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ON THE REFINEMENTS OF JENSEN-MERCER'S INEQUALITY[‡]

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Abstract. In this paper we give refinements of Jensen-Mercer's inequality and its generalizations and give applications for means. We prove n-exponential convexity of the functions constructed from these refinements. At the end we discuss some examples.

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1. INTRODUCTION

In paper [8] A. McD. Mercer proved the following variant of Jensen's inequality, to which we will refer as to the Jensen-Mercer inequality.

THEOREM 1. *Let $[a,b]$ be an interval in $\mathbb{R}$, and $x_1,...,x_n \in [a,b]$. Let $w_1,w_2,...,w_n$ be nonnegative real numbers such that $\sum_{i=1}^n w_i = 1$. If ϕ is a convex function on $[a,b]$, then*

$$\phi\left(a+b-\sum_{i=1}^n w_i x_i\right) \leq \phi(a)+\phi(b)-\sum_{i=1}^n w_i\phi(x_i). \tag{1}$$

Given two real row n-tuples $\mathbf{x}=(x_1,...,x_n)$ and $\mathbf{y}=(y_1,...,y_n)$, $\mathbf{y}$ is said to majorize $\mathbf{x}$, if

$$\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$$

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holds for $k = 1, 2, ..., n-1$ and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i,$$

where $x_{[1]} \geq ... \geq x_{[n]}$, and $y_{[1]} \geq ... \geq y_{[n]}$, are the entries of $\mathbf{x}$ and $\mathbf{y}$, respectively, in nonincreasing order (see [6, p. 10]).

The following extension of (1) is given in [9].

THEOREM 2. *Let $\phi : [a,b] \to \mathbb{R}$ be a continuous convex function on $[a,b]$. Suppose that $\boldsymbol{a} = (a_1, ..., a_m)$ with $a_j \in [a,b]$, and $\boldsymbol{X} = (x_{ij})$ is a real $n \times m$ matrix such that $x_{ij} \in [a,b]$ for all $i = 1, \ldots, n;\ j = 1, \ldots, m$.*

If $\boldsymbol{a}$ majorizes each row of $\boldsymbol{X}$, that is

$$\boldsymbol{x}_{i.} = (x_{i1}, ..., x_{im}) \prec (a_1, ..., a_m) = \boldsymbol{a} \text{ for each } i = 1, ..., n;$$

then we have the inequality

$$\phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i x_{ij}\right) \leq \sum_{j=1}^{m} \phi(a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i \phi(x_{ij}), \tag{2}$$

where $\sum_{i=1}^{n} w_i = 1$ with $w_i \geq 0$.

In this paper we give refinements of (1), (2) and give applications for means. We construct functionals from these refinements and prove mean value theorems. The notion of n-exponential convexity is introduced in [10]. The class of n-exponential convex functions is more general than the class of log-convex functions. We follow the method illustrated in [10] to give the n-exponential convexity and exponential convexity for these functionals.

2. MAIN RESULTS

Let $\phi : [a,b] \to \mathbb{R}$ be a convex function. If $x_i \in [a,b]$ and $w_i > 0$, $i \in \{1, 2, ..., n\}$ with $\sum_{i=1}^{n} w_i = 1$. Throughout the paper we assume that $I \subset \{1, 2, ..., n\}$ with $I \neq \emptyset$ and $I \neq \{1, 2, ..., n\}$ unless stated. We define $W_I = \sum_{i \in I} w_i$ and $W_{\bar{I}} = 1 - \sum_{i \in I} w_i$. For the convex function ϕ and the n-tuple $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{w} = (w_1, ..., w_n)$ as above, we can define the following functional

(3)

$$D(\mathbf{w}, \mathbf{x}, \phi; I) := W_I \phi\left(a + b - \tfrac{1}{W_I}\sum_{i \in I} w_i x_i\right) + W_{\bar{I}} \phi\left(a + b - \tfrac{1}{W_{\bar{I}}}\sum_{i \in \bar{I}} w_i x_i\right).$$

It is worth to observe that for $I = \{k\}$, $k \in \{1, ..., n\}$ we have the functional

$$\begin{aligned} D_k(\mathbf{w}, \mathbf{x}, \phi) &:= D(\mathbf{w}, \mathbf{x}, \phi; \{k\}) \\ &= w_k \phi(a + b - x_k) + (1 - w_k)\phi\left(a + b - \tfrac{\sum_{i=1}^{n} w_i x_i - w_k x_k}{1 - w_k}\right). \end{aligned}$$

The following refinement of (1) is valid.

THEOREM 3. *Let $[a,b]$ be an interval in $\mathbb{R}$, and $x_1,...,x_n \in [a,b]$. Let $w_1,w_2,...,w_n$ be positive real numbers such that $\sum_{i=1}^n w_i = 1$. If $\phi : [a,b] \to \mathbb{R}$ is a convex function, then for any non empty subset I of $\{1,...,n\}$ we have*

$$\phi\left(a+b-\sum_{i=1}^n w_i x_i\right) \le D(\mathbf{w},\mathbf{x},\phi;I) \le \phi(a)+\phi(b)-\sum_{i=1}^n w_i\phi(x_i). \tag{4}$$

Proof. By the convexity of the function ϕ we have

$$\begin{aligned}
&\phi\Big(a+b-\sum_{i=1}^n w_i x_i\Big) = \phi\Big(\sum_{i=1}^n w_i\Big(a+b-x_i\Big)\Big)\\
&= \phi\Big(W_I\Big(\tfrac{1}{W_I}\sum_{i\in I} w_i\Big(a+b-x_i\Big)\Big) + W_{\overline{I}}\Big(\tfrac{1}{W_{\overline{I}}}\sum_{i\in \overline{I}} w_i\Big(a+b-x_i\Big)\Big)\Big)\\
&\le W_I\phi\Big(\tfrac{1}{W_I}\sum_{i\in I} w_i\Big(a+b-x_i\Big)\Big) + W_{\overline{I}}\phi\Big(\tfrac{1}{W_{\overline{I}}}\sum_{i\in \overline{I}} w_i\Big(a+b-x_i\Big)\Big)\\
&= D(\mathbf{w},\mathbf{x},\phi;I)
\end{aligned}$$

for any I, which proves the first inequality in (4).

By the Jensen-Mercer inequality (1) we also have

$$\begin{aligned}
&D(\mathbf{w},\mathbf{x},\phi;I) = W_I\phi\Big(a+b-\tfrac{1}{W_I}\sum_{i\in I} w_i x_i\Big) + W_{\overline{I}}\phi\Big(a+b-\tfrac{1}{W_{\overline{I}}}\sum_{i\in \overline{I}} w_i x_i\Big)\\
&\le W_I\Big(\phi(a)+\phi(b)-\tfrac{1}{W_I}\sum_{i\in I} w_i\phi(x_i)\Big) + W_{\overline{I}}\Big(\phi(a)+\phi(b)-\tfrac{1}{W_{\overline{I}}}\sum_{i\in \overline{I}} w_i\phi(x_i)\Big)\\
&= \phi(a)+\phi(b)-\sum_{i=1}^n w_i\phi(x_i)
\end{aligned}$$

for any I, which proves the second inequality in (4). □

REMARK 4. In [7] from the proof of Theorem 2.3 we have left inequality of (4). □

REMARK 5. We observe that the inequality (4) can be written in an equivalent form as

$$\phi\left(a+b-\sum_{i=1}^n w_i x_i\right) \le \min_I D(\mathbf{w},\mathbf{x},\phi;I) \tag{5}$$

and

$$\max_I D(\mathbf{w},\mathbf{x},\phi;I) \le \phi(a)+\phi(b)-\sum_{i=1}^n w_i\phi(x_i). \tag{6}$$

The following special cases of (5) and (6) can be given:

$$\phi\left(a+b-\sum_{i=1}^{n} w_i x_i\right) \leq \min_{k\in\{1,\ldots,n\}} D_k(\mathbf{w},\mathbf{x},\phi)$$

and

$$\max_{k\in\{1,\ldots,n\}} D_k(\mathbf{w},\mathbf{x},\phi) \leq \phi(a)+\phi(b)-\sum_{i=1}^{n} w_i\phi(x_i). \quad \square$$

The case of uniform distribution, namely, when $w_i = \frac{1}{n}$ for all $i = 1, 2, ..., n$ is of interest as well. If we consider a natural number m with $m \in \{1, 2, \ldots, n-1\}$ and if we define

$$D_m(\mathbf{x},\phi) := \tfrac{m}{n}\phi\left(a+b-\tfrac{1}{m}\sum_{i=1}^{m} x_i\right)+\tfrac{n-m}{n}\phi\left(a+b-\tfrac{1}{n-m}\sum_{j=m+1}^{n} x_j\right)$$

then we can state the following result:

Corollary 6. *If $\phi : [a, b] \to \mathbb{R}$ is a convex function, $x_i \in [a, b]$, $i \in \{1, 2, ..., n\}$, then for any $m \in \{1, 2, ..., n-1\}$ we have*

$$\phi\left(a+b-\tfrac{1}{n}\sum_{i=1}^{n} x_i\right) \leq D_m(\mathbf{x},\phi) \leq \phi(a)+\phi(b)-\tfrac{1}{n}\sum_{i=1}^{n}\phi(x_i).$$

In particular, we have the bounds

$$\phi\left(a+b-\tfrac{1}{n}\sum_{i=1}^{n} x_i\right) \leq \min_{m\in\{1,\ldots,n-1\}} D_m(\mathbf{x},\phi)$$

and

$$\max_{m\in\{1,\ldots,n-1\}} D_m(\mathbf{x},\phi) \leq \phi(a)+\phi(b)-\tfrac{1}{n}\sum_{i=1}^{n}\phi(x_i).$$

The following refinement of (2) is valid.

Theorem 7. *Let $\phi : [a, b] \to \mathbb{R}$ be a continuous convex function on $[a, b]$. Suppose that $\boldsymbol{a} = (a_1, ..., a_m)$ with $a_j \in [a, b]$, and $\boldsymbol{X} = (x_{ij})$ is a real $n \times m$ matrix such that $x_{ij} \in [a, b]$ for all $i = 1, \ldots, n;\ j = 1, \ldots, m$.*

If $\boldsymbol{a}$ majorizes each row of $\boldsymbol{X}$, then for any non empty subset I of $\{1, ..., n\}$ we have

(7)

$$\phi\left(\sum_{j=1}^{m} a_j-\sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i x_{ij}\right) \leq \tilde{D}(\mathbf{w},\mathbf{X},\phi;I) \leq \sum_{j=1}^{m}\phi(a_j)-\sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i\phi(x_{ij}),$$

where

(8)

$$\tilde{D}(\mathbf{w},\mathbf{X},\phi;I) := = W_I\phi\left(\sum_{j=1}^{m}a_j - \frac{1}{W_I}\sum_{j=1}^{m-1}\sum_{i\in I}w_ix_{ij}\right) + W_{\bar{I}}\phi\left(\sum_{j=1}^{m}a_j - \frac{1}{W_{\bar{I}}}\sum_{j=1}^{m-1}\sum_{i\in \bar{I}}w_ix_{ij}\right),$$

$W_I = \sum_{i\in I} w_i, W_{\bar{I}} = \sum_{i\in\bar{I}} w_i,\ \sum_{i=1}^{n} w_i = 1$ *with* $w_i > 0$.

Proof. The proof is similar to the proof of Theorem 3 but use (2) instead of (1). □

As above we can give the following remark.

Remark 8.

$$\phi\left(\sum_{j=1}^{m}a_j - \sum_{j=1}^{m-1}\sum_{i=1}^{n}w_ix_{ij}\right) \le \min_I \tilde{D}(\mathbf{w},\mathbf{X},\phi;I)$$

and

$$\max_I \tilde{D}(\mathbf{w},\mathbf{X},\phi;I) \le \sum_{j=1}^{m}\phi(a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n}w_i\phi(x_{ij}). \quad \square$$

Remark 9. If in (2) we set $m = 2, a_1 = a, a_2 = b$ and $x_{i1} = x_i$ for $i = 1, ..., n$ we get (4). □

An $m\times m$ matrix $\mathbf{A} = (a_{jk})$ is said to be doubly stochastic, if $a_{jk} \ge 0$ and $\sum_{j=1}^{m} a_{jk} = \sum_{k=1}^{m} a_{jk} = 1$ for all $j, k = 1, ..., m$. It is well known [6, p. 20] that if $\mathbf{A}$ is an $m\times m$ doubly stochastic matrix, then

(9) $$\mathbf{aA} \prec \mathbf{a} \text{ for each real } m\text{-tuple } \mathbf{a} = (a_1, a_2, ..., a_m).$$

By applying Theorem 7 and (9), one obtains:

Corollary 10. *Let* $\phi : [a, b] \to \mathbb{R}$ *be a continuous convex function on* $[a, b]$. *Suppose that* $\boldsymbol{a} = (a_1, ..., a_m)$ *with* $a_j \in [a, b]$ $j = 1, ..., m$ *and* $\boldsymbol{A}_1, \boldsymbol{A}_2, ..., \boldsymbol{A}_n$ *are* $m\times m$ *doubly stochastic matrices. Set*

$$X = (x_{ij}) = \begin{pmatrix} \boldsymbol{aA}_1 \\ \cdot \\ \cdot \\ \boldsymbol{aA}_n \end{pmatrix}.$$

Then inequalities in (7) *hold.*

Remark 11. In [4] Dragomir has given related refinements of Jensen's inequality. □

3. APPLICATIONS

For $\emptyset \neq I \subseteq \{1, ..., n\}$ let A_I, G_I, H_I and $M_I^{[r]}$ be the arithmetic, geometric, harmonic means, and power mean of order $r \in \mathbb{R}$, respectively of $x_i \in [a, b]$, where $0 < a < b$, formed with the positive weights w_i, $i \in I$. For $I = \{1, ..., n\}$ we denote the arithmetic, geometric, harmonic and power means by A_n, G_n, H_n and $M_n^{[r]}$ respectively.

If we define

$$\begin{aligned}
\tilde{A}_I : &= a + b - \tfrac{1}{W_I}\sum_{i\in I} w_i x_i = a + b - A_I \\
\tilde{G}_I : &= \frac{ab}{\left(\prod\limits_{i\in I} x_i^{w_i}\right)^{\frac{1}{W_I}}} = \frac{ab}{G_I} \\
\tilde{H}_I : &= \left(a^{-1} + b^{-1} - \tfrac{1}{W_I}\sum_{i\in I} w_i x_i^{-1}\right)^{-1} = \left(a^{-1} + b^{-1} - H_I^{-1}\right)^{-1} \\
\tilde{M}_I^{[r]} : &= \begin{cases} \left(a^r + b^r - \left(M_I^{[r]}\right)^r\right)^{\frac{1}{r}}, & r \neq 0; \\ \tilde{G}_I, & r = 0, \end{cases}
\end{aligned}$$

where

$$M_I^{[r]} := \begin{cases} \left(\frac{1}{W_I}\sum_{i\in I} w_i x_i^r\right)^{\frac{1}{r}}, & r \neq 0; \\ \left(\prod\limits_{i\in I} x_i^{w_i}\right)^{\frac{1}{W_I}}, & r = 0, \end{cases}$$

then the following inequalities hold.

THEOREM 12.

(10)(*i*) $\tilde{G}_n \leq \min\limits_I \tilde{A}_I^{W_I} \tilde{A}_{\overline{I}}^{W_{\overline{I}}}$ *and* $\tilde{A}_n \geq \max\limits_I \tilde{A}_I^{W_I} \tilde{A}_{\overline{I}}^{W_{\overline{I}}}$.

(11) (*ii*) $\tilde{G}_n \leq \min\limits_I \left[W_I \tilde{G}_I + W_{\overline{I}} \tilde{G}_{\overline{I}}\right]$ *and* $\tilde{A}_n \geq \max\limits_I \left[W_I \tilde{G}_I + W_{\overline{I}} \tilde{G}_{\overline{I}}\right]$.

Proof. (i) Applying Theorem 3 to the convex function $\phi(x) = -\ln x$, we obtain

$$-\ln \tilde{A}_n \leq -W_I \ln \tilde{A}_I - W_{\overline{I}} \ln \tilde{A}_{\overline{I}} \leq -\ln \tilde{G}_n. \tag{12}$$

Now (10) follows from Remark 5 and (12).

(ii) Applying Theorem 3 to the convex function $\phi(x) = \exp(x)$, and replacing a, b, and x_i with $\ln a, \ln b$, and $\ln x_i$ respectively and using Remark 5, we obtain (11). □

The following particular case of Theorem 12 is of interest.

Corollary 13.

$$(i)\quad \frac{1}{\tilde{G}_n} \le \min_I \frac{1}{\tilde{H}_I^{W_I} \tilde{H}_{\bar{I}}^{W_{\bar{I}}}} \quad \text{and} \quad \frac{1}{\tilde{H}_n} \ge \max_I \frac{1}{\tilde{H}_I^{W_I} \tilde{H}_{\bar{I}}^{W_{\bar{I}}}}.$$

$$(ii)\quad \frac{1}{\tilde{G}_n} \le \min_I \left[\frac{W_I}{\tilde{G}_I} + \frac{W_{\bar{I}}}{\tilde{G}_{\bar{I}}}\right] \quad \text{and} \quad \frac{1}{\tilde{H}_n} \ge \max_I \left[\frac{W_I}{\tilde{G}_I} + \frac{W_{\bar{I}}}{\tilde{G}_{\bar{I}}}\right].$$

Proof. Directly from Theorem 12 by the substitutions $a \to \frac{1}{a}, b \to \frac{1}{b}, x_i \to \frac{1}{x_i}$. □

Theorem 14. *For $r \le 1$, we have the following inequalities*

$$\tilde{M}_n^{[r]} \le \min_I \left[W_I \tilde{M}_I^{[r]} + W_{\bar{I}} \tilde{M}_{\bar{I}}^{[r]}\right],$$

$$\tilde{A}_n \ge \max_I \left[W_I \tilde{M}_I^{[r]} + W_{\bar{I}} \tilde{M}_{\bar{I}}^{[r]}\right]. \tag{13}$$

For $r \ge 1$, the inequalities in (13) *are reversed.*

Proof. For $r \le 1$, $r \ne 0$, use Theorem 3 for the convex function $\phi(x) = x^{\frac{1}{r}}$, and replacing a, b, and x_i with a^r, b^r, and x_i^r respectively and for $r = 0$ use Theorem 3 for the convex function $\phi(x) = \exp(x)$; replacing a, b, and x_i with $\ln a, \ln b$, and $\ln x_i$ respectively, we obtain (13) by Remark 5.

If $r \ge 1$, then the function $\phi(x) = x^{\frac{1}{r}}$ is concave, so the inequalities in (13) are reversed. □

Corollary 15.

$$\tilde{H}_n \le \min_I \left[W_I \tilde{H}_I + W_{\bar{I}} \tilde{H}_{\bar{I}}\right],$$

$$\tilde{A}_n \ge \max_I \left[W_I \tilde{H}_I + W_{\bar{I}} \tilde{H}_{\bar{I}}\right].$$

Remark 16. Obviously, part (ii) of Theorem 12 is also a direct consequences of Theorem 14. □

Theorem 17. *Let $r, s \in \mathbb{R}$, $r \le s$.*

(i) If $s \ge 0$, then

$$\left(\tilde{M}_n^{[r]}\right)^s \le \min_I \left[W_I \left(\tilde{M}_I^{[r]}\right)^s + W_{\bar{I}} \left(\tilde{M}_{\bar{I}}^{[r]}\right)^s\right],$$

$$\left(\tilde{M}_n^{[r]}\right)^s \ge \max_I \left[W_I \left(\tilde{M}_I^{[r]}\right)^s + W_{\bar{I}} \left(\tilde{M}_{\bar{I}}^{[r]}\right)^s\right]. \tag{14}$$

(ii) If $s < 0$, then inequalities in (14) *are reversed.*

Proof. Let $s \ge 0$. Using Theorem 3 and Remark 5 to the convex function $\phi(x) = x^{\frac{s}{r}}$, and replacing a, b, and x_i with a^r, b^r, and x_i^r respectively, we obtain (14).

If $s < 0$, then the function $\phi(x) = x^{\frac{s}{r}}$, is concave so inequalities in (14) are reversed. □

Let $\phi : [a,b] \to \mathbb{R}$ be a strictly monotonic and continuous function. Then for a given n- tuple $\mathbf{x} = (x_1, ..., x_n) \in [a,b]^n$ and positive n- tuple $\mathbf{w} = (w_1, ..., w_n)$ with $\sum_{i=1}^n w_i = 1$, the value

$$M_\phi^{[n]} = \phi^{-1}\left(\sum_{i=1}^n w_i \phi(x_i)\right)$$

is well defined and is called $quasi - arithmetic\ mean$ of $\mathbf{x}$ with wight $\mathbf{w}$ (see for example [2, p. 215]). If we define

$$\tilde{M}_\phi^{[n]} = \phi^{-1}\left(\phi(a) + \phi(b) - \sum_{i=1}^n w_i \phi(x_i)\right),$$

then we have the following results.

Theorem 18. *Let $\phi, \psi : [a,b] \to \mathbb{R}$ be strictly monotonic and continuous functions. If $\psi \circ \phi^{-1}$ is convex on $[a,b]$, then*

$$\begin{aligned}
&\psi\left(\tilde{M}_\phi^{[n]}\right) \le \min_I \left[W_I \psi\left(\tilde{M}_\phi^{[I]}\right) + W_{\bar{I}} \psi\left(\tilde{M}_\phi^{[\bar{I}]}\right)\right], \\
(15) \qquad &\psi\left(\tilde{M}_\phi^{[n]}\right) \ge \max_I \left[W_I \psi\left(\tilde{M}_\phi^{[I]}\right) + W_{\bar{I}} \psi\left(\tilde{M}_\phi^{[\bar{I}]}\right)\right], \\
&\text{where } \tilde{M}_\phi^{[J]} = \phi^{-1}\left(\phi(a) + \phi(b) - \tfrac{1}{W_J} \sum_{i \in J} w_i \phi(x_i)\right).
\end{aligned}$$

Proof. Applying Theorem 3 to the convex function $f = \psi \circ \phi^{-1}$ and replacing a, b, and x_i with $\phi(a), \phi(b)$, and $\phi(x_i)$ respectively and then using Remark 5, we obtain (15). □

Remark 19. Theorems 12, 14 and 17 follow from Theorem 18, by choosing adequate functions ϕ,ψ and appropriate substitutions. □

4. FURTHER GENERALIZATION

Let E be a nonempty set, $\mathfrak{A}$ be an algebra of subsets of E, and L be a linear class of real valued functions $f : E \to \mathbb{R}$ having the properties:

L1 : $f, g \in L \Rightarrow (\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
L2 : $\mathbf{1} \in L$, i.e., if $f(t) = 1$ for all $t \in E$, then $f \in L$;
L3 : $f \in L$, $E_1 \in \mathfrak{A} \Rightarrow f.\chi_{E_1} \in \mathfrak{A}$,

where χ_{E_1} is the indicator function of E_1. It follows from L_2, L_3 that $\chi_{E_1} \in L$ for every $E_1 \in \mathfrak{A}$.

An isotonic linear functional $A : L \to \mathbb{R}$ is a functional satisfying the following properties:

A1 : $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for $f, g \in L, \alpha, \beta \in \mathbb{R}$;
A2 : $f \in L, f(t) \ge 0$ on $E \Rightarrow A(f) \ge 0$;

It follows from L_3 that for every $E_1 \in \mathfrak{A}$ such that $A(\chi_{E_1}) > 0$, the functional A_1 defined for all $f \in L$ as $A_1(f) = \frac{A(f.\chi_{E_1})}{A(\chi_{E_1})}$ is an isotonic linear functional with $A(\mathbf{1}) = 1$. Furthermore, we observe that

$$A(\chi_{E_1}) + A(\chi_{E \setminus E_1}) = 1,$$

$$A(f) = A(f.\chi_{E_1}) + A(f.\chi_{E \setminus E_1}).$$

Let $\phi : [a, b] \to \mathbb{R}$ be a continuous function. In [3], under the above assumptions, the following variant of the Jessen inequality is proved, if ϕ is convex, then

$$\phi(a + b - A(f)) \leq \phi(a) + \phi(b) - A(\phi(f)); \tag{16}$$

if ϕ is concave then the inequality (16) is reversed.

The following refinement of (16) holds.

Theorem 20. *Under the above assumptions, if ϕ is convex, then*

$$\phi(a + b - A(f)) \leq \overline{D}(A, f, \phi; E_1) \leq \phi(a) + \phi(b) - A(\phi(f)); \tag{17}$$

where

$$\begin{aligned} &\overline{D}(A, f, \phi; E_1) := \\ &= A(\chi_{E_1})\phi\left(a + b - \frac{A(f.\chi_{E_1})}{A(\chi_{E_1})}\right) + A(\chi_{E \setminus E_1})\phi\left(a + b - \frac{A(f.\chi_{E \setminus E_1})}{A(\chi_{E \setminus E_1})}\right) \end{aligned} \tag{18}$$

for all $E_1 \in \mathfrak{A}$ such that $0 < A(\chi_{E_1}) < 1$

Proof. The first inequality follows by using definition of convex function and the second follows by using (16) for $A_1(f)$ instead of $A(f)$. □

Remark 21. In [7] from the proof of Theorem 4.1 we have left inequality of (18). □

Remark 22. We observe that the inequality (17) can be written in an equivalent form as

$$\phi(a + b - A(f)) \leq \min_{E_1 \in \mathfrak{A}} \overline{D}(A, f, \phi; E_1)$$

and

$$\phi(a) + \phi(b) - A(\phi(f)) \geq \max_{E_1 \in \mathfrak{A}} \overline{D}(A, f, \phi; E_1). \quad \square$$

The following particular case of Theorem 20 is of interest:

Corollary 23. *Let (Ω, P, μ) be a probability measure space, and let $f : \Omega \to [a, b]$ be a measurable function. Then for any continuous convex function*

$\phi : [a,b] \to \mathbb{R}$, *and for any set* E_1 *in* P *with* $\mu(E_1), \mu(\Omega \backslash E_1) > 0$ *we have*

$$\phi\left(a+b-\int_\Omega f\mathrm{d}\mu\right) \leq \min_{E_1\in P}\left[\mu(E_1)\phi\left(a+b-\tfrac{1}{\mu(E_1)}\int_{E_1} f\mathrm{d}\mu\right)\right.$$
$$\left.+\mu(\Omega\setminus E_1)\phi\left(a+b-\tfrac{1}{\mu(\Omega\backslash E_1)}\int_{\Omega\backslash E_1} f\mathrm{d}\mu\right)\right]$$

and

$$\phi(a)+\phi(b)-\int_\Omega \phi(f)\mathrm{d}\mu \geq \max_{E_1\in P}\left[\mu(E_1)\phi\left(a+b-\tfrac{1}{\mu(E_1)}\int_{E_1} f\mathrm{d}\mu\right)\right.$$
$$\left.+\mu(\Omega\setminus E_1)\phi\left(a+b-\tfrac{1}{\mu(\Omega\backslash E_1)}\int_{\Omega\backslash E_1} f\mathrm{d}\mu\right)\right].$$

Proof. This is a special case of Theorem 20 for the functional A defined on the class $L^1(\mu)$ as $A(f) = \int_\Omega f\mathrm{d}\mu$. □

REMARK 24. We also may obtain similar results as in Theorem 18 for the generalized quasi-arithmetic means of Mercers type defined in [3], as

$$\tilde{M}_\phi(f,A) = \phi^{-1}(\phi(a)+\phi(b)-A(\phi(f))).$$

□

5. n-EXPONENTIAL CONVEXITY OF THE JENSEN-MERCER DIFFERENCES

Under the assumptions of Theorem 3 using (4) we define the following functionals:

$$(19)\quad \Psi_1(\mathbf{w},\mathbf{x},\phi) = D(\mathbf{w},\mathbf{x},\phi;I) - \phi\big(a+b-\sum_{i=1}^n w_i x_i\big) \geq 0,$$

$$(20)\quad \Psi_2(\mathbf{w},\mathbf{x},\phi) = \phi(a)+\phi(b)-\sum_{i=1}^n w_i\phi(x_i) - D(\mathbf{w},\mathbf{x},\phi;I) \geq 0,$$

$$(21)\quad \Psi_3(\mathbf{w},\mathbf{x},\phi) = \phi(a)+\phi(b)-\sum_{i=1}^n w_i\phi(x_i) - \phi\big(a+b-\sum_{i=1}^n w_i x_i\big) \geq 0.$$

Also, under the assumptions of Theorem 7 using (7) we define the functionals as follows:

$$(22)\quad \Psi_4(\mathbf{w},\mathbf{X},\phi) = \tilde{D}(\mathbf{w},\mathbf{X},\phi;I) - \phi\big(\sum_{j=1}^m a_j - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i x_{ij}\big) \geq 0,$$

$$(23)\quad \Psi_5(\mathbf{w},\mathbf{X},\phi) = \sum_{j=1}^m \phi(a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i\phi(x_{ij}) - \tilde{D}(\mathbf{w},\mathbf{X},\phi;I) \geq 0,$$

(24)
$$\Psi_6(\mathbf{w},\mathbf{X},\phi)=\sum_{j=1}^{m}\phi(a_j)-\sum_{j=1}^{m-1}\sum_{i=1}^{n}w_i\phi(x_{ij})-\phi\Big(\sum_{j=1}^{m}a_j-\sum_{j=1}^{m-1}\sum_{i=1}^{n}w_ix_{ij}\Big)\geq 0.$$

Similarly, under the assumptions of Theorem 20 using (17) we define the following functionals:

$$\Psi_7(A,f,\phi) = \overline{D}(A,f,\phi;E_1)-\phi(a+b-A(f))\geq 0, \tag{25}$$
$$\Psi_8(A,f,\phi) = \phi(a)+\phi(b)-A(\phi(f))-\overline{D}(A,f,\phi;E_1)\geq 0, \tag{26}$$
$$\Psi_9(A,f,\phi) = \phi(a)+\phi(b)-A(\phi(f))-\phi(a+b-A(f))\geq 0. \tag{27}$$

Now we are in position to give mean value theorems for $\Psi_j(.,.,\phi)$, $j=1,2,\dots,9$.

Theorem 25. *Let $\phi\in C^2([a,b])$, $\boldsymbol{x}=(x_1,...,x_n)\in[a,b]^n$ and $\boldsymbol{w}=(w_1,...,w_n)$ be n-tuple of positive real numbers such that $\sum_{i=1}^{n}w_i=1$. Then there exists $c_j\in[a,b]$ such that*
$$\Psi_j(\boldsymbol{w},\boldsymbol{x},\phi)=\tfrac{\phi''(c_j)}{2}\Psi_j(\boldsymbol{w},\boldsymbol{x},\phi_0),\ \ \textit{where } \phi_0(x)=x^2; j=1,2,3.$$

Proof. Fix $j=1,2,3$.
Since the functions
$$\phi_1=\tfrac{\Gamma}{2}x^2-\phi(x),\ \phi_2(x)=\phi(x)-\tfrac{\gamma}{2}x^2$$
are convex, where $\Gamma=\max_{x\in[a,b]}\phi''(x)$ and $\gamma=\min_{x\in[a,b]}\phi''(x)$, we have
$$\Psi_j(\mathbf{w},\mathbf{x},\phi_1)\geq 0 \tag{28}$$
$$\Psi_j(\mathbf{w},\mathbf{x},\phi_2)\geq 0. \tag{29}$$
From (28) and (29) we get
$$\tfrac{\gamma}{2}\Psi_j(\mathbf{w},\mathbf{x},\phi_0)\leq\Psi_j(\mathbf{w},\mathbf{x},\phi)\leq\tfrac{\Gamma}{2}\Psi_j(\mathbf{w},\mathbf{x},\phi_0).$$
If $\Psi_j(\mathbf{w},\mathbf{x},\phi_0)=0$ then there is nothing to prove. Suppose $\Psi_j(\mathbf{w},\mathbf{x},\phi_0)>0$. We have
$$\gamma\leq\tfrac{2\Psi_j(\mathbf{w},\mathbf{x},\phi)}{\Psi_j(\mathbf{w},\mathbf{x},\phi_0)}\leq\Gamma.$$
Hence, there exists $c_j\in[a,b]$ such that
$$\Psi_j(\mathbf{w},\mathbf{x},\phi)=\tfrac{\phi''(c_j)}{2}\Psi_j(\mathbf{w},\mathbf{x},\phi_0).$$
□

Theorem 26. *Let $\phi,\psi\in C^2([a,b])$, $\boldsymbol{x}=(x_1,...,x_n)\in[a,b]^n$ and $\boldsymbol{w}=(w_1,...,w_n)$ be n-tuple of positive real numbers such that $\sum_{i=1}^{n}w_i=1$. Then there exists $c_j\in[a,b]$ such that*
$$\tfrac{\Psi_j(\boldsymbol{w},\boldsymbol{x},\phi)}{\Psi_j(\boldsymbol{w},\boldsymbol{x},\psi)}=\tfrac{\phi''(c_j)}{\psi''(c_j)},\ j=1,2,3.$$
provided that the denominators are non-zero.

Proof. Let us define

$$g_j = a_j\phi - b_j\psi, \;\; j = 1, 2, 3,$$
$$\text{where } a_j = \Psi_j(\mathbf{w}, \mathbf{x}, \psi), \qquad b_j = \Psi_j(\mathbf{w}, \mathbf{x}, \phi).$$

Obviously $g_j \in C^2([a, b])$, by using Theorem 25 there exists $c_j \in [a, b]$ such that

$$\left(\tfrac{a_j\phi''(c_j)}{2} - \tfrac{b_j\psi''(c_j)}{2}\right)\Psi_j(\mathbf{w}, \mathbf{x}, \phi_0) = 0.$$

Since $\Psi_j(\mathbf{w}, \mathbf{x}, \phi_0) \neq 0$ (otherwise we have a contradiction with $\Psi_j(\mathbf{w}, \mathbf{x}, \psi) \neq 0$ by Theorem 25), we get

$$\tfrac{\Psi_j(\mathbf{w},\mathbf{x},\phi)}{\Psi_j(\mathbf{w},\mathbf{x},\psi)} = \tfrac{\phi''(c_j)}{\psi''(c_j)}, \;\; j = 1, 2, 3.$$

□

THEOREM 27. *Let $\phi \in C^2([a, b])$, $\boldsymbol{a} = (a_1, ..., a_m)$ with $a_j \in [a, b]$, and $\boldsymbol{X} = (x_{ij})$ is a real $n \times m$ matrix such that $x_{ij} \in [a, b]$ for all $i = 1, \dots, n;\ j = 1, \dots, m$ and $\boldsymbol{a}$ majorizes each row of $\boldsymbol{X}$. Then there exists $c_k \in [a, b]$ such that*

$$\Psi_k(\boldsymbol{w}, \boldsymbol{X}, \phi) = \tfrac{\phi''(c_j)}{2}\Psi_k(\boldsymbol{w}, \boldsymbol{X}, \phi_0), \;\; \textit{where } \phi_0(x) = x^2; k = 4, 5, 6.$$

THEOREM 28. *Let $\phi, \psi \in C^2([a, b])$. Suppose that $\boldsymbol{a} = (a_1, ..., a_m)$ with $a_j \in [a, b]$, and $\boldsymbol{X} = (x_{ij})$ is a real $n \times m$ matrix such that $x_{ij} \in [a, b]$ for all $i = 1, \dots, n;\ j = 1, \dots, m$ and $\boldsymbol{a}$ majorizes each row of $\boldsymbol{X}$. Then there exists $c_k \in [a, b]$ such that*

$$\tfrac{\Psi_k(\boldsymbol{w},\boldsymbol{X},\phi)}{\Psi_k(\boldsymbol{w},\boldsymbol{X},\psi)} = \tfrac{\phi''(c_k)}{\psi''(c_k)}; \;\; k = 4, 5, 6,$$

provided that the denominators are non-zero.

THEOREM 29. *Suppose $\phi \in C^2([a, b])$ and L satisfy properties L_1, L_2, on a nonempty set E. Assume that A is an isotonic linear functional on L with $A(\mathbf{1}) = 1$. Let $f \in L$ be such that $\phi(f) \in L$. Then there exists $c_j \in [a, b]$ such that*

$$\Psi_j(A, f, \phi) = \tfrac{\phi''(c)}{2}\Psi_j(A, f, \phi_0), \;\; \textit{where } \phi_0(x) = x^2; j = 7, 8, 9.$$

THEOREM 30. *Suppose $\phi, \psi \in C^2([a, b])$ and L satisfy properties L_1, L_2, on a nonempty set E. Assume that A is an isotonic linear functional on L with $A(\mathbf{1}) = 1$. Let $f \in L$ be such that $\phi(f), \psi(f) \in L$. Then there exists $c_j \in [a, b]$ such that*

$$\tfrac{\Psi_j(A,f,\phi)}{\Psi_j(A,f,\psi)} = \tfrac{\phi''(c_j)}{\psi''(c_j)}, \;\; j = 7, 8, 9.$$

provided that the denominators are non-zero.

REMARK 31. If the inverse of $\frac{\phi''}{\psi''}$ exists, then from the above mean value theorems we can give generalized means

$$c_j = \left(\tfrac{\phi''}{\psi''}\right)^{-1}\left(\tfrac{\Psi_j(.,.,\phi)}{\Psi_j(.,.,\psi)}\right), \;\; j = 1, 2, \dots, 9. \quad □ \tag{30}$$

Definition 32 ([10]). *A function* $\phi : J \to \mathbb{R}$ *is* n-exponentially convex *in the Jensen sense on the interval* J *if*

$$\sum_{k,l=1}^{n} \alpha_k \alpha_l \phi\left(\tfrac{x_k+x_l}{2}\right) \geq 0$$

holds for $\alpha_k \in \mathbb{R}$ *and* $x_k \in J$, $k = 1, 2, ..., n$.

A function $\phi : J \to \mathbb{R}$ *is* n-exponentially convex *if it is* n-exponentially convex *in the Jensen sense and continuous on* J.

Remark 33. From the definition it is clear that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are m-exponentially convex in the Jensen sense for every $m \in \mathbb{N}, m \leq n$. □

Proposition 34. *If* $\phi : J \to \mathbb{R}$ *is an* n*-exponentially convex function, then the matrix* $\left[\phi\left(\frac{x_k+x_l}{2}\right)\right]_{k,l=1}^{m}$ *is a positive semi-definite matrix for all* $m \in \mathbb{N}, m \leq n$. *Particularly,*

$$\det\left[\phi\left(\tfrac{x_k+x_l}{2}\right)\right]_{k,l=1}^{m} \geq 0$$

for all $m \in \mathbb{N}$, $m = 1, 2, ..., n$.

Definition 35. *A function* $\phi : J \to \mathbb{R}$ *is exponentially convex in the Jensen sense on* I *if it is* n*-exponentially convex in the Jensen sense for all* $n \in \mathbb{N}$. *A function* $\phi : J \to \mathbb{R}$ *is exponentially convex if it is exponentially convex in the Jensen sense and continuous.*

Remark 36. It is easy to show that $\phi : [a, b] \to \mathbb{R}^+$ is log-convex in the Jensen sense if and only if

$$\alpha^2\phi(x) + 2\alpha\beta\phi\left(\tfrac{x+y}{2}\right) + \beta^2\phi(y) \geq 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in [a, b]$. It follows that a function is log-convex in the Jensen-sense if and only if it is 2-exponentially convex in the Jensen sense.

Also, using basic convexity theory it follows that a function is log-convex if and only if it is 2-exponentially convex. □

When dealing with functions with different degree of smoothness divided differences are found to be very useful.

Definition 37. *The second order divided difference of a function* $\phi : [a, b] \to \mathbb{R}$ *at mutually different points* $y_0, y_1, y_2 \in [a, b]$ *is defined recursively by*

$$[y_i; \phi] = \phi(y_i), \ i = 0, 1, 2$$

$$[y_i, y_{i+1}; \phi] = \tfrac{\phi(y_{i+1})-\phi(y_i)}{y_{i+1}-y_i}, \ i = 0, 1$$

$$[y_0, y_1, y_2; \phi] = \tfrac{[y_1,y_2;\phi]-[y_0,y_1;\phi]}{y_2-y_0}. \tag{31}$$

REMARK 38. The value $[y_0, y_1, y_2; \phi]$ is independent of the order of the points y_0, y_1, and y_2. By taking limits this definition may be extended to include the cases in which any two or all three points coincide as follows: $\forall y_0,\, y_1,\, y_2 \in [a, b]$

$$\lim_{y_1 \to y_0} [y_0, y_1, y_2; \phi] = [y_0, y_0, y_2; \phi] = \frac{f(y_2)-f(y_0)-\phi^{'}(y_0)(y_2-y_0)}{(y_2-y_0)^2}, \quad y_2 \neq y_0$$

provided that ϕ' exists, and furthermore, taking the limits $y_i \to y_0, i = 1, 2$ in (31), we get

$$[y_0, y_0, y_0; \phi] \;=\; \lim_{y_i \to y_0} [y_0, y_1, y_2; \phi] = \frac{\phi^{''}(y_0)}{2} \text{ for } i = 1, 2$$

provided that $\phi^{''}$ exists on [a,b]. □

We use an idea from [5] to give an elegant method of producing an n-exponentially convex functions and exponentially convex functions applying the functionals $\Psi_j(.,.,\phi)$, $j = 1, \ldots, 9$, on a given family with the same property.

THEOREM 39. *Let $\Lambda = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval $[a, b]$, such that the function $t \to [y_0, y_1, y_2; \phi_t]$ is n-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in [a, b]$. Let $\Psi_j(.,.,\phi_t)$ $(j = 1, 2, \ldots, 9)$ be linear functionals defined as in* (19)–(27). *Then $t \to \Psi_j(.,.,\phi_t)$ is an n-exponentially convex function in the Jensen sense on J. If the function $t \to \Psi_j(.,.,\phi_t)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. Fix $1 \leq j \leq 9$.
Let us define the function

$$\omega(y) = \sum_{k,l=1}^{n} b_k b_l \phi_{t_{kl}}(y),$$

where $t_{kl} = \frac{t_k+t_l}{2}$, $t_k \in J, b_k \in \mathbb{R}$, $k = 1, 2, ..., n$.
Since the function $t \to [y_0, y_1, y_2; \phi_t]$ is n-exponentially convex in the Jensen sense, we have

$$[y_0, y_1, y_2; \omega] = \sum_{k,l=1}^{n} b_k b_l [y_0, y_1, y_2; \phi_{t_{kl}}] \geq 0,$$

which implies that ω is a convex function on $[a, b]$ and therefore we have $\Psi_j(.,.,\omega) \geq 0$; $j = 1, 2, ..., 9$. Hence

$$\sum_{k,l=1}^{n} b_k b_l \Psi_j(.,.,\phi_{t_{kl}}) \geq 0.$$

We conclude that the function $t \to \Psi_j(.,.,\phi_t)$ is an n-exponentially convex function in the Jensen sense on J.

If the function $t \to \Psi_j(.,.,\phi_t)$ is continuous on J, then it is n-exponentially convex on J by definition. $\square$

As a consequence of the above theorem we can give the following corollary.

Corollary 40. *Let $\Lambda = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval $[a,b]$, such that the function $t \to [y_0, y_1, y_2; \phi_t]$ is exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in [a,b]$. Let $\Psi_j(.,.,\phi_t)$ $(j = 1,2,\dots,9)$ be linear functionals defined as in* (19)–(27). *Then $t \to \Psi_j(.,.,\phi_t)$ is an exponentially convex function in the Jensen sense on J. If the function $t \to \Psi_j(.,.,\phi_t)$ is continuous on J, then it is exponentially convex on J.*

Corollary 41. *Let $\Lambda = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval $[a,b]$, such that the function $t \to [y_0, y_1, y_2; \phi_t]$ is 2-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in [a,b]$. Let $\Psi_j(.,.,\phi_t)$ $(j = 1,2,\dots,9)$ be linear functionals defined as in* (19)–(27). *Then the following statements hold:*
(i) *If the function $t \to \Psi_j(.,.,\phi_t)$ is continuous on J, then it is 2-exponentially convex on J, and thus log convex on J.*
(ii) *If the function $t \to \Psi_j(.,.,\phi_t)$ is strictly positive and differentiable on J, then for every $s,t,u,v \in J$, such that $s \le u$ and $t \le v$, we have*

$$\mathfrak{B}_{s,t}(.,.,\Psi_j,\Lambda) \le \mathfrak{B}_{u,v}(.,.,\Psi_j,\Lambda) \tag{32}$$

where

$$\mathfrak{B}^j_{s,t}(\Lambda) = \mathfrak{B}_{s,t}(.,.,\Psi_j,\Lambda) = \begin{cases} \left(\frac{\Psi_j(.,.,\phi_s)}{\Psi_j(.,.,\phi_t)}\right)^{\frac{1}{s-t}}, & s \ne t, \\ \exp\left(\frac{\frac{d}{ds}\Psi_j(.,.,\phi_s)}{\Psi_j(.,.,\phi_s)}\right), & s = t, \end{cases} \tag{33}$$

for $\phi_s, \phi_t \in \Lambda$.

Proof. (i) See Remark 36 and Theorem 39.

(ii) From the definition of convex function ϕ, we have the following inequality [11, p.2]

$$\frac{\phi(s)-\phi(t)}{s-t} \le \frac{\phi(u)-\phi(v)}{u-v}, \tag{34}$$

$\forall\, s,t,u,v \in J$ such that $s \le u$, $t \le v$, $s \ne t$, $u \ne v$.
Since by (i), $\Psi_j(.,.,\phi_s)$ is log-convex, so set $\phi(x) = \ln \Psi_j(.,.,\phi_x)$ in (34) we have

$$\frac{\ln \Psi_j(.,.,\phi_s) - \ln \Psi_j(.,.,\phi_t)}{s-t} \le \frac{\ln \Psi_j(.,.,\phi_u) - \ln \Psi_j(.,.,\phi_v)}{u-v} \tag{35}$$

for $s \le u$, $t \le v$, $s \ne t$, $u \ne v$, which equivalent to (32). The cases for $s = t$, $u = v$ follow from (34) by taking limit.

$\square$

Remark 42. In [1] authors gave related results for the Jensen Mercer inequality. $\square$

6. EXAMPLES

In this section we will vary on choice of family of functions in order to give some examples of exponentially convex functions and to construct some means in the same way as given in [5] and [10]. For simplicity we assume that $J(\mathbf{a}, X, \mathbf{w}) = \sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i x_{ij}$. Let ϕ_t be any function, $t \in J$ where J is an interval in $\mathbb{R}$, we apply the conditions:

$$\lim_{t \to t_0} A(\phi_t) = A(\lim_{t \to t_0} \phi_t),$$

$$\lim_{t \to t_0} \frac{A(\phi_{t+\Delta t}) - A(\phi_t)}{\Delta t} = A\left(\lim_{t \to t_0} \frac{\phi_{t+\Delta t} - \phi_t}{\Delta t}\right).$$

Example 43. Let

$$\Lambda_1 = \{\psi_t : \mathbb{R} \to [0, \infty) : t \in \mathbb{R}\}$$

be the family of functions defined by

$$\psi_t(x) = \begin{cases} \frac{1}{t^2}\, \mathrm{e}^{tx}, & \mathrm{t} \neq 0, \\ \frac{1}{2}\, x^2, & \mathrm{t} = 0. \end{cases}$$

Since, $\psi_t(x)$ is a convex function on $\mathbb{R}$ and $\psi_t''(x)$ is exponentially convex function [5], using analogous arguing as in the proof of Theorems 39 we have that $t \mapsto [y_0, y_1, y_2; \psi_t]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 40 we conclude that $t \mapsto \Psi_j(.,.,\psi_t)$; $j = 1, ..., 9$ are exponentially convex in the Jensen sense. It is easy to see that these mappings are continuous, so they are exponentially convex.

Assume that $t \mapsto \Psi_j(.,.,\psi_t) > 0$ $(j = 1, 2, ..., 9)$. By using this family of convex functions in (30) for $j = 1, 2, \ldots, 9$, we obtain the following means:

$$\Gamma_{s,t}^{j} = \begin{cases} \frac{1}{s-t} \ln\left(\frac{\Psi_j(.,.,\psi_s)}{\Psi_j(.,.,\psi_t)}\right), & s \neq t, \\ \frac{\Psi_j(.,.,id.\psi_s)}{\Psi_j(.,.,\psi_s)} - \frac{2}{s}, & s = t \neq 0, \\ \frac{\Psi_j(.,.,id.\psi_0)}{3\Psi_j(.,.,\psi_0)}, & s = t = 0. \end{cases}$$

In particular for $j = 6$ we have

$$\Gamma_{s,t}^{6} = \frac{1}{s-t} \ln\left(\frac{t^2\left(\sum_{j=1}^{m} e^{sa_j} - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i e^{sx_{ij}} - e^{sJ(\mathbf{a},X,\mathbf{w})}\right)}{s^2\left(\sum_{j=1}^{m} e^{ta_j} - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i e^{tx_{ij}} - e^{tJ(\mathbf{a},X,\mathbf{w})}\right)}\right),\ s \neq t;\ s, t \neq 0,$$

$$\Gamma_{s,s}^{6} = \frac{\sum_{j=1}^{m} a_j e^{sa_j} - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i x_{ij} e^{sx_{ij}} - J(\mathbf{a},X,\mathbf{w}) e^{sJ(\mathbf{a},X,\mathbf{w})}}{\sum_{j=1}^{m} e^{sa_j} - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i e^{sx_{ij}} - e^{sJ(\mathbf{a},X,\mathbf{w})}} - \frac{2}{s},\ s \neq 0,$$

$$\Gamma_{s,0}^{6} = \frac{1}{s} \ln\left(\frac{2\left(\sum_{j=1}^{m} e^{sa_j} - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i e^{sx_{ij}} - e^{sJ(\mathbf{a},X,\mathbf{w})}\right)}{s^2\left(\sum_{j=1}^{m} a_j^2 - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i x_{ij}^2 - J^2(\mathbf{a},X,\mathbf{w})\right)}\right),\ s \neq 0,$$

$$\Gamma_{0,0}^{6} = \frac{\sum_{j=1}^{m} a_j^3 - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i x_{ij}^3 - J^3(\mathbf{a},X,\mathbf{w})}{3(\sum_{j=1}^{m} a_j^2 - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i x_{ij}^2 - J^2(\mathbf{a},X,\mathbf{w}))}.$$

Since $\Gamma^j_{s,t} = \ln \mathfrak{B}^j_{s,t}(\Lambda_1)$ $(j = 1, 2, ..., 9)$, so by (32) these means are monotonic. $\square$

Example 44. Let

$$\Lambda_2 = \{\varphi_t : (0, \infty) \to \mathbb{R} : t \in \mathbb{R}\}$$

be the family of functions defined by

$$\varphi_t(x) = \begin{cases} \frac{x^t}{t(t-1)}, & t \neq 0,1, \\ -\ln x, & t=0, \\ x \ln x, & t=1. \end{cases}$$

Since $\varphi_t(x)$ is a convex function for $x \in \mathbb{R}^+$ and $t \to \varphi_t''(x)$ is exponentially convex, so by the same arguments given in previous example we conclude that $\Psi_j(.,.,\varphi_t)$; $j = 1, ..., 9$ are exponentially convex. We assume that $[a, b] \subset \mathbb{R}^+$ and $\Psi_j(.,.,\varphi_t) > 0 (j = 1, ..., 9)$. By using this family of convex functions in (30) for $j = 1, 2, \ldots, 9$ we have the following means:

$$\tilde{\Gamma}^j_{s,t} = \begin{cases} \left(\frac{\Psi_j(.,.,\varphi_s)}{\Psi_j(.,.,\varphi_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{1-2s}{s(s-1)} - \frac{\Psi_j(.,.,\varphi_0\varphi_s)}{\Psi_j(.,.,\varphi_s)}\right), & s = t \neq 0, 1, \\ \exp\left(1 - \frac{\Psi_j(.,.,\varphi_0{}^2)}{2\Psi_j(.,.,\varphi_0)}\right), & s = t = 0, \\ \exp\left(-1 - \frac{\Psi_j(.,.,\varphi_0\varphi_1)}{2\Psi_j(.,.,\varphi_1)}\right), & s = t = 1. \end{cases}$$

In particular for $j = 6$ we have

$$\tilde{\Gamma}^6_{s,t} = \left(\frac{t(t-1)}{s(s-1)} . \frac{\sum_{j=1}^m a_j^s - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i x_{ij}^s - J^s(\mathbf{a}, X, \mathbf{w})}{\sum_{j=1}^m a_j^t - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i x_{ij}^t - J^t(\mathbf{a}, X, \mathbf{w})}\right)^{\frac{1}{s-t}}, s \neq t;\ s, t \neq 0, 1,$$

$$\tilde{\Gamma}^6_{s,s} = \exp\left(\frac{1-2s}{s(s-1)}\right.$$
$$\left. - \frac{\sum_{j=1}^m \ln a_j a_j^s - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i \ln x_{ij} x_{ij}^s - \ln J(\mathbf{a}, X, \mathbf{w}) J^s(\mathbf{a}, X, \mathbf{w})}{\sum_{j=1}^m a_j^s - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i x_{ij}^s - J^s(\mathbf{a}, X, \mathbf{w})}\right), s \neq 0, 1,$$

$$\tilde{\Gamma}^6_{s,0} = \left(\frac{\sum_{j=1}^m a_j^s - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i x_{ij}^s - J^s(\mathbf{a}, X, \mathbf{w})}{s(1-s)\sum_{j=1}^m \ln a_j - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i \ln x_{ij} - \ln J(\mathbf{a}, X, \mathbf{w})}\right)^{\frac{1}{s}}, s \neq 0,$$

$$\tilde{\Gamma}^6_{s,1} = \left(\frac{\sum_{j=1}^m a_j^s - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i x_{ij}^s - J^s(\mathbf{a}, X, \mathbf{w})}{s(s-1)\sum_{j=1}^m a_j \ln a_j - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i x_{ij} \ln x_{ij} - J(\mathbf{a}, X, \mathbf{w}) \ln J(\mathbf{a}, X, \mathbf{w})}\right)^{\frac{1}{s-1}} s \neq 0,$$

$$\tilde{\Gamma}^6_{0,0} = \exp\left(1 - \frac{\sum_{j=1}^m \ln^2 a_j - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i \ln^2 x_{ij} - \ln^2 J(\mathbf{a}, X, \mathbf{w})}{2(\sum_{j=1}^m \ln a_j - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i \ln x_{ij} - \ln J(\mathbf{a}, X, \mathbf{w}))}\right),$$

$$\tilde{\Gamma}^6_{1,1} = \exp\left(-1 - \frac{\sum_{j=1}^m a_j \ln^2 a_j - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i x_{ij} \ln^2 x_{ij} - J(\mathbf{a}, X, \mathbf{w}) \ln^2 J(\mathbf{a}, X, \mathbf{w})}{2(\sum_{j=1}^m a_j \ln a_j - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i x_{ij} \ln x_{ij} - J(\mathbf{a}, X, \mathbf{w}) \ln J(\mathbf{a}, X, \mathbf{w}))}\right).$$

Since $\tilde{\Gamma}^j_{s,t} = \mathfrak{B}^j_{s,t}(\Lambda_2)$ $(j = 1, 2, ..., 9)$, so by (32) these means are monotonic. $\square$

Example 45. Let

$$\Lambda_3 = \{\theta_t : (0,\infty) \to (0,\infty) : t \in (0,\infty)\}$$

be the family of functions defined by

$$\theta_t(x) = \frac{\mathrm{e}^{-x\sqrt{t}}}{t}.$$

Since $t \to \frac{d^2}{dx^2}\theta_t(x) = \mathrm{e}^{-x\sqrt{t}}$ is exponentially convex, being the Laplace transform of a non-negative function [5], so by same argument given in Example 43 we conclude that $\Psi_j(.,.,\theta_t)$; $j = 1,...,9$ are exponentially convex. We assume that $[a,b] \subset \mathbb{R}^+$ and $\Psi_j(.,.,\theta_t) > 0 (j = 1,...,9)$. For this family of convex functions $\mathfrak{B}^j_{s,t}(\Lambda_3)$ $(j = 1,2,\ldots,9)$ from (33) become

$$\mathfrak{B}^j_{s,t}(\Lambda_3) = \begin{cases} \left(\frac{\Psi_j(.,.,\theta_s)}{\Psi_j(.,.,\theta_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(-\frac{\Psi_j(.,.,id.\theta_s)}{2\sqrt{s}(\Psi_j(.,.,\theta_s))} - \frac{1}{s}\right), & s = t. \end{cases}$$

In particular for $j = 6$ we have

$$\mathfrak{B}^6_{s,t}(\Lambda_3) = \left(\frac{t}{s}.\frac{\sum_{j=1}^m e^{-a_j\sqrt{s}} - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i e^{-x_{ij}\sqrt{s}} - e^{-J(\mathbf{a},X,\mathbf{w})\sqrt{s}}}{\sum_{j=1}^m e^{-a_j\sqrt{t}} - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i e^{-x_{ij}\sqrt{t}} - e^{-J(\mathbf{a},X,\mathbf{w})\sqrt{t}}}\right)^{\frac{1}{s-t}}, \ s \neq t,$$

$$\mathfrak{B}^6_{s,s}(\Lambda_3) =$$

$$= \exp\left(-\frac{1}{2\sqrt{s}}\frac{\sum_{j=1}^m a_j e^{-a_j\sqrt{s}} - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i x_{ij} e^{-x_{ij}\sqrt{s}} - J(\mathbf{a},X,\mathbf{w})e^{-J(\mathbf{a},X,\mathbf{w})\sqrt{s}}}{\sum_{j=1}^m e^{-a_j\sqrt{s}} - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i e^{-x_{ij}\sqrt{s}} - e^{-J(\mathbf{a},X,\mathbf{w})\sqrt{s}}} - \frac{1}{s}\right).$$

Monotonicity of $\mathfrak{B}^j_{s,t}(\Lambda_3)$ follows from (32). By (30)

$$\bar{\Gamma}^j_{s,t} = -(\sqrt{s} + \sqrt{t}) \ln \mathfrak{B}^j_{s,t}(\Lambda_3) \quad (j = 1,2,...,9)$$

defines a class of means. □

Example 46. Let

$$\Lambda_4 = \{\phi_t : (0,\infty) \to (0,\infty) : t \in (0,\infty)\}$$

be the family of functions defined by

$$\phi_t(x) = \begin{cases} \frac{t^{-x}}{(\ln t)^2}, & t \neq 1, \\ \frac{x^2}{2}, & t = 1. \end{cases}$$

Since $\frac{d^2}{dx^2}\phi_t(x) = t^{-x} = \mathrm{e}^{-xlnt} > 0$, for $x > 0$, so by same argument given in Example 43 we conclude that $t \to \Psi_j(.,.,\phi_t)$; $j = 1,...,9$ are exponentially convex. We assume that $[a,b] \subset \mathbb{R}^+$ and $\Psi_j(.,.,\phi_t) > 0 (j = 1,...,9)$. For this family of convex functions $\mathfrak{B}^j_{s,t}(\Lambda_4)$ $(j = 1, 2, \ldots, 9)$ from (33) become

$$\mathfrak{B}^j_{s,t}(\Lambda_4) = \begin{cases} \left(\frac{\Psi_j(.,.,\phi_s)}{\Psi_j(.,.,\phi_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(-\frac{\Psi_j(.,.,id.\phi_s)}{s\Psi_j(.,.,\phi_s)} - \frac{2}{s\ln s}\right), & s = t \neq 1, \\ \exp\left(\frac{1}{3}\frac{\Psi_j(.,.,id.\phi_1)}{\Psi_j(.,.,\phi_1)}\right), & s = t = 1, \end{cases}$$

In particular for $j = 6$ we have

$$\mathfrak{B}^6_{s,t}(\Lambda_4) = \left(\frac{(\ln t)^2}{(\ln s)^2} \cdot \frac{\sum_{j=1}^m s^{-a_j} - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i s^{-x_{ij}} - s^{-J(\mathbf{a},X,\mathbf{w})}}{\sum_{j=1}^m t^{-a_j} - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i t^{-x_{ij}} - t^{-J(\mathbf{a},X,\mathbf{w})}}\right)^{\frac{1}{s-t}}, s \neq t,$$

$$\mathfrak{B}^6_{s,s}(\Lambda_4) = \exp\left(-\frac{1}{s}\frac{\sum_{j=1}^m a_j s^{-a_j} - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i x_{ij} s^{-x_{ij}} - J(\mathbf{a},X,\mathbf{w})s^{-J(\mathbf{a},X,\mathbf{w})}}{\sum_{j=1}^m s^{-a_j} - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i s^{-x_{ij}} - s^{-J(\mathbf{a},X,\mathbf{w})}} - \frac{2}{s\ln s}\right), s \neq 1,$$

$$\mathfrak{B}^6_{s,1}(\Lambda_4) = \left(\frac{2(\sum_{j=1}^m s^{-a_j} - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i s^{-x_{ij}} - s^{-J(\mathbf{a},X,\mathbf{w})})}{(\ln s)^2[\sum_{j=1}^m a_j^2 - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i x_{ij}^2 - J^2(\mathbf{a},X,\mathbf{w})]}\right)^{\frac{1}{s-1}},$$

$$\mathfrak{B}^6_{1,1}(\Lambda_4) = \frac{\sum_{j=1}^m a_j^3 - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i x_{ij}^3 - J^3(\mathbf{a},X,\mathbf{w})}{3(\sum_{j=1}^m a_j^2 - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i x_{ij}^2 - J^2(\mathbf{a},X,\mathbf{w}))}.$$

Monotonicity of $\mathfrak{B}^j_{s,t}(\Lambda_4)$ follows from (32). By (30)

$$\hat{\Gamma}^j_{s,t} = -L(s,t)\ln\mathfrak{B}^j_{s,t}(\Lambda_4) \quad (j = 1, 2, ..., 9)$$

defines a class of means, where $L(s,t)$ is Logarithmic mean defined as:

$$L(s,t) = \begin{cases} \frac{s-t}{\ln s - \ln t}, & \text{s} \neq \text{t}, \\ s, & \text{s=t}. \end{cases}$$

□

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