have $l_0 = 0$, then we would obtain $j_1/(n+1) \leq x_0 < y_0 \leq (j_1+1)/n \leq (j_1+2)/(n+1)$, which would imply $y_0 - x_0 \leq 2/(n+1) < 2/n$, in contradiction with the supposition that $n > 2/(y_0 - x_0)$.

The inequality $\frac{j_1}{n+l_0+1} \leq x_0 \leq \frac{j_1}{n+l_0}$ and (2.1) also implies that

$$|f((j_1/(n+l_0)) - f(x_0)| < \varepsilon. \tag{2.5}$$

Since $j_1 \leq n/2$, applying successively relation (2.3) we obtain

$$\begin{aligned}
f(j_1/n) - f(j_1/(n+1)) &\leq \tfrac{\varepsilon}{n}, \\
f(j_1/(n+1)) - f(j_1/(n+2)) &\leq \tfrac{\varepsilon}{n+1}, \\
&\vdots \\
f(j_1/(n+l_0-1)) - f(j_1/(n+l_0)) &\leq \tfrac{\varepsilon}{n+l_0-1}.
\end{aligned}$$

Taking the sum of all these inequalities we get

$$\begin{aligned}
f(j_1/n) - f(j_1/(n+l_0)) &\leq \tfrac{\varepsilon}{n} + \tfrac{\varepsilon}{n+1} + ... + \tfrac{\varepsilon}{n+l_0-1} \\
&\leq \tfrac{l_0\varepsilon}{n}.
\end{aligned}$$

Then, by relations (2.4)–(2.5) we obtain

$$\begin{aligned}
&f(y_0) - f(x_0) = \\
&= (f(y_0) - f(j_1/n)) + (f(j_1/n) - f(j_1/(n+l_0))) + (f(j_1/(n+l_0)) - f(x_0)) \\
&\leq |f(y_0) - f(j_1/n)| + f(j_1/n) - f(j_1/(n+l_0)) + |f(j_1/(n+l_0)) - f(x_0)| \\
&\leq 2\varepsilon + \tfrac{l_0\varepsilon}{n}
\end{aligned}$$

and since $0 \leq f(y_0) - f(x_0)$, we obtain

$$0 \leq f(y_0) - f(x_0) \leq 2\varepsilon + \frac{l_0 \varepsilon}{n}. \tag{2.6}$$

On the other hand, since $0 < x_0 \leq j_1/(n+l_0)$, after some simple calculations we get (note that $j_1 \leq n/2$)

$$l_0 \leq j_1/x_0 - n \leq n(1/(2x_0) - 1)$$

Using this information in relation (2.6) we obtain

$$0 \leq f(y_0) - f(x_0) \leq \varepsilon(2 + 1/(2x_0) - 1)$$

where $\varepsilon > 0$ was chosen arbitrary. Therefore, passing in the previous inequality with $\varepsilon \searrow 0$, we obtain $f(x_0) = f(y_0)$ (here, it is important that $x_0 > 0$). Since on the interval $[a, b]$ the maximum value and the minimum value of the function f coincide, we obtain that f is a constant function on the interval $[a, b]$ and hence we obtained the desired conclusion for this case.

Subcase A$_2$) Let us choose arbitrary $n \in \mathbb{N}$, $n > \max\{n_0, n_1, 2/(x_0 - y_0)\}$. By relation (2.1) it follows that

$$L_{n+1}^{(M)}(f)(j/n) - f(j/n) \leq \frac{\varepsilon}{n+1} \text{ for all } j \in \{0, 1, ..., n\}.$$

Moreover, combining the inequality in Lemma 2.2 (ii) with the above inequality, we get

$$f(j/(n+1)) - f(j/n) \leq \frac{\varepsilon}{n+1} \text{ for all } j \in \{0, 1, ..., n\}, j \leq n/2. \tag{2.7}$$

Let j_1 and l_0 be chosen as in the previous case, with the difference that now we have $j_1/(n + l_0 + 1) \leq y_0 \leq j_1/(n + l_0)$ and $j_1/n \leq x_0 \leq (j_1 + 1)/n$. Applying successively the above inequality (2.7) we get

$$\begin{aligned} f(j_1/(n+1)) - f(j_1/n) &\leq \frac{\varepsilon}{n+1}, \\ f(j_1/(n+2)) - f(j_1/(n+1)) &\leq \frac{\varepsilon}{n+2}, \\ &\vdots \\ f(j_1/(n+l_0)) - f(j_1/(n+l_0-1)) &\leq \frac{\varepsilon}{n+l_0}. \end{aligned}$$

Taking the sum of all these inequalities and then reasoning as in the previous case we obtain that

$$f(j_1/(n+l_0)) - f(j_1/n) \leq \frac{l_0 \varepsilon}{n+1}.$$

Now, reasoning again as in the previous case we obtain

$$0 \leq f(y_0) - f(x_0) \leq \frac{l_0 \varepsilon}{n+1} + 2\varepsilon \leq \frac{l_0 \varepsilon}{n} + 2\varepsilon \leq \varepsilon(2 + 1/(2y_0) - 1).$$

Again, we easily obtain that $f(x_0) = f(y_0)$ which implies that f is constant on $[a, b]$. Summarizing, we obtain that (A) holds.

(B) Let us choose arbitrary $a, b \in \mathbb{R}$ such that $1/2 < a < b < 1$. Further one, let x_0 and y_0 be the points where f attaints its minimum and maximum respectively on the interval $[a, b]$. Without any loss of generality we may suppose that $x_0 \neq y_0$ (contrariwise it follows that f is constant on $[a, b]$ and there is nothing to prove). We have two subcases: B_1) $x_0 < y_0$ and B_2) $x_0 > y_0$.

Subcase B_1) Let us choose arbitrary $n \in \mathbb{N}$, $n > \max\{n_0, n_1, 2/(y_0 - x_0)\}$. By relation (2.1) it follows that

$$L_{n+1}^{(M)}(f)((j/n) - f(j/n) \leq \frac{\varepsilon}{n+1} \text{ for all } j \in \{0, 1, ..., n\}.$$

Moreover, combining the inequality in Lemma 2.3 (ii) with the above inequality, we get

$$f((j+1)/(n+1)) - f(j/n)) \leq \frac{\varepsilon}{n+1} \text{ for all } j \in \{0, 1, ..., n\}, j \geq n/2. \tag{2.8}$$

Further one, let us choose $j_1 \in \{1, 2, ..., n\}$ such that $(j_1 - 1)/n \leq x_0 \leq j_1/n$ and $j_1/n \leq y_0$. Note that there exists such an index j_1, because the previous inequalities are equivalent to $nx_0 \leq j_1 \leq nx_0 + 1$, $nx_0 \leq j_1 \leq ny_0$, while the condition $n > 2/(y_0 - x_0)$ is equivalent to the condition $ny_0 - nx_0 > 2$.

Also, from $1/2 < x_0 \leq \frac{j_1}{n}$, it easily follows that $j_1 \geq n/2$.

As a first consequence, from relation (2.2) we obtain

$$|f(j_1/n) - f(x_0)| < \varepsilon. \tag{2.9}$$

Then, since $\lim\limits_{l\to\infty} \frac{j_1+l}{n+l} = 1$, by $y_0 < 1$ and $j_1/n \leq y_0$ it follows that there exists $l_0 \in \mathbb{N}$ such that $\frac{j_1+l_0}{n+l_0} \leq y_0 \leq \frac{j_1+l_0+1}{n+l_0+1}$.

It is worth noting here that the above l_0 cannot be equal to 0, because if we would have $l_0 = 0$ then we would obtain $(j_1 - 1)/n \leq x_0 \leq j_1/n \leq y_0 \leq (j_1+1)/(n+1) \leq (j_1+1)/n$, which would imply $y_0 - x_0 \leq 2/n$, in contradiction with the supposition that $n > 2/(y_0 - x_0)$.

The inequality $\frac{j_1+l_0}{n+l_0} \leq y_0 \leq \frac{j_1+l_0+1}{n+l_0+1}$ and (2.1) also implies that

$$|f((j_1 + l_0)/(n + l_0)) - f(y_0)| < \varepsilon. \tag{2.10}$$

Since by $j_1 \geq n/2$ it is very easy to verify that for $l \in \{0, 1, ..., l_0\}$ we have $j_1 + l \geq (n + l)/2$, applying successively relation (2.8) we obtain

$$\begin{aligned}
f((j_1 + l_0)/(n + l_0)) - f((j_1 + l_0 - 1)/(n + l_0 - 1))) &\leq \frac{\varepsilon}{n+l_0}, \\
f((j_1 + l_0 - 1)/(n + l_0 - 1)) - f((j_1 + l_0 - 2)/(n + l_0 - 2))) &\leq \frac{\varepsilon}{n+l_0-1} \\
&\vdots \\
f((j_1 + 1)/(n + 1)) - f(j_1/n) &\leq \frac{\varepsilon}{n+1}.
\end{aligned}$$

Taking the sum of all these inequalities and then reasoning as in the previous cases we obtain that

$$f((j_1 + l_0)/(n + l_0)) - f(j_1/n) \leq \frac{l_0\varepsilon}{n+1},$$

and then

$$0 \leq f(y_0) - f(x_0) \leq \frac{l_0\varepsilon}{n+1} + 2\varepsilon. \tag{2.11}$$

On the other hand, by $\frac{j_1+l_0}{n+l_0} \le y_0$ it follows (note that $y_0 < 1$)

$$l_0 \le \frac{ny_0-j_1}{1-y_0} \le \frac{ny_0}{1-y_0}.$$

Using the above inequality in relation (2.11) we easily obtain $0 \le f(y_0) - f(x_0) \le \varepsilon\left(y_0/(1-y_0)+2\right)$. Now reasoning as in the subcase A$_1$) we obtain $f(x_0) = f(y_0)$ and we immediately conclude that f is constant on $[a, b]$.

Subcase B$_2$) Let us choose arbitrary $n \in \mathbb{N}$, $n > \max\{n_0, n_1, 2/(x_0 - y_0)\}$. By relation (2.1) it follows that

$$L_n^{(M)}(f)((j+1)/(n+1)) - f((j+1)/(n+1)) \le \frac{\varepsilon}{n} \text{ for all } j \in \{0, 1, ..., n\}.$$

Moreover, combining the inequality in Lemma 2.3 (i) with the above inequality, we get

$$f(j/n) - f((j+1)/(n+1)) \le \frac{\varepsilon}{n} \text{ for all } j \in \{0, 1, ..., n\}, j \ge n/2. \tag{2.12}$$

Let j_1 and l_0 be chosen as in the previous case, with the difference that now we have $(j_1-1)/n \le y_0 \le j_1/n$ and $\frac{j_1+l_0}{n+l_0} \le x_0 \le \frac{j_1+l_0+1}{n+l_0+1}$. Applying successively the above inequality (2.12) we get

$$\begin{aligned} f((j_1+l_0-1)/(n+l_0-1))) - f((j_1+l_0)/(n+l_0)) &\le \frac{\varepsilon}{n+l_0-1}, \\ f((j_1+l_0-2)/(n+l_0-2))) - f((j_1+l_0-1)/(n+l_0-1)) &\le \frac{\varepsilon}{n+l_0-2}, \\ &\vdots \\ f(j_1/n) - f((j_1+1)/(n+1)) &\le \frac{\varepsilon}{n}. \end{aligned}$$

Taking the sum of all these inequalities and then reasoning as in the previous case we obtain that

$$f(j_1/n) - f((j_1+l_0)/(n+l_0)) \le \frac{l_0\varepsilon}{n}.$$

Now, reasoning again as in the previous case we obtain

$$0 \le f(y_0) - f(x_0) \le \frac{l_0\varepsilon}{n} + 2\varepsilon$$

and since by the same method like in the previous case we have $l_0 \le \frac{nx_0}{1-x_0}$, we easily obtain $0 \le f(y_0) - f(x_0) \le \varepsilon\left(x_0/(1-x_0)+2\right)$. This easily implies that $f(x_0) = f(y_0)$, which means that f is constant on $[a, b]$. Summarizing, we obtain that (B) holds.

Now, by the discussion just before the beginning of the case (A), we conclude that f is constant on the whole interval $[0, 1]$.

At the end, we discuss now the general case when the Lagrange max-prod operator is attached to functions defined on an interval $[a, b]$ with $a < b$. To make distinction between the general case and the particular case of the interval $[0, 1]$ in what follows we denote with $\overline{L}_n^{(M)}$ the Lagrange max-product operator attached to functions defined on the interval $[0, 1]$. In addition, in what follows, for all all $n \in \mathbb{N}$ and $k \in \{0, 1, ..., n\}$ we denote with $l_{n,k}^1$ the fundamental Lagrange polynomials defined on the interval.$[0, 1]$. Suppose now that for a function $f \in C([a, b])$ we have $\|L_n^{(M)}(f) - f\| = o(1/n)$. Let us define the function $g : [0,1] \to [a,b]$, $g(y) = a + (b-a)y$. It is immediate that for any $x \in [a,b]$ there exists an unique $y(x) \in [0,1]$ such that $f(x) = (f \circ g)(y(x))$. Then we observe that for any $x \in [a,b]$ we have

$$l_{n,k}(x) = (b-a)^n \cdot l^1_{n,k}(y(x)),\ n \in \mathbb{N},\ k \in \{0,1,...,n\}.$$

The above equalities imply

$$\begin{aligned} L_n^{(M)}(f)(x) &= \frac{\bigvee_{k=0}^{n} l_{n,k}(x) f\big(x_{n,k}\big)}{\bigvee_{k=0}^{n} l_{n,k}(x)} = \frac{(b-a)^n \cdot \bigvee_{k=0}^{n} l^1_{n,k}(y(x))(f \circ g)\Big(\frac{k}{n}\Big)}{(b-a)^n \cdot \bigvee_{k=0}^{n} l^1_{n,k}(y(x))} \\ &= \overline{L}_n^{(M)}(f \circ g)(y(x)). \end{aligned}$$

for all $x \in [a,b]$. This last formula together with the previous relation

$$\|L_n^{(M)}(f) - f\| = o(1/n),$$

easily implies that

$$\|\overline{L}_n^{(M)}(f \circ g)(y(x)) - (f \circ g)(y(x))\| \le \|L_n^{(M)}(f) - f\| = o(1/n)$$

for all $x \in [a,b]$ which now easily implies that $\|\overline{L}_n^{(M)}(f \circ g) - (f \circ g)\| = o(1/n)$. Consequently, we can apply the conclusion of the particular case considered at the beginning of the proof and we thus conclude that $f \circ g$ is a constant function. This easily implies that f is a constant function and now the proof is complete. □

Remark 2.5. Because it is easy to check that $L_n^{(M)}$ reproduces the constant functions in $C_+[a,b]$, it follows that the special saturation class in $C_+[a,b]$ for $L_n^{(M)}$ is exactly the class of positive constant functions.

Note that in fact Theorem 2.4 holds for any $f \in C[a,b] = \{f : [a,b] \to \mathbb{R}; f$ continuous on $[a,b]\}$. We have considered $f \in C_+[a,b]$ only because the Jackson-type estimate in the approximation of f by $L_n^{(M)}(f)$ (mentioned in Introduction) holds for all $f \in C_+[a,b]$. □

3. LOCAL INVERSE RESULT

According to Corollary 3.2, (i) in [13], the saturation order $\frac{1}{n}$ in the above Theorem 2.4 is attained for positive Lipschitz functions on $[a,b]$.

Conversely, we can present the following local inverse result.

Theorem 3.1. *Let $f : [a,b] \to [0,+\infty)$ and $a < \alpha < \beta < b$ be such that f is continuous on $[\alpha,\beta]$. If there exists a constant $M > 0$ (independent of n but depending on f, α and β) such that*

$$\|L_n^{(M)}(f) - f\|_{[\alpha,\beta]} \le M/n, \text{ for all } n \in \mathbb{N},$$

then $f|_{[\alpha,\beta]} \in \mathrm{Lip}\,[\alpha,\beta]$, that is f is a Lipschitz function on $[\alpha,\beta]$. Here $\|f\|_{[\alpha,\beta]} = \sup\{|f(x)|; x \in [\alpha,\beta]\}$ and

$$\mathrm{Lip}\,[\alpha,\beta] = \{g : [\alpha,\beta] \to \mathbb{R}; |g(x) - g(y)| \le C|x-y|, \text{ for all } x,y \in [\alpha,\beta]\}.$$

The proof of Theorem 3.1 requires the following three lemmas.

LEMMA 3.2. *Let $f:[0,1]\to\mathbb{R}$, $n\in\mathbb{N}$ and $0<\alpha<\beta\le 1/2$ be such that f is continuous on $[\alpha,\beta]$. Also, denote*

$$M_n(\alpha,\beta)=\max\left\{\left|f\left(\tfrac{k}{n}\right)-f\left(\tfrac{k}{n+1}\right)\right|:\ k\in\{0,...,n\},\alpha\le\tfrac{k}{n+1}\le\tfrac{k}{n}\le\beta\right\}.$$

Then

$$\limsup_{n\to\infty} n\cdot\omega_1(f,/n)_{[\alpha,\beta]}=\infty \quad \textit{if and only if} \quad \limsup_{n\to\infty} n\cdot M_n(\alpha,\beta)=\infty.$$

where

$$\omega_1(f,\delta)_{[\alpha,\beta]}=\sup\{|f(x)-f(y)|;x,y\in[\alpha,\beta],|x-y|\le\delta\}.$$

Proof. We prove only the direct implication since the converse one is immediate. Since f is continuous on the interval $[\alpha,\beta]$, it easily follows that for each $n\in\mathbb{N}$, $n\ge 2$, $1/n\le\beta-\alpha$, there exist $x_n,y_n\in[\alpha,\beta]$ satisfying $|x_n-y_n|\le 1/n$ and $\omega_1(f,1/n)_{[\alpha,\beta]}=|f(x_n)-f(y_n)|$. Clearly that by hypothesis and without any loss of generality, we may suppose that $x_n\ne y_n$ and $x_n<y_n$, for all $n\in\mathbb{N}$.

Let us consider the sequences $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$, $a_n=n\omega_1(f,1/n)_{[\alpha,\beta]}=n\,|f(x_n)-f(y_n)|$ and $b_n=n\cdot M_n(\alpha,\beta)$.

Let us fix $n\in\mathbb{N}$. Since f is uniformly continuous on $[\alpha,\beta]$, it follows that there exists $m\in\mathbb{N}$ such that for all $x,y\in[0,1]$ satisfying $|x-y|\le 1/m$ we have $|f(x)-f(y)|\le 1/n$. In addition, we may choose sufficiently large $m\in\mathbb{N}$ such that $y_n-x_n>2/m$, that is $m>2/(y_n-x_n)$.

Since $0<\alpha<y_n\le\beta<1/2$, clearly there exists $j\in\{1,...,m-1\}$ (depending on m and n) such that $j/m\le y_n\le(j+1)/m$.

Since $\lim_{l\to\infty} j/(m+l)=0$ and since $x_n\ge\alpha>0$, it results the existence of $l_0\in\mathbb{N}$ (depending on j and m) such that $j/(m+l_0+1)\le x_n\le j/(m+l_0)$.

By the inequalities $x_n\le j/(m+l_0)<j/m\le y_n$, we get

$$\begin{aligned}
&|f(x_n)-f(y_n)|\le\\
&\le|f(x_n)-f(j/(m+l_0))|+|f(j/(m+l_0))-f(j/(m+l_0-1))|\\
&\quad+...+|f(j/(m+1))-f(j/m)|+|f(j/m)-f(y_n)|\\
&\le|f(x_n)-f(j/(m+l_0))|+|f(j/m)-f(y_n)|\\
&\quad+l_0\,|f(j/(m+p+1))-f(j/(m+p))|
\end{aligned}$$

where $p\in\{0,1,...,l_0\}$ is such that

$$\begin{aligned}
&|f(j/(m+p+1))-f(j/(m+p))|=\\
&=\max\{|f(j/(m+k))-f(j/(m+k+1))|:\ k\in\{0,1,...,l_0-1\}\}.
\end{aligned}$$

On the other hand, we observe that $\max\{|j/(m+l_0)-x_n|,|j/m-y_n|\}\le 1/m$, which implies $|f(x_n)-f(j/(m+l_0))|\le 1/n$ and $|f(j/m)-f(y_n)|\le 1/n$. We thus obtain that

$$|f(x_n)-f(y_n)|\le 2/n+l_0\,|f(j/(m+p))-f(j/(m+p+1))|. \tag{3.1}$$

By the inequalities $x_n \leq j/(m+l_0) \leq j/m \leq y_n$ we get $j/m - j/(m+l_0) \leq y_n - x_n \leq 1/n$ and this implies $jl_0/(m(m+l_0)) \leq 1/n$ and then $l_0 \leq m/j \cdot (m+l_0)/n \leq 1/\alpha \cdot (m+l_0)/n$. (Here we used that $\alpha \leq x_n < j/m$).

Then, by the inequalities $0 < \alpha \leq x_n \leq A := j/(m+l_0) \leq B := j/m \leq y_n \leq \beta$ we easily get $B/A \leq \beta/\alpha$, which immediately implies $j/(m+l_0) \geq j/m \cdot \alpha/\beta$. From here we get $m + l_0 \leq m\beta/\alpha$, that is $l_0 \leq m(\beta/\alpha - 1)$. Replacing this last inequality in the inequality $l_0 \leq 1/\alpha \cdot (m+l_0)/n$ just proved above, we conclude that $l_0 \leq \beta/\alpha^2 \cdot m/n$.

Replacing now in relation (3.1) and then multiplying with n, we get

$$n \cdot |f(x_n) - f(y_n)| \leq 2 + \beta/\alpha^2 \cdot m \cdot |f(j/(m+p)) - f(j/(m+p+1))| \leq$$
$$\leq 2 + \beta/\alpha^2 \cdot (m+p) \cdot |f(j/(m+p)) - f(j/(m+p+1))|$$

and clearly this implies that $a_n \leq 2 + \beta/\alpha^2 \cdot M_{m+p}(\alpha, \beta)$. Summarizing, for any $n \in \mathbb{N}$ there exist $m + p \in \mathbb{N}$ such that $a_n \leq \beta/\alpha^2 \cdot b_{m+p} + 2$. Since $m > 2/(y_n - x_n)$ and $y_n - x_n < 1/n$, we get $m > 2n$. Therefore, by $\limsup\limits_{n\to\infty} a_n = \infty$, it easily follows that $\limsup\limits_{n\to\infty} b_n = \infty$ and the lemma is proved. $\square$

In an absolutely similar manner we obtain the following.

Lemma 3.3. *Let $f : [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$ and $1/2 \leq \alpha < \beta < 1$ be such that f is continuous on $[\alpha, \beta]$. Also, denote*

$$P_n(\alpha, \beta) = \max\left\{\left|f\left(\tfrac{k}{n}\right) - f\left(\tfrac{k+1}{n+1}\right)\right| : \ k \in \{0, ..., n\}, \alpha \leq \tfrac{k}{n} \leq \tfrac{k+1}{n+1} \leq \beta\right\}.$$

Then

$$\limsup_{n\to\infty} n \cdot \omega_1(f, 1/n)_{[\alpha,\beta]} = \infty \quad \textit{if and only if} \quad \limsup_{n\to\infty} n \cdot P_n(\alpha, \beta) = \infty.$$

where

$$\omega_1(f, \delta)_{[\alpha,\beta]} = \sup\{|f(x) - f(y)|; \ x, y \in [\alpha, \beta], |x - y| \leq \delta\}.$$

Also, we can prove:

Lemma 3.4. *Let $f : [0,1] \to [0, \infty)$ and $0 < \alpha < \beta < 1$ be such that f is continuous on $[\alpha, \beta]$. If*

$$\limsup_{n\to\infty} n \cdot \omega_1(f, 1/n)_{[\alpha,\beta]} = \infty,$$

then

$$\limsup_{n\to\infty} n \cdot \left\|L_n^{(M)}(f) - f\right\|_{[\alpha,\beta]} = \infty.$$

Here $\|f\|_{[\alpha,\beta]} = \sup\{|f(x)|; \ x \in [\alpha, \beta]\}$.

Proof. If $\alpha < 1/2 < \beta$ then by the hypothesis it is elementary to prove that either $\limsup\limits_{n\to\infty} n \cdot \omega_1(f, 1/n)_{[\alpha,1/2]} = \infty$ or $\limsup\limits_{n\to\infty} n \cdot \omega_1(f, 1/n)_{[1/2,\beta]} = \infty$. Therefore, without any loss of generality we may suppose that we have only two cases: (i) $0 < \alpha < \beta \leq 1/2$ and (ii) $1/2 \leq \alpha < \beta < 1$.

Case (i) For fixed $n \in \mathbb{N}$ with $n \geq 1/(\beta-\alpha)$, let us choose $k(n) \in \{1, ..., n\}$ such that $\alpha \leq \frac{k(n)}{n+1} \leq \frac{k(n)}{n} \leq \beta$ and

$$M_n(\alpha,\beta) = \left| f\left(\frac{k(n)}{n}\right) - f\left(\frac{k(n)}{n+1}\right)\right|.$$

Note that such an index $k(n)$ exists, because the inequalities $\alpha \leq k(n)/(n+1) \leq k(n)/n \leq \beta$ imply $\alpha(n+1) \leq k(n) \leq \beta n$, where $\beta n - \alpha(n+1) \geq 1$.

Since $\beta \leq 1/2$, it results that $k(n) \leq n/2$ and hence we can use the conclusion of Lemma 2.2. This means that we have

$$L_n^{(M)}(f)(k(n)/(n+1)) \geq f(k(n)/n)$$

and

$$L_{n+1}^{(M)}(f)(k(n)/n) \geq f(k(n)/(n+1)).$$

If $f\,(k(n)/n) \geq f(k(n)/(n+1))$ then

$$\begin{aligned} &n \cdot \left(L_n^{(M)}(f)(k(n)/(n+1)) - f(k(n)/(n+1))\right) \\ &\geq n \cdot (f(k(n)/n) - f(k(n)/(n+1))) = n \cdot M_n(\alpha,\beta) \end{aligned}$$

and this implies

$$n \cdot M_n(\alpha,\beta) \leq n \cdot \left\| L_n^{(M)}(f) - f \right\|_{[\alpha,\beta]}.$$

If $f\,(k(n)/n) < f(k(n)/(n+1))$ then

$$\begin{aligned} &(n+1) \cdot \left(L_{n+1}^{(M)}(f)(k(n)/n) - f\,(k(n)/n)\right) \\ &\geq (n+1) \cdot (f(k(n)/(n+1)) - f\,(k(n)/n)) \geq n \cdot M_n(\alpha,\beta). \end{aligned}$$

and this implies

$$n \cdot M_n(\alpha,\beta) \leq (n+1) \cdot \left\| L_{n+1}^{(M)}(f) - f \right\|_{[\alpha,\beta]}.$$

In conclusion, for any $n \in \mathbb{N}$ with $n \geq 1/(\beta-\alpha)$, we have

$$n \cdot M_n(\alpha,\beta) \leq \max\left\{ n \cdot \left\| L_n^{(M)}(f) - f \right\|_{[\alpha,\beta]}, (n+1) \cdot \left\| L_{n+1}^{(M)}(f) - f \right\|_{[\alpha,\beta]} \right\}.$$

Since by Lemma 3.2 we have $\limsup_{n\to\infty} n \cdot M_n(\alpha,\beta) = \infty$, it easily follows now that $\limsup_{n\to\infty} n \cdot \left\| L_n^{(M)}(f) - f \right\|_{[\alpha,\beta]} = \infty$.

Case (ii) The proof is similar with that of the Case (i), which proves the lemma. □

Now we are in position to prove Theorem 3.1.

Proof of Theorem 3.1. Using the same type of reasoning as in the proof of Theorem 2.4 it suffices to deal only with the particular case when $a = 0$ and $b = 1$. Firstly we prove that f is a Lipschitz function on $[\alpha,\beta]$ if and only if $\limsup_{n\to\infty} n \cdot \omega_1(f, 1/n)_{[\alpha,\beta]} < \infty$. Indeed, if f is a Lipschitz function on $[\alpha,\beta]$ then evidently that there exists $M > 0$ such that we have $n \cdot \omega_1(f, 1/n)_{[\alpha,\beta]} \leq M$, which implies $\limsup_{n\to\infty} n \cdot \omega_1(f, 1/n)_{[\alpha,\beta]} \leq M < \infty$.

Conversely, $\limsup_{n\to\infty} n \cdot \omega_1(f, 1/n)_{[\alpha,\beta]} \leq M < \infty$ implies $\omega_1(f, 1/n)_{[\alpha,\beta]} \leq \frac{M}{n}$, for all $n \in \mathbb{N}$. For arbitrary $h \in (0, 1)$, let $n \in \mathbb{N}$ be such that $\frac{1}{n+1} \leq h \leq \frac{1}{n}$. It follows $\omega_1(f; h)_{[\alpha,\beta]} \leq \omega_1(f, 1/n)_{[\alpha,\beta]} \leq \frac{M}{n} \leq \frac{2M}{n+1} \leq 2Mh$, that is $\omega_1(f; h)_{[\alpha,\beta]} \leq 2Mh$, for all $h \in [0, 1]$, which obviously is equivalent with the fact that f is a Lipschitz function on $[\alpha, \beta]$ (indeed, for fixed $x, y \in [\alpha, \beta]$ we have $|f(x) - f(y)| \leq \omega_1(f; |x - y|)_{[\alpha,\beta]} \leq 2M|x - y|$).

Now, by the hypothesis it follows $n \cdot \|L_n^{(M)}(f) - f\|_{[\alpha,\beta]} \leq M$, for all $n \in \mathbb{N}$. Supposing that f is not a Lipschitz function on $[\alpha, \beta]$, by the above considerations it follows that $\limsup_{n\to\infty} n \cdot \omega_1(f, 1/n)_{[\alpha,\beta]} = \infty$. But then by Lemma 3.4 we get

$$\limsup_{n\to\infty} n \cdot \left\| L_n^{(M)}(f) - f \right\|_{[\alpha,\beta]} = \infty,$$

which is a contradiction. The theorem is proved. □

REFERENCES

[1] B. Bede and S.G. Gal, *Approximation by nonlinear Bernstein and Favard-Szász-Mirakjan operators of max-product kind*, J. Concrete and Applicable Mathematics, **8**, no. 2, pp. 193–207, 2010.

[2] B. Bede, L. Coroianu and S.G. Gal, *Approximation and shape preserving properties of the Bernstein operator of max-product kind*, Intern. J. Math. Math. Sci., **2009**, Article ID 590589, 26 pages, doi:10.1155/2009/590589.

[3] B. Bede, L. Coroianu and S.G. Gal, *Approximation by truncated Favard-Szász-Mirakjan operator of max-product kind*, Demonstratio Mathematica, **XLIV**, no. 1, pp. 105–122, 2011.

[4] B. Bede, L. Coroianu and S.G. Gal, *Approximation and shape preserving properties of the nonlinear Bleimann-Butzer-Hahn operators of max-product kind*, Comment. Math. Univ. Carol., **51**, no. 3, pp. 397–415, 2010.

[5] B. Bede, L. Coroianu and S.G. Gal *Approximation and shape preserving properties of the nonlinear Meyer-Konig and Zeller operator of max-product kind*, Numerical Functional Analysis and Optimization, **31**, no. 3, pp. 232–253, 2010.

[6] B. Bede, L. Coroianu and S.G. Gal, *Approximation and shape preserving properties of the truncated Baskakov operator of max-product kind*, Revista de la Union Matematica Argentina, **52**, no. 1, pp. 89–107, 2011.

[7] B. Bede, L. Coroianu and S.G. Gal, *Approximation and shape preserving properties of the nonlinear Baskakov operator of max-product kind*, Studia Univ. Babeş-Bolyai, ser. Math., **LV**, pp. 193–218, 2010.

[8] S. Bernstein, *Quelques remarques sur l'interpolation*, Math. Ann., **79**, no.1-2, pp. 1–12, 1918.

[9] E. Borel, *Sur l'interpolation*, C.R. Acad. Sci. Paris, **124**, pp. 673–676, 1897.

[10] S. Cobzas and I. Muntean, *Condensation of singularities and divergence results in approximation theory*, J. Approx. Theory, **31**, no. 2, pp. 138–153, 1980.

[11] L. Coroianu and S.G. Gal, *Approximation by nonlinear Lagrange interpolation operators of max-product kind on Chebyshev knots of second kind*, J. Comp. Anal. Appl., **13**, no. 2, pp. 211–224, 2010.

[12] L. Coroianu and S.G. Gal, *Approximation by nonlinear Hermite-Fejér interpolation operators of max-product kind on Chebyshev nodes*, Revue d'Anal. Numér. Théor. Approx. (Cluj), **39**, no.1, pp. 29–39, 2010.

[13] L. Coroianu and S.G. Gal, *Approximation by max-product Lagrange interpolation operators*, Studia Univ. "Babeş-Bolyai", ser. Math., **LVI**, no. 2, pp. 1–11, 2011.

[14] L. Coroianu and S.G. Gal, *Classes of functions with improved estimates in approximation by the max-product Bernstein operator*, Analysis and Applications, **9**, no. 3, pp. 249–274, 2011.

[15] S.G. Gal, *Shape-Preserving Approximation by Real and Complex Polynomials*, Birkhäuser, Boston-Basel-Berlin, 2008.

[16] I. Muntean, *The Lagrange interpolation operators are densely divergent*, Studia Univ. "Babes-Bolyai" (Cluj), ser. math. **21**, pp. 28–30, 1976.

[17] J. Szabados and P. Vértesi, *Interpolation of Functions*, World Scientific, Singapore, New Jersey, London, Hong Kong, 1990.

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