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LOGARITHMIC MEAN AND WEIGHTED SUM OF GEOMETRIC AND ANTI-HARMONIC MEANS

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Abstract. We consider the problem of finding the optimal values $\alpha,\ \beta \in \mathbb{R}$ for which the inequality

$$\alpha G(a,b) + (1-\alpha)C(a,b) < L(a,b) < \beta G(a,b) + (1-\beta)C(a,b)$$

holds for all $a, b > 0,\ a \neq b$, where $G(a,b), L(a,b)$ and $C(a,b)$ are respectively the geometric, logarithmic and anti-harmonic means of a and b.

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1. INTRODUCTION

Given $a, b > 0$, $a \neq b$, the *geometric*, *logarithmic* and *anti-harmonic* means are defined by

$$G = \sqrt{ab}, \quad L = \tfrac{b-a}{\ln b - \ln a}, \quad C = \tfrac{a^2+b^2}{a+b}.$$

It is well-known that

$$G < L < C. \tag{1}$$

In this paper we find the values of the parameters $\alpha,\ \beta \in \mathbb{R}$ for which the inequality

$$\alpha G(a,b) + (1-\alpha)C(a,b) < L(a,b) < \beta G(a,b) + (1-\beta)C(a,b)$$

holds for all positive numbers $a \neq b$.

Recently, results of this type have been obtained for various triplets of means. Not being exhaustive, we mention Alzer and Qiu [1] for geometric, exponential (identric) and arithmetic means, Xia and Chu [4] for harmonic, logarithmic respectively identric and arithmetic means, and Chu et al. [3] for harmonic, Seiffert and arithmetic means. Several theorems concerning three means chosen from

$$H < G < L < I < A < Q < S < C \tag{2}$$

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are proved in [2]. For those means Symbolic Algebra Program *Maple* was used to find the interval where the parameters α and β can vary, and then the proofs were given.

We can use *Maple* also to understand the expected degree of difficulty of the proof. Doing so, we found that the problem involving the means G, L and C is among the more difficult ones.

2. MAIN RESULT

THEOREM 1. *The inequality*

$$\alpha G(a,b) + (1-\alpha)C(a,b) < L(a,b) < \beta G(a,b) + (1-\beta)C(a,b) \tag{3}$$

holds for all positive numbers $a \neq b$ *if and only if* $\alpha \geq 1$ *and* $\beta < \beta_0$*, where* $\beta_0 = g(x_0) = 0.87002762...$*, with* x_0 *the unique root of* (7) *which is greater than* 1*, and with* g *defined in* (6)*.*

Proof. The double inequality (3) is equivalent to

$$\beta < \frac{C(a,b)-L(a,b)}{C(a,b)-G(a,b)} < \alpha. \tag{4}$$

Without loss of generality, we can consider $0 < a < b$. Denoting by $t = b/a$, $t > 1$, due to the homogeneity of the means, the problem reduces to find $\inf f$ and $\sup f$, where

$$f(t) = \frac{C(1,t)-L(1,t)}{C(1,t)-G(1,t)} = \frac{\left(t^2+1\right)\ln t - t^2 + 1}{\left(\sqrt{t}-1\right)^2\left(t+\sqrt{t}+1\right)\ln t}. \tag{5}$$

The function f is obviously bounded, $0 \leq f(t) \leq 1$. We shall find $\inf f$ and $\sup f$ for $t > 1$.

Define

$$g(x) = f(x^2) = \frac{2\left(x^4+1\right)\ln x - x^4 + 1}{2(x-1)^2(x^2+x+1)\ln x}, \quad x > 1. \tag{6}$$

In order to find $\inf f = \inf g$ and $\sup f = \sup g$ we shall show first that

$$\boxed{g' \text{ has a unique root in } (1,\infty).} \tag{*}$$

Suppose for a moment that this is true and denote by x_0 this root. We have $\lim_{x\to 1} g(x) = 8/9$, $\lim_{x\to\infty} g(x) = 1$, $g(7) = 0.87003995... < 8/9$. It follows that g has a minimal point in $(1,\infty)$, so this point must be x_0. Furthermore, g must be monotonic in $(1, x_0)$ and (x_0,∞) and so $\beta_0 = \inf g = g(x_0)$, $\sup g = \max(1, 8/9) = 1$.

So, it remains to prove (*).

The derivative of g is given by

$$g'(x) = \frac{h(x)}{2\,x(x-1)^3(x^2+x+1)^2(\ln x)^2},$$

where

$$h(x) = \begin{array}{l} -2x\,(x+1)\left(x^4+4x^2+1\right)(\ln x)^2 \\ +x\,(x-1)\left(x^4+2x^3+6x^2+2x+1\right)\ln x \\ +\,(x+1)\left(x^2+x+1\right)\left(x^2+1\right)(x-1)^2 . \end{array} \tag{7}$$

The equation $g'(x) = 0$ is equivalent to $h(x) = 0$, hence to

$$\ln x - \frac{\left(x^5+2\,x^4+6\,x^3+2\,x^2+x+\sqrt{p}\right)(x-1)}{4x(x^5+x^4+4\,x^3+4\,x^2+x+1)} = 0, \tag{8}$$

where

$$p = \begin{array}{l} 8\,x^{11}+25\,x^{10}+76\,x^9+160\,x^8+236\,x^7+286\,x^6 \\ +236\,x^5+160\,x^4+76\,x^3+25\,x^2+8\,x. \end{array} \tag{9}$$

We have considered in (8) the positive root of the quadratic in $\ln x$ equation $h(x) = 0$. Let us denote the left hand side of (8) by $k(x)$.

We have to show that k has a unique root in $(1, \infty)$. To this aim we compute $k'(x)$. The Computer Algebra System *Maple* will help us to do and organize the computations.

We are interested in the numerator of $k'(x)$ expressed in terms of $d = \sqrt{p}$, where p is the polynomial given in (9), i.e.

```
> p:=8*x^11+25*x^10+76*x^9+160*x^8+236*x^7+286*x^6+236*x^5
+160*x^4+76*x^3+25*x^2+8*x;
```

The numerator of $k'(x)$ is given by

```
> numer(normal(subs(p=d^2, normal(diff(k(x),x)))))
        assuming d > 0:
> ndk := collect(%,d);
```

$$\begin{aligned} ndk := \big(&9\,x^2+10\,x^3+20\,x^4+2\,x+2\,x^9+20\,x^6+9\,x^8+10\,x^7+x^{10}+24\,x^5 \\ &+1\big)\,d-1-4\,x-658\,x^8-306\,x^5-586\,x^7-478\,x^6-478\,x^{10}-586\,x^9 \\ &-306\,x^{11}-154\,x^{12}-64\,x^{13}-64\,x^3-22\,x^2-154\,x^4-x^{16}-22\,x^{14}-4\,x^{15} \end{aligned}$$

Therefore the numerator ndk of $k'(x)$ is of the form $p_1 d + p_0$ (p_0 and p_1 being polynomials) and a root of $k'(x)$ must be a root of the polynomial $p_1^2 p - p_0^2$.

We can factorize this polynomial using *Maple*:

```
> p0:= coeff(ndk,d,0):
> p1:= coeff(ndk,d,1):
> factor(p1^2*p-p0^2);
```

$$\begin{gathered} -(x-1)^4\,(x+1)^4\left(x^2+1\right)^2\left(x^2+x+1\right)\left(x^4+4\,x^2+1\right)^2 \\ \left(x^{10}-x^9-3\,x^8-44\,x^7-94\,x^6-150\,x^5-94\,x^4-44\,x^3-3\,x^2-x+1\right) \end{gathered}$$

It follows that any root of $k'(x)$ in $(1, \infty)$ must be a root of the 10th degree polynomial

$$P = x^{10}-x^9-3\,x^8-44\,x^7-94\,x^6-150\,x^5-94\,x^4-44\,x^3-3\,x^2-x+1.$$

But the polynomial P has a unique root in $(1,\infty)$. This can be verified using the Sturm sequence.

Indeed, *Maple* gives:

> `sturm(P,x,0,infinity);`

$$1$$

We conclude that k' has a unique root $r \in (1,\infty)$; actually $r \in (4,5)$ because $k'(4) > 0$, $k'(5) < 0$. So, $k' > 0$ in $(1,r)$ and $k' < 0$ in (r,∞). Since $k(1) = 0$ and $\lim_{x\to\infty} k(x) = -\infty$ it follows that k has a unique root in $(1,\infty)$, actually in (r,∞). So, we have proved (*).

The unique solution x_0 of $g'(x) = 0$ can be easily approximated by using the command

> `Digits:=30:`
> `x0:=fsolve(h(x),x=4..infinity);`

$$x0 := 7.27177296398582281915348781959$$

giving $g(x_0) = 0.87002762....$ □

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