REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 41 (2012) no. 1, pp. 48–61
ictp.acad.ro/jnaat

MINMAX FRACTIONAL PROGRAMMING PROBLEM INVOLVING GENERALIZED CONVEX FUNCTIONS

ANURAG JAYSWAL *, I.M. STANCU-MINASIAN† and DILIP KUMAR‡

Abstract. In the present study we focus our attention on a minmax fractional programming problem and its second order dual problem. Duality results are obtained for the considered dual problem under the assumptions of second order (F, α, ρ, d)-type I functions.

MSC 2000. 26A51; 49J35; 90C32.
Keywords. Minmax fractional programming, (F, α, ρ, d)-type I functions; second order duality.

1. INTRODUCTION

We consider the following minmax fractional programming problem:

$$\text{(P)} \qquad \text{Minimize } \psi(x) = \sup_{y \in Y} \frac{f(x,y)}{h(x,y)}$$

subject to

$$g(x) \leq 0, \; x \in \mathbb{R}^n,$$

where Y is a compact subset of $\mathbb{R}^l$, $f(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}$, $h(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}$ are C^2 mappings on $\mathbb{R}^n \times \mathbb{R}^l$ and $g(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is C^2 mapping on $\mathbb{R}^n$. It is assumed that for each (x, y) in $\mathbb{R}^n \times \mathbb{R}^l$, $f(x,y) \geq 0$ and $h(x,y) > 0$.

In recent years, optimality conditions and duality for generalized minmax fractional programming have received much attention by many authors (see, for example, [1, 3, 8, 10–12, 14–17]). In particular, Crouzeix *et al.* [5] showed that the minmax fractional programming reduces to solving a minmax nonlinear parametric programming. In [3], Bector *et al.* used a parametric approach

*Department of Applied Mathematics, Indian School of Mines, Dhanbad-826004, Jharkhand, India, e-mail: anurag_jais123@yahoo.com.
The research of this author was partially supported by the Indian School of Mines, Dhanbad under FRS(17)/2010-2011/AM.
†Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, 13 Septembrie Street, No. 13, 050711 Bucharest, Romania, e-mail: stancu_minasian@yahoo.com.
‡Department of Applied Mathematics, Birla Institute of Technology, Mesra, Ranchi-835215, Jharkhand, India.

to obtain duality for the generalized minmax fractional programming involving differentiable pseudoconvex and quasiconvex functions.

Mangasarian [13] first formulated the second order dual for a nonlinear programming problem. Hanson [7] established second order duality theorems for nonlinear mathematical programming problem under defined second order type-I functions.

Zhang and Mond [18] introduced the concept of second order (F,ρ)-convexity and obtained some duality results concerning with nonlinear multiobjective programming problems. Ahmad and Husain [1] extended (F,α,ρ,d)-convex functions which were introduced by Liang *et al.* [9] to second order (F,α,ρ,d)-convex functions. Hachimi and Aghezzaf [6] further extended it to second order (F,α,ρ,d)-type I functions.

Husain *et al.* [8] established duality theorems for two types of second order dual models related to minmax fractional programming problem (P) under the assumptions of η-bonvexity/generalized η-bonvexity.

Motivated by the earlier works and importance of the second order generalized convexity, in this paper we establish the second order duality theorems for the dual problem related to minmax fractional programming problem (P) under the assumption of generalized second order (F,α,ρ,d)-type I functions.

The paper is organized as follows. Some definitions and notation are given in Section 2. Under the assumptions of generalized second order (F,α,ρ,d)-type I functions, second order weak, strong and strict converse duality theorems related to problem (P) are given in Section 3. Concluding remarks are presented in Section 4.

2. NOTATION AND PRELIMINARIES

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}^n_+$ its non-negative orthant. Let X be a nonempty open subset of $\mathbb{R}^n$. For $x,\, y \in \mathbb{R}^n$, we let $x \leq y \Leftrightarrow y - x \in \mathbb{R}^n_+$; $x < y \Leftrightarrow y - x \in \mathbb{R}^n_+ \setminus \{0\}$.

Throughout this paper, we denote by $S = \{x \in X : g(x) \leq 0\}$ the set of all feasible solutions of problem (P). For each $(x,y) \in \mathbb{R}^n \times \mathbb{R}^l$, we define

$$J(x) = \{j \in M = \{1,2,...,m\} : \; g_j(x) = 0\},$$

$$Y(x) = \left\{ y \in Y : \; f(x,y)/h(x,y) = \sup_{z\in Y} f(x,z)/h(x,z) \right\},$$

and $K(x) = \big\{(s,t,\bar{y}) \in \mathbb{N} \times \mathbb{R}^s_+ \times \mathbb{R}^{ls} : \; 1 \leq s \leq n+1,\, t = (t_1,t_2,...,t_s) \in \mathbb{R}^s_+$ with

$$\sum_{i=1}^{s} t_i = 1, \bar{y} = (\bar{y}_1, \bar{y}_2 ..., \bar{y}_s), \text{ with } \bar{y}_i \in Y(x), i = 1,2,...,s\}.$$

Definition 2.1. *A functional $F : X \times X \times \mathbb{R}^n \to \mathbb{R}$ is said to be sublinear in its third argument, if for any $x, \bar{x} \in X$,*

(i) $F(x,\bar{x};a_1+a_2) \leq F(x,\bar{x};a_1) + F(x,\bar{x};a_2) \;\; \forall\, a_1,\, a_2 \in \mathbb{R}^n$;

(ii) $F(x, \bar{x}; \alpha\, a) = \alpha\, F(x, \bar{x}; a) \ \ \forall \alpha \in \mathbb{R}_+, \forall\, a \in \mathbb{R}^n.$

By (ii) it is clear that $F(x, \bar{x};\, 0\,) = 0$.

Now, we let F be a sublinear functional and $d(\cdot, \cdot) : X \times X \to \mathbb{R}$. Let $\alpha = (\alpha^1, \alpha^2)$, where α^1, $\alpha^2 : X \times X \to \mathbb{R}_+ \setminus \{0\}$, $\rho = (\rho^1, \rho^2)$, where $\rho^1 = (\rho_1^1, \rho_2^1, ..., \rho_s^1) \in \mathbb{R}^s$ and $\rho^2 = (\rho_1^2, \rho_2^2, ..., \rho_m^2) \in \mathbb{R}^m$. Let $f(\cdot, \cdot) : X \times Y(x) \to \mathbb{R}$ and $g(\cdot) : X \to \mathbb{R}^m$ be two twice differentiable functions.

Definition 2.2. [2] *For each* $j \in M$, (f, g_j) *is said to be second-order* (F, α, ρ, d)*-type I at* $\bar{x} \in X$ *if for all* $x \in S$ *and* $y_i \in Y(x)$, *we have*

$$f(x, y_i) - f(\bar{x}, y_i) + \tfrac{1}{2} p^T \nabla^2 f(\bar{x}, y_i)\, p \geq$$

$$\geq F\left(x, \bar{x}; \alpha^1(x, \bar{x}) \left[\nabla f(\bar{x}, y_i) + \nabla^2 f(\bar{x}, y_i)\, p\right]\right) + \rho_i^1 d^2(x, \bar{x}),\ i = 1, 2, ..., s,$$

$$- g_j(\bar{x}) + \tfrac{1}{2} p^T \nabla^2 g_j(\bar{x})\, p \geq$$

$$\geq F\left(x, \bar{x}; \alpha^2(x, \bar{x}) \left[\nabla g_j(\bar{x}) + \nabla^2 g_j(\bar{x})\, p\right]\right) + \rho_j^2 d^2(x, \bar{x}),\ \ j = 1, 2, ..., m,$$

where $p \in \mathbb{R}^n$.

If the first inequality in the above definition is satisfied under the form

$$f(x, y_i) - f(\bar{x}, y_i) + \tfrac{1}{2} p^T \nabla^2 f(\bar{x}, y_i)\, p >$$
$$> F\left(x, \bar{x}; \alpha^1(x, \bar{x}) \left[\nabla f(\bar{x}, y_i) + \nabla^2 f(\bar{x}, y_i)\, p\right]\right) + \rho_i^1 d^2(x, \bar{x}),\ \ i = 1, 2, ..., s,$$

then we say that for each $j \in M, (f, g_j)$ is *second-order strictly* (F, α, ρ, d)*-type I at* $\bar{x}$.

Definition 2.3. [2] *For each* $j \in M$, (f, g_j) *is said to be second-order pseudoquasi* (F, α, ρ, d)*-type I at* $\bar{x} \in X$ *if for all* $x \in S$ *and* $y_i \in Y(x)$, *we have*

$$f(x, y_i) < f(\bar{x}, y_i) - \tfrac{1}{2} p^T \nabla^2 f(\bar{x}, y_i)\, p$$
$$\Rightarrow F\left(x, \bar{x}; \alpha^1(x, \bar{x}) \left[\nabla f(\bar{x}, y_i) + \nabla^2 f(\bar{x}, y_i)\, p\right]\right) < -\rho_i^1 d^2(x, \bar{x}),\ \ i = 1, 2, ..., s,$$
$$-g_j(\bar{x}) + \tfrac{1}{2} p^T \nabla^2 g_j(\bar{x})\, p \ \leq\ 0$$
$$\Rightarrow\ F\left(x, \bar{x}; \alpha^2(x, \bar{x}) \left[\nabla g_j(\bar{x}) + \nabla^2 g_j(\bar{x})\, p\right]\right) \leq -\rho_j^2 d^2(x, \bar{x}),\ \ j = 1, 2, ..., m,$$

where $p \in \mathbb{R}^n$.

If the first implication in the above definition is satisfied under the form

$$F\left(x, \bar{x}; \alpha^1(x, \bar{x}) \left[\nabla f(\bar{x}, y_i) + \nabla^2 f(\bar{x}, y_i)\, p\right]\right) \geq -\rho_i^1 d^2(x, \bar{x}),$$

$$\Rightarrow f(x, y_i) > f(\bar{x}, y_i) - \tfrac{1}{2} p^T \nabla^2 f(\bar{x}, y_i)\, p,\ \ i = 1, 2, ..., s,$$

then we say that for each $j \in M$, (f, g_j) *is second-order strictly pseudoquasi* (F, α, ρ, d)*-type* I *at* $\bar{x}$.

The following result will be needed in the sequel.

THEOREM 2.1. [4] *Let x^* be a solution of problem (P) and let $\nabla g_j(x^*), j \in J(x^*)$, be linearly independent. Then there exist $(s^*, t^*, \bar{y}^*) \in K(x^*)$, $\lambda^* \in \mathbb{R}_+$, and $\mu^* \in \mathbb{R}_+^m$ such that*

$$\nabla \sum_{i=1}^{s^*} t_i^* \left(f(x^*, \bar{y}_i^*) - \lambda^* h(x^*, \bar{y}_i^*)\right) + \nabla \sum_{j=1}^{m} \mu_j^* g_j(x^*) = 0,$$

$$f(x^*, \bar{y}_i^*) - \lambda^* h(x^*, \bar{y}_i^*) = 0,\ i = 1, 2, ..., s^*,$$

$$\sum_{j=1}^{m} \mu_j^* g_j(x^*) = 0,$$

$$t_i^* \geq 0, \sum_{i=1}^{s^*} t_i^* = 1, \quad \bar{y}_i^* \in Y(x^*),\ i = 1, 2, ..., s^*.$$

3. DUALITY

In this section, we consider the following dual model [8] for (P).

$$\text{(MD)} \qquad \max_{(s,t,\bar{y}) \in K(z)} \ \sup_{(z,\mu,\lambda,p) \in H_1(s,t,\bar{y})} \lambda,$$

where $H_1(s, t, \bar{y})$ denotes the set of all $(z, \mu, \lambda, p) \in \mathbb{R}^n \times \mathbb{R}_+^m \times \mathbb{R}_+ \times \mathbb{R}^n$ satisfying

$$\nabla \sum_{i=1}^{s} \left(f(z, \bar{y}_i) - \lambda h(x, \bar{y}_i)\right) + \nabla^2 \sum_{i=1}^{s} t_i \left(f(z, \bar{y}_i) - \lambda h(x, \bar{y}_i)\right) p + + \nabla \sum_{j=1}^{m} \mu_j g_j(z) + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z) p = 0, \tag{3.1}$$

$$\sum_{i=1}^{s} t_i \left(f(z, \bar{y}_i) - \lambda h(z, \bar{y}_i)\right) - \tfrac{1}{2} p^T \nabla^2 \sum_{i=1}^{s} t_i \left(f(z, \bar{y}_i) - \lambda h(z, \bar{y}_i)\right) p \geq 0, \tag{3.2}$$

$$\sum_{j=1}^{m} \mu_j g_j(z) - \tfrac{1}{2} p^T \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z) p \geq 0. \tag{3.3}$$

If, for a triplet $(s, t, \bar{y}) \in K(z)$, the set $H_1(s, t, \bar{y})$ is empty, we define the supremum over it to be $-\infty$.

REMARK 3.1. If $p = 0$, then (MD) becomes the dual problem considered in [11]. □

THEOREM 3.1. *(Weak duality) Let x and $(z, \mu, \lambda, s, t, \bar{y}, p)$ be feasible solutions to* (P) *and* (MD)*, respectively. Assume that*

(i) $\left(\sum\limits_{i=1}^{s} t_i\left(f\left(\cdot,\bar{y}_i\right) \ -\lambda h\left(\cdot,\bar{y}_i\right)\right), \sum\limits_{j=1}^{m} \mu_j g_j\left(\cdot\right)\right)$ *is second order* (F,α,ρ,d)-*type I at* z,

(ii) $\frac{\rho_1^1}{\alpha^1(x,z)} + \frac{\rho_1^2}{\alpha^2(x,z)} \geq 0.$

Then

$$\sup_{y\in Y} \tfrac{f(x,y)}{h(x,y)} \geq \lambda.$$

Proof. Suppose contrary to the result that

$$\sup_{y\in Y} \tfrac{f(x,y)}{h(x,y)} < \lambda.$$

Therefore, we have

$$f\left(x,\bar{y}_i\right) - \lambda h\left(x,\bar{y}_i\right) < 0 \text{ for all } \bar{y}_i \in Y\left(x\right), \ \ i = 1,2,...,s.$$

It follows from $t_i \geq 0, \ i = 1,2,...,s$, that

$$t_i\left(f\left(x,\bar{y}_i\right) - \lambda h\left(x,\bar{y}_i\right)\right) \leq 0,$$

with at least one strict inequality, since $t = (t_1,t_2,...,t_s) \neq 0$. Taking summation over i, we have

$$\sum_{i=1}^{s} t_i\left(f\left(x,\bar{y}_i\right) - \lambda h\left(x,\bar{y}_i\right)\right) < 0,$$

which together with (3.2) gives

$$\sum_{i=1}^{s} t_i\left(f\left(x,\bar{y}_i\right) - \lambda h\left(x,\bar{y}_i\right)\right) < 0 \leq$$
$$\leq \sum_{i=1}^{s} t_i\left(f\left(z,\bar{y}_i\right) - \lambda h\left(z,\bar{y}_i\right)\right) - \tfrac{1}{2}p^T\nabla^2\sum_{i=1}^{s} t_i\left(f\left(z,\bar{y}_i\right) - \lambda h\left(z,\bar{y}_i\right)\right)p.$$

That is,

$$\sum_{i=1}^{s} t_i\left(f\left(x,\bar{y}_i\right) - \lambda h\left(x,\bar{y}_i\right)\right) - \sum_{i=1}^{s} t_i\left(f\left(z,\bar{y}_i\right) - \lambda h\left(z,\bar{y}_i\right)\right) + \tag{3.4}$$
$$+ \tfrac{1}{2}p^T\nabla^2\sum_{i=1}^{s} t_i\left(f\left(z,\bar{y}_i\right) - \lambda h\left(z,\bar{y}_i\right)\right)p < 0.$$

Using (3.3), (3.4) and hypothesis (i), we obtain

$$0 > \sum_{i=1}^{s} t_i \left(f\left(x, \bar{y}_i\right) - \lambda h\left(x, \bar{y}_i\right)\right) - \sum_{i=1}^{s} t_i \left(f\left(z, \bar{y}_i\right) - \lambda h\left(z, \bar{y}_i\right)\right)$$
$$+ \tfrac{1}{2} p^T \nabla^2 \sum_{i=1}^{s} t_i \left(f\left(z, \bar{y}_i\right) - \lambda h\left(z, \bar{y}_i\right)\right) p$$
$$\geq F\left(x,z;\alpha^1(x,z)\left(\nabla \sum_{i=1}^{s} t_i(f\left(z, \bar{y}_i\right) - \lambda h\left(z, \bar{y}_i\right)) + \nabla^2 \sum_{i=1}^{s} t_i\left(f\left(z, \bar{y}_i\right) - \lambda h\left(z, \bar{y}_i\right)\right)p\right)\right)$$
$$+ \rho_1^1 d^2\left(x, z\right),$$

and

$$0 \geq -\sum_{j=1}^{m} \mu_j g_j\left(z\right) + \tfrac{1}{2} p^T \nabla^2 \sum_{j=1}^{m} \mu_j g_j\left(z\right) p$$
$$\geq F\left(x, z; \alpha^2\left(x, z\right)\left(\nabla \sum_{j=1}^{m} \mu_j g_j\left(z\right) + \nabla^2 \sum_{j=1}^{m} \mu_j g_j\left(z\right) p\right)\right) + \rho_1^2 d^2\left(x, z\right).$$

Since $\alpha^1\left(x, z\right) > 0$ and $\alpha^2\left(x, z\right) > 0$, by using the sublinearity of F, the above two inequalities imply

$$F\left(x, z; \nabla \sum_{i=1}^{s} t_i \left(f\left(z, \bar{y}_i\right) - \lambda h\left(z, \bar{y}_i\right)\right) + \nabla^2 \sum_{i=1}^{s} t_i \left(f\left(z, \bar{y}_i\right) - \lambda h\left(z, \bar{y}_i\right)\right) p\right) < -\tfrac{\rho_1^1 d^2(x,z)}{\alpha^1(x,z)} \tag{3.5}$$

and

$$F\left(x, z; \nabla \sum_{j=1}^{m} \mu_j g_j\left(z\right) + \nabla^2 \sum_{j=1}^{m} \mu_j g_j\left(z\right) p\right) \leq -\tfrac{\rho_1^2 d^2(x,z)}{\alpha^2(x,z)}. \tag{3.6}$$

From (3.1), (3.5), (3.6) and the sublinearity of F, we get

$$0 = F\left(x, z; \nabla \sum_{i=1}^{s} t_i \left(f\left(z, \bar{y}_i\right) - \lambda h\left(z, \bar{y}_i\right)\right) + \nabla^2 \sum_{i=1}^{s} t_i \left(f\left(z, \bar{y}_i\right) - \lambda h\left(z, \bar{y}_i\right)\right) p\right.$$
$$\left. + \nabla \sum_{j=1}^{m} \mu_j g_j\left(z\right) + \nabla^2 \sum_{j=1}^{m} \mu_j g_j\left(z\right) p\right)$$

$$\leq F\left(x,z;\nabla\sum_{i=1}^{s}t_i\left(f\left(z,\bar{y}_i\right)-\lambda h\left(z,\bar{y}_i\right)\right)+\nabla^2\sum_{i=1}^{s}t_i\left(f\left(z,\bar{y}_i\right)-\lambda h\left(z,\bar{y}_i\right)\right)p\right)$$
$$+F\left(x,z;\nabla\sum_{j=1}^{m}\mu_j g_j\left(z\right)+\nabla^2\sum_{j=1}^{m}\mu_j g_j\left(z\right)p\right)$$
$$<-\left(\tfrac{\rho_1^1}{\alpha^1(x,z)}+\tfrac{\rho_1^2}{\alpha^2(x,z)}\right)d^2\left(x,z\right)<0.\ (\text{by(ii)})$$

Thus, we have a contradiction. Hence, the proof is complete. □

THEOREM 3.2. *(Weak duality) Let x and $(z,\mu,\lambda,s,t,\bar{y},p)$ be feasible solutions to (P) and (MD), respectively, Assume that*

(i) $\left(\sum_{i=1}^{s}t_i\left(f\left(\cdot,\bar{y}_i\right)-\lambda h\left(\cdot,\bar{y}_i\right)\right),\sum_{j=1}^{m}\mu_j g_j\left(\cdot\right)\right)$ *is second order pseudoquasi (F,α,ρ,d)-type I at z,*

(ii) $\frac{\rho_1^1}{\alpha^1(x,z)}+\frac{\rho_1^2}{\alpha^2(x,z)}\geq 0.$

Then

$$\sup_{y\in Y}\tfrac{f(x,y)}{h(x,y)}\geq\lambda.$$

Proof. Following the lines of proof of Theorem 3.1, we have:

$$\sum_{i=1}^{s}t_i\left(f\left(x,\bar{y}_i\right)-\lambda h\left(x,\bar{y}_i\right)\right)< \tag{3.7}$$
$$<\sum_{i=1}^{s}t_i\left(f\left(z,\bar{y}_i\right)-\lambda h\left(z,\bar{y}_i\right)\right)-\tfrac{1}{2}p^T\nabla^2\sum_{i=1}^{s}t_i\left(f\left(z,\bar{y}_i\right)-\lambda h\left(z,\bar{y}_i\right)\right)p.$$

Using (3.3), (3.7) and hypothesis (i), we obtain

$$F\left(x,z;\alpha^1(x,z)\left(\nabla\sum_{i=1}^{s}t_i\left(f\left(z,\bar{y}_i\right)-\lambda h\left(z,\bar{y}_i\right)\right)+\nabla^2\sum_{i=1}^{s}t_i\left(f\left(z,\bar{y}_i\right)-\lambda h\left(z,\bar{y}_i\right)\right)p\right)\right)$$
$$<-\rho_1^1 d^2\left(x,z\right)$$

and

$$F\left(x,z;\alpha^2\left(x,z\right)\left(\nabla\sum_{j=1}^{m}\mu_j g_j\left(z\right)+\nabla^2\sum_{j=1}^{m}\mu_j g_j\left(z\right)p\right)\right)\leq-\rho_1^2 d^2\left(x,z\right).$$

Since $\alpha^1(x,z) > 0$ and $\alpha^2(x,z) > 0$, and the sublinearity of F in the above inequalities, we summarize to get

$$F\Bigg(x,z;\nabla\sum_{i=1}^{s}t_i\left(f\left(z,\bar{y}_i\right)-\lambda h\left(z,\bar{y}_i\right)\right)+\nabla^2\sum_{i=1}^{s}t_i\left(f\left(z,\bar{y}_i\right)-\lambda h\left(z,\bar{y}_i\right)\right)p+ +\nabla\sum_{j=1}^{m}\mu_j g_j(z)+\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)\,p\Bigg) < -\left(\frac{\rho_1^1}{\alpha^1(x,z)}+\frac{\rho_1^2}{\alpha^2(x,z)}\right)d^2(x,z) < 0. \tag{3.8}$$

Since $\frac{\rho_1^1}{\alpha^1(x,z)}+\frac{\rho_1^2}{\alpha^2(x,z)} \geq 0$, by inequality (3.8), we have

$$F\Bigg(x,z;\nabla\sum_{i=1}^{s}t_i\left(f\left(z,\bar{y}_i\right)-\lambda h\left(z,\bar{y}_i\right)\right)+\nabla^2\sum_{i=1}^{s}t_i\left(f\left(z,\bar{y}_i\right)-\lambda h\left(z,\bar{y}_i\right)\right)p+ +\nabla\sum_{j=1}^{m}\mu_j g_j(z)+\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)\,p\Bigg) < 0,$$

which contradicts (3.1), as $F(x,z;0) = 0$. This completes the proof. $\square$

Theorem 3.3. *(Strong duality) Assume that x^* is an optimal solution to* (P) *and $\nabla g_j(x^*), j \in J(x^*)$, are linearly independent. Then there exist $(s^*,t^*,\bar{y}^*) \in K(x^*)$ and $(x^*,\mu^*,\lambda^*,p^*=0) \in H_1(s^*,t^*,\bar{y}^*)$ such that $(x^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*=0)$ is a feasible solution to* (MD) *and the two objectives have the same values. Further, if the hypotheses of weak duality Theorems* 3.1 *or* 3.2 *hold for all feasible solutions $(z,\mu,\lambda,s,t,\bar{y},p)$ to* (MD)*, then $(x^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*=0)$ is an optimal solution to* (MD).

Proof. Since x^* is an optimal solution to (P) and $\nabla g_j(x^*), j \in J(x^*)$, are linearly independent, then by Theorem 2.1, there exist $(s^*,t^*,\bar{y}^*) \in K(x^*)$ and $(x^*,\mu^*,\lambda^*,p^*=0) \in H_1(s^*,t^*,\bar{y}^*)$ such that $(x^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*=0)$ is a feasible solution to (MD) and the two objectives have the same values. Optimality of $\left(x^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*=0\right)$ for (MD) thus follows from weak duality Theorems 3.1 or 3.2. $\square$

Theorem 3.4. *(Strict converse duality) Let x^* be an optimal solution to* (P) *and $\left(z^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*\right)$ be optimal solution to* (MD). *Assume that are satisfied the conditions:*

(i) $\nabla g_j(x^*), j \in J(x^*)$, *are linearly independent,*

(ii) $\left(\sum_{i=1}^{s^*}t_i^*\left(f\left(\cdot,\bar{y}_i^*\right)-\lambda^* h\left(\cdot,\bar{y}_i^*\right)\right),\sum_{j=1}^{m}\mu_j^* g_j(\cdot)\right)$ *is second order (F,α,ρ,d)-type I at z^*,*

(iii) $\frac{\rho_1^1}{\alpha^1(x^*,z^*)}+\frac{\rho_1^2}{\alpha^2(x^*,z^*)} > 0$.

Then $z^* = x^*$*, that is,* z^**is an optimal solution to* (P)*.*

Proof. Suppose to contrary that $z^* \neq x^*$ and exhibit a contradiction. Since x^* and $(z^*, \mu^*, \lambda^*, s^*, t^*, \bar{y}^*, p^*)$ are optimal solutions to (P) and (MD), respectively, and $\nabla g_j(x^*), j \in J(x^*)$, are linearly independent, by Theorem 3.3 we have

$$\sup_{y^* \in Y} \tfrac{f(x^*,y^*)}{h(x^*,y^*)} = \lambda^*.$$

Therefore, we have

$$f(x^*, \bar{y}_i^*) - \lambda^* h(x^*, \bar{y}_i^*) \leq 0 \text{ for all } \bar{y}_i^* \in Y(x^*), \ \ i = 1, 2, ..., s^*.$$

It follows from $t_i^* \geq 0, \ i = 1, 2, ..., s$, that

$$t_i^* \left(f(x^*, \bar{y}_i^*) - \lambda^* h(x^*, \bar{y}_i^*)\right) \leq 0,$$

with at least one strict inequality, since $t = (t_1, t_2, ..., t_s) \neq 0$. Taking summation over i, we have

$$\sum_{i=1}^{s^*} t_i^* \left(f(x^*, \bar{y}_i^*) - \lambda^* h(x^*, \bar{y}_i^*)\right) < 0,$$

which together with (3.2) gives

$$\sum_{i=1}^{s^*} t_i^* \left(f(x^*, \bar{y}_i^*) - \lambda^* h(x^*, \bar{y}_i^*)\right) < 0 \leq$$
$$\leq \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right) - \tfrac{1}{2} p^{*T} \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right) p^*.$$

That is,

$$\sum_{i=1}^{s^*} t_i^* \left(f(x^*, \bar{y}_i^*) - \lambda^* h(x^*, \bar{y}_i^*)\right) - \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right) + \tfrac{1}{2} p^{*T} \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right) p^* < 0. \tag{3.9}$$

Using (3.3), (3.8) and hypothesis (ii), we obtain

$$\begin{aligned}0 > &\sum_{i=1}^{s^*} t_i^* \left(f\left(x^*, \bar{y}_i^*\right) - \lambda^* h\left(x^*, \bar{y}_i^*\right)\right) - \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \bar{y}_i^*\right) - \lambda^* h\left(z^*, \bar{y}_i^*\right)\right) \\ &+ \tfrac{1}{2} p^{*T} \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \bar{y}_i^*\right) - \lambda^* h\left(z^*, \bar{y}_i^*\right)\right) p^* \\ \geq\ & F\left(x^*, z^*; \alpha^1\left(x^*, z^*\right)\left(\nabla \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \bar{y}_i^*\right) - \lambda^* h\left(z^*, \bar{y}_i^*\right)\right)\right.\right. \\ &\left.\left. + \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \bar{y}_i^*\right) - \lambda^* h\left(z^*, \bar{y}_i^*\right)\right) p^*\right)\right) + \rho_1^1 d^2\left(x^*, z^*\right),\end{aligned}$$

and

$$\begin{aligned}0 \geq & -\sum_{j=1}^{m} \mu_j^* g_j\left(z^*\right) + \tfrac{1}{2} p^{*T} \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j\left(z^*\right) p^* \\ \geq & F\left(x^*, z^*; \alpha^2\left(x^*, z^*\right)\left(\nabla \sum_{j=1}^{m} \mu_j^* g_j\left(z^*\right) + \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j\left(z^*\right) p^*\right)\right) \\ & + \rho_1^2 d^2\left(x^*, z^*\right).\end{aligned}$$

Since $\alpha^1(x, z) > 0$ and $\alpha^2(x, z) > 0$, by using the sublinearity of F, the above two inequalities imply

$$\begin{aligned}&F\left(x^*, z^*; \left(\nabla \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \bar{y}_i^*\right) - \lambda^* h\left(z^*, \bar{y}_i^*\right)\right)\right.\right. \\ &\left.\left. + \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \bar{y}_i^*\right) - \lambda^* h\left(z^*, \bar{y}_i^*\right)\right) p^*\right)\right) < -\tfrac{\rho_1^1 d^2\left(x^*, z^*\right)}{\alpha^1\left(x^*, z^*\right)},\end{aligned} \tag{3.10}$$

and

$$F\left(x^*, z^*; \left(\nabla \sum_{j=1}^{m} \mu_j^* g_j\left(z^*\right) + \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j\left(z^*\right) p^*\right)\right) \leq -\tfrac{\rho_1^2 d^2\left(x^*, z^*\right)}{\alpha^2\left(x^*, z^*\right)}. \tag{3.11}$$

From (3.1), (3.10), (3.11) and the sublinearity of F, we get

$$\begin{aligned}0 = F\Bigg(x^*, z^*; &\nabla \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \bar{y}_i^*\right) - \lambda^* h\left(z^*, \bar{y}_i^*\right)\right) \\ &+ \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f\left(z^*, \bar{y}_i^*\right) - \lambda^* h\left(z^*, \bar{y}_i^*\right)\right) p^*\end{aligned}$$

$$+\nabla \sum_{j=1}^{m} \mu_j^* g_j(z^*) + \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j(z^*)\, p^* \Bigg)$$
$$\leq F\left(x^*, z^*; \nabla \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right) \right.$$
$$\left. +\nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right) p^* \right)$$
$$+ F\left(x, z; \nabla \sum_{j=1}^{m} \mu_j^* g_j(z^*) + \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j(z^*)\, p^* \right)$$
$$< -\left(\tfrac{\rho_1^1}{\alpha^1(x^*, z^*)} + \tfrac{\rho_1^2}{\alpha^2(x^*, z^*)}\right) d^2(x^*, z^*) < 0 \text{ (by (iii))}.$$

Thus, we have a contradiction. Hence $z^* = x^*$. □

THEOREM 3.5 (Strict converse duality) *Let x^* be an optimal solution to* (P) *and* $\left(z^*, \mu^*, \lambda^*, s^*, t^*, \bar{y}^*, p^*\right)$ *be optimal solution to* (MD). *Assume that are satisfied the conditions:*

(i) $\nabla g_j(x^*), j \in J(x^*)$, *are linearly independent,*

(ii) $\left(\sum\limits_{i=1}^{s^*} t_i^* \left(f(\cdot, \bar{y}_i^*) - \lambda^* h(\cdot, \bar{y}_i^*)\right), \sum\limits_{j=1}^{m} \mu_j^* g_j(\cdot)\right)$ *is second order strictly pseudoquasi* (F, α, ρ, d)*-type I at* z^*,

(iii) $\frac{\rho_1^1}{\alpha^1(x^*, z^*)} + \frac{\rho_1^2}{\alpha^2(x^*, z^*)} \geq 0$.

Then, $z^* = x^*$; that is, $z^*$is an optimal solution to (P).

Proof. We proceed as in the proof of Theorem 3.4 and obtain

$$\sum_{i=1}^{s^*} t_i^* f(x^*, \bar{y}_i^*) - \lambda^* h(x^*, \bar{y}_i^*) < \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right)$$
$$- \tfrac{1}{2} p^{*T} \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right) p^*. \tag{3.12}$$

From (3.3), and the second part of the hypothesis on $\left(\sum\limits_{i=1}^{s^*} t_i^* \left(f(\cdot, \bar{y}_i^*) - \lambda^* h(\cdot, \bar{y}_i^*)\right), \sum\limits_{j=1}^{m} g_j(\cdot)\right)$ at z^*, we have

$$F\left(x^*, z^*; \alpha^1(x^*, z^*) \left(\nabla \sum_{j=1}^{m} \mu_j^* g_j(z^*) + \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j(z^*)\, p^*\right)\right) \leq -\rho_1^2 d^2(x^*, z^*).$$

As $\alpha^2(x^*, z^*) > 0$ and as F is sublinear, it follows that

$$F\left(x^*, z^*; \left(\nabla \sum_{j=1}^{m} \mu_j^* g_j(z^*) + \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j(z^*) p^*\right)\right) \leq -\tfrac{\rho_1^2 d^2(x^*,z^*)}{\alpha^2(x^*,z^*)}. \tag{3.13}$$

From relation (3.1), (3.13) and the sublinearity of F, we obtain

$$F\left(x^*, z^*; \left(\nabla \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right)\right.\right.$$
$$\left.\left. + \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right) p^*\right)\right) \geq \tfrac{\rho_1^2 d^2(x^*,z^*)}{\alpha^2(x^*,z^*)}.$$

In view of $\frac{\rho_1^1}{\alpha^1(x^*,z^*)} + \frac{\rho_1^2}{\alpha^2(x^*,z^*)} \geq 0$, $\alpha^1(x^*, z^*) > 0$ and the sublinearity of F, the above inequality becomes

$$F\left(x^*, z^*; \alpha^1(x^*, z^*) \left(\nabla \sum_{i=1}^{s^*} t_i^* \left(f(z^*, y_i^*) - \lambda^* h(z^*, y_i^*)\right)\right.\right.$$
$$\left.\left. + \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f(z^*, y_i^*) - \lambda^* h(z^*, y_i^*)\right) p^*\right)\right) \geq -\rho_1^1 d^2(x^*, z^*).$$

Using the first part of the hypothesis on $\left(\sum_{i=1}^{s^*} t_i^* \left(f(\cdot, \bar{y}_i) - \lambda^* h(\cdot, \bar{y}_i)\right), \sum_{j=1}^{m} g_j(\cdot)\right)$ at z^*, it follows that

$$\sum_{i=1}^{s^*} t_i^* \left(f(x^*, \bar{y}_i^*) - \lambda^* h(x^*, \bar{y}_i^*)\right) > \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right)$$
$$- \tfrac{1}{2} p^{*T} \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right) p^*.$$

which is a contradiction to (3.12). Hence, $z^* = x^*$. □

4. CONCLUSIONS

In this paper, we have discussed the second order duality for dual model of minmax fractional programming problems under the assumptions of generalized (F, α, ρ, d)-type I convexity. It will be interesting to see whether or not the second order duality results developed in this paper still hold for the following nondifferentiable minmax fractional programming problems:

$$\textbf{(P1)} \qquad \operatorname{Min} \sup_{y \in Y} \frac{\phi(x,y) + \left(x^T B x\right)^{1/2}}{\psi(x,y) - \left(x^T D x\right)^{1/2}} \quad \text{subject to } g(x) \leq 0, \quad x \in \mathbb{R}^n,$$

where Y is a compact subset of $\mathbb{R}^m, \phi(.,.)$, $\psi(.,.)$: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $g(.,.)$: $\mathbb{R}^n \to \mathbb{R}$ are continuously differentiable functions, and B and D are two positive semidefinite $n \times n$ symmetric matrices.

$$(\mathbf{P2}) \qquad \operatorname{Min} \sup_{v \in W} \frac{\operatorname{Re}\left[\phi(\xi,v)+\left(z^T Bz\right)^{1/2}\right]}{\operatorname{Re}\left[\psi(\xi,v)-\left(z^T Dz\right)^{1/2}\right]}, \quad \text{subject to } -g(\xi) \in S^0, \quad \xi \in C^{2n},$$

where $\xi = (z, \bar{z})$, $\quad v = (w, \bar{w})$ for $z \in C^n$, $\quad w \in C^l$, $\phi(\cdot,\cdot) : C^{2n} \times C^{2l} \to C$ and $\psi(\cdot,\cdot) : C^{2n} \times C^{2l} \to C$ are analytic with respect to ξ, W is a specified compact subset in C^{2l}, S^0 is a polyhedral cone in C^m and $g : C^{2n} \to C^m$ is analytic. Also $B, D \in C^{n \times n}$ are positive semidefinite Hermitian matrices.

This would be task of some of our forthcoming works.

REFERENCES

[1] I. Ahmad and Z. Husain, *Duality in nondifferentiable minimax fractional programming with generalized convexity*, Appl. Math. Comp., **176**, pp. 545–551, 2006.

[2] I. Ahmad, Z. Husain and S. Sharma, *Second-order duality in nondifferentiable minmax programming involving type*-I *functions*, J. Comp. Appl. Math., **215**, pp. 91–102, 2008.

[3] C.R. Bector, S. Chandra and I. Husain, *Second order duality for a minimax programming problem*, Opsearch, **28**, pp. 249–263, 1991.

[4] S. Chandra and V. Kumar, *Duality in fractional minimax programming*, J. Austral. Math. Soc. Ser. A, **58**, pp. 376–386, 1995.

[5] J.P. Crouzeix, J.A. Ferland and S. Schaible, *Duality in generalized fractional programming*, Math. Programming, **27**, pp. 342–354, 1983.

[6] M. Hachimi and B. Aghezzaf, *Second order duality in multiobjective programming involving generalized type-I functions*, Numer. Funct. Anal. Optim., **25**, pp. 725–736, 2004.

[7] M.A. Hanson, *Second order invexity and duality in mathematical programming*, Opsearch **30**, pp. 313–320, 1993.

[8] Z. Husain, I. Ahmad and S. Sharma *Second order duality for minmax fractional programming*, Optim. Lett., **3**, pp. 277–286, 2009.

[9] Z.A. Liang, H.X. Huang and P. Pardalos, P.M., *Optimality conditions and duality for a class of nonlinear fractional programming problems*, J. Optim. Theory Appl., **110**, pp. 611–619, 2001.

[10] Z. Liang and Z. Shi, *Optimality conditions and duality for a minimax fractional programming with generalized convexity*, J. Math. Anal. Appl., **277**, pp. 474–488, 2003.

[11] J.C. Liu and C.S. Wu, *On minimax fractional optimality conditions with* (F, ρ)*-convexity*, J. Math. Anal. Appl, **219**, pp. 36–51 1998.

[12] X.J. Long and J. Quan, *Optimality conditions and duality for minimax fractional programming involving nonsmooth generalized univexity*, Numer. Algebra Control Optim. **1**, pp. 361–370, 2011.

[13] O.L. Mangasarian, *Second and higher order duality in nonlinear programming*, J. Math. Anal. Appl. **51**, pp. 607–620, 1975.

[14] Q. Hu, G. Yang and J. Jian, *On second order duality for minimax fractional programming*, Nonlinear Anal. Real World Appl., **12**, pp. 3509–3514, 2011.

[15] Q. Hu, Y. Chen and J. Jian, *Second-order duality for nondifferentiable minimax fractional programming*, Int. J. Comput. Math., **89**, pp. 11–16, 2012.

[16] S.R. Yadav and R.N. Mukherjee, *Duality for fractional minimax programming problems*, J. Austral. Math. Soc. Ser. B, **31**, pp. 482–492, 1990.

[17] X.M. Yang and S.H. Hou, *On minimax fractional optimality and duality with generalized convexity*, J. Global Optim., **31**, pp. 235–252, 2005.

[18] J. Zhang and B. Mond, *Second order duality for multiobjective nonlinear programming involving generalized convexity*, in: B.M. Glover, B.D. Craven, D. Ralph (eds.), Proceeding of Optimization Miniconference III, University of Ballarat, pp. 79–95, 1997.

Received by the editors: March 23, 2012.