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ON NEWTON'S METHOD USING RECURRENT FUNCTIONS UNDER HYPOTHESES UP TO THE SECOND FRÉCHET DERIVATIVE

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Abstract. We provide semilocal result for the convergence of Newton method to a locally unique solution of an equation in a Banach space setting using hypotheses up to the second Fréchet–derivatives and our new idea of recurrent functions. The advantages of such conditions over earlier ones in some cases are: finer bounds on the distances involved, and a better information on the location of the solution.

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1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where F is a twice Fréchet–differentiable operator defined on a open convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

Computational sciences have received substantial and significant interest of researchers in recent years in several areas such as engineering sciences, dynamical systems, economic equilibrium theory and mathematical programming. Various problems can be solved using the computational sciences by passing first through mathematical modeling and then later looking for the solution iteratively [4], [5]. For example, dynamic systems are mathematically modeled by difference or differential equations and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = Q(x)$, for some suitable operator Q, where x is the state. Then the equilibrium states are determined

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by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative–when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand.

The famous Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \quad (n \geq 0), \quad (x_0 \in \mathcal{D}) \tag{1.2}$$

has long played a central role in approximating solutions $x^\star$ of nonlinear equations and systems. Here $F'(x_n)$ denotes the Fréchet-derivative of operator F evaluated at $x = x_n$ $(n \geq 0)$ [4], [10]. The geometric interpretation of Newton's method is well known, if F is a real function. In such a case x_{n+1} is the point where the tangential line $y - F(x_n) = F'(x_n)\ (x - x_n)$ of function $F(x_n)$ at the point $(x_n, F(x_n))$ intersects the x–axis.

Local and semilocal convergence theorems for the quadratic convergence of Newton's method to $x^\star$ have been given under several assumptions by various authors [1]–[11]. For example, Lipschitz conditions have been used on the Fréchet–derivative $F'(x)$ of F $(x \in \mathcal{D})$ [1], [2], [9]–[11], or center-Lipschitz conditions on the second-derivative $F''(x)$ of F $(x \in \mathcal{D})$ [1]–[11]. Here, we use center-Lipschitz conditions on both first and second Fréchet-derivatives of F and recurrent functions. This particular combination has several advantages over the previously mentioned works. That is why we provide new semilocal convergence theorems for Newton's method.

In particular, assume the Lipschitz condition:

$$\parallel F'(x_0)^{-1}\ (F'(x) - F'(y)) \parallel \leq d\ \parallel x - y \parallel, \tag{1.3}$$

for all x, y in $\mathcal{D}$ and

$$\parallel F'(x_0)^{-1}\ F(x_0) \parallel \leq \eta.$$

Then, we arrive at the famous for its simplicity and clarity Kantorovich hypothesis for the semilocal convergence of Newton's method (see [4], [5], [10]):

$$h_K = d\ \eta \leq \tfrac{1}{2}. \tag{1.4}$$

In view of (1.3), there exists $b \geq 0$, such that center-Lipschitz condition

$$\parallel F'(x_0)^{-1}\ (F'(x) - F'(x_0)) \parallel \leq b\ \parallel x - x_0 \parallel$$

for all $x \in \mathcal{D}$.

Clearly,

$$b \leq d$$

holds in general and $\frac{d}{b}$ can be arbitrarily large [4], [6].

Note that in practice the computation of Lipschitz constant d requires that of b. Therefore, the introduction of the center-Lipschitz condition is not an additional hypothesis. Condition (1.3) is exclusively used in the literature to obtain upper bounds on the norms $\parallel F'(x_n)^{-1} \ F'(x_0) \parallel$ $(n \geq 1)$. In particular, if $x \in \mathcal{D}_0 = U(x_0, \frac{1}{d}) \subseteq \mathcal{D}$, $d \neq 0$, we obtain using (1.3):

$$\parallel F'(x_0)^{-1} \ (F'(x) - F'(x_0)) \parallel \leq d \ \parallel x - x_0 \parallel < 1. \tag{1.5}$$

In view of (1.5) and the Banach lemma on invertible operator [4], we conclude that $F'(x)^{-1}$ exists on $\mathcal{D}_0$ and

$$\parallel F'(x)^{-1} \ (F'(x_0) \parallel \leq \frac{1}{1 - d \ \|x - x_0\|}. \tag{1.6}$$

The usage of estimate (1.6) and several majorizing techniques all lead to condition (1.4) [4]–[7].

It is clear that (1.3) is overused, when it comes to obtaining estimate (1.6). Using the needed center-Lipschitz condition, we arrive at the more precise estimate (for $b < d$):

$$\parallel F'(x)^{-1} \ F'(x_0) \parallel \leq \frac{1}{1 - b \ \|x - x_0\|}. \tag{1.7}$$

Using (1.7) instead of (1.6) and our new idea of recurrent functions, we showed that (1.4) can always be replaced by

$$h_A = d_0 \ \eta \leq \frac{1}{2}, \tag{1.8}$$

where,

$$d_0 = \frac{1}{8} \ (d + 4 \ b + \sqrt{d^2 + 8 \ b \ d}).$$

Note that

$$h_K \leq \frac{1}{2} \Longrightarrow h_A \leq \frac{1}{2}$$

but not necessarily vice versa, unless $b = d$.

In this case, finer errors bounds on the distances $\parallel x_{n+1} - x_n \parallel$, $\parallel x_n - x^\star \parallel$ $(n \geq 0)$ and an at least as precise information on the location of the solution than the ones provided by the Newton-Kantorovich theorem [4], are also obtained.

Hence, the applicability of Newton's method for solving nonlinear equations has been extended and under the same computational cost.

The idea of introducing recurrent functions is a logical consequence of proving the convergence of the majorizing sequence $\{\varsigma_n\}$ by showing:

$$0 \leq \varsigma_{n+1} - \varsigma_n \leq \epsilon \ (\varsigma_n - \varsigma_{n-1}), \quad (n \geq 1)$$

where, $\{\varsigma_n\}$ is given by

$$\varsigma_0 = 0, \qquad \varsigma_1 = \eta, \quad \varsigma_{n+1} = \varsigma_n + \frac{d \ (\varsigma_n - \varsigma_{n-1})^2}{2 \ (1 - b \ \varsigma_n)}, \quad (n \geq 1)$$

and

$$\epsilon = \frac{2 \ d}{d + \sqrt{d^2 + 8 \ b \ d}}.$$

It turns out that the idea of using the center-Lipschitz condition in combination with recurrent functions can be used when (1.7) also replaces the corresponding estimates by Huang [9] and Gutiérrez [8], (see (2.68), (2.71) and Proposition 6), which are using hypotheses on the second Fréchet–derivative $F''(x)$ $(x \in \mathcal{D})$ of operator F.

The advantages of this approach over the works by Huang [9] and Gutiérrez [8] can be (as in the case already stated above) under the same or weaker hypotheses:

(a) Finer error bounds on the distances $\parallel x_{n+1} - x_n \parallel$, $\parallel x_n - x^\star \parallel$ $(n \geq 0)$;
(b) Better information on the location of the solution $x^\star$.

2. SEMILOCAL CONVERGENCE ANALYSIS OF NEWTON'S METHOD

Let $a > 0$, $b > 0$, $c > 0$ and $\eta > 0$ be given constants. It is convenient for us to define the polynomial g on $[0, +\infty)$ by

$$g(s) = 2\ b\ s^2 + c\ s - (c + a\ n). \tag{2.1}$$

Assume:

$$\eta < \delta_1 = \tfrac{2\ b}{a}. \tag{2.2}$$

Then, the polynomial g has a unique root $\delta/2$ in $(0, 1)$. We have

$$g(0) = -(c + a\ \eta) < 0 \tag{2.3}$$

and

$$g(1) = 2\ b - a\ \eta > 0 \quad \text{(by (2.2))}. \tag{2.4}$$

It follows from the Intermediate Value Theorem, (2.3) and (2.4) that there exists a root $\frac{\delta}{2} \in (0, 1)$ of the polynomial g, given by

$$\delta = \tfrac{4(c + a\ \eta)}{c + \sqrt{c^2 + 8\ b\ (c + a\ \eta)}}. \tag{2.5}$$

We also have

$$g'(s) = 4\ b\ s + c > 0 \quad (s > 0). \tag{2.6}$$

That is, the polynomial g crosses the positive-axis only once. Hence, δ is the only root of the polynomial g on $(0, 1)$. Let us define the polynomial p on $[0, +\infty)$ by

$$p(s) = \tfrac{1}{3}\ a\ s^2 + (c + \delta\ b) s - \delta. \tag{2.7}$$

The polynomial p has also a unique positive root, given by

$$\delta_2 = \tfrac{2\ \delta}{c + \delta\ b + \sqrt{(c + \delta\ b)^2 + \frac{4}{3}\ a\ \delta}}. \tag{2.8}$$

Let us also define

$$\delta_3 = \tfrac{1}{b}(1 - \tfrac{\delta}{2}) \tag{2.9}$$

and set

$$\eta_0 = \min\{\delta_1, \delta_2, \delta_3\}. \tag{2.10}$$

We can show the following result on majorizing sequences for Newton's method (1.2).

Lemma 1. *Let $a > 0$, $b > 0$, $c > 0$ and $\eta > 0$ be given constants. Assume:*

$$\eta \le \eta_0, \tag{2.11}$$

where η_0 is defined in (2.10). *Inequality* (2.11) *is strict if $\eta_0 = \delta_1$.*
Then, scalar sequence $\{t_n\}$ ($n \ge 0$), generated by

$$\begin{aligned} &t_0 = 0, \qquad t_1 = \eta, \\ &t_{n+2} = t_{n+1} + \frac{a\left(\frac{1}{3}\,(t_{n+1}-t_n)+t_n\right)+c}{2\,(1-b\,t_{n+1})}\,(t_{n+1}-t_n)^2 \end{aligned} \tag{2.12}$$

is increasing, bounded from above by

$$t^{\star\star} = \frac{2\,\eta}{2-\delta} \tag{2.13}$$

and converges to its unique least upper bound $t^\star$, with

$$t^\star \in [0, t^{\star\star}]. \tag{2.14}$$

Moreover the following estimates hold for all $n \ge 0$:

$$0 < t_{n+2} - t_{n+1} \le \frac{\delta}{2}\,(t_{n+1}-t_n) \le \left(\frac{\delta}{2}\right)^{n+1} \eta \tag{2.15}$$

and

$$t^\star - t_n \le \frac{2\,\eta}{2-\delta}\left(\frac{\delta}{2}\right)^n. \tag{2.16}$$

Proof. We shall show using induction on m:

$$\begin{aligned} 0 &< t_{m+2} - t_{m+1} \\ &= \frac{a\left(\frac{1}{3}\,(t_{m+1}-t_m)^2 + t_m\,(t_{m+1}-t_m)\right)+c\,(t_{m+1}-t_m)}{2\,(1-b\,t_{m+1})}\,(t_{m+1}-t_m) \\ &\le \frac{\delta}{2}\,(t_{m+1}-t_m) \end{aligned} \tag{2.17}$$

and

$$b\,t_{m+1} < 1. \tag{2.18}$$

Estimates (2.17) and (2.18) hold for $m = 0$, since

$$\frac{a\left(\frac{1}{3}\,(t_1-t_0)^2 + t_0\,(t_1-t_0)\right)+c\,(t_1-t_0)}{1-b\,t_1} = \frac{\frac{1}{3}\,a\,\eta^2 + c\,\eta}{1-b\,\eta} = \delta_0 \le \delta \tag{2.19}$$

and

$$b\ t_1 = b\ \eta < 1, \tag{2.20}$$

by the choice of δ and (2.11).

Let us assume (2.17)–(2.18) for all $n \leq m+1$. Then, we get from (2.15):

$$t_{m+2} \leq \frac{1-\left(\frac{\delta}{2}\right)^{m+2}}{1-\frac{\delta}{2}}\ \eta < \frac{2\ \eta}{2-\delta} = t^{\star\star}. \tag{2.21}$$

We shall show (2.19) and (2.20), if

$$\begin{aligned} &a\left\{\frac{1}{3}\left(\left(\frac{\delta}{2}\right)^m \eta\right)^2 + \frac{1-\left(\frac{\delta}{2}\right)^m}{1-\frac{\delta}{2}}\left(\frac{\delta}{2}\right)^m \eta^2\right\} \\ &\quad + c\left(\frac{\delta}{2}\right)^m \eta + b\ \delta\ \frac{1-\left(\frac{\delta}{2}\right)^{m+1}}{1-\frac{\delta}{2}}\ \eta - \delta \leq 0. \end{aligned} \tag{2.22}$$

Estimates (2.22) motivates us to define polynomials f_m ($m \geq 1$), for $s = \frac{\delta}{2}$ and show instead:

$$\begin{aligned} f_m(s) \ =\ & \frac{a\ \eta^2}{3}\ s^{2\,m-1} + a\ (1+s+\cdots+s^{m-1})\ s^{m-1}\ \eta^2 \\ & + c\ s^{m-1}\ \eta + 2\ b\ (1+s+\cdots+s^m)\ \eta - 2 \leq 0. \end{aligned} \tag{2.23}$$

We need a relationship between two consecutive functions f_m:

(2.24)

$$\begin{aligned} f_{m+1}(s) \ =\ & \frac{a\ \eta^2}{3}\ s^{2\,m+1} + a\ (1+s+\cdots+s^{m-1}+s^m)\ s^m\ \eta^2 \\ & + c\ s^m\ \eta + 2\ b\ (1+s+\cdots+s^m+s^{m+1})\ \eta - 2 \\ =\ & \frac{a\ \eta^2}{3}\ s^{2\,m+1} + \frac{a\ \eta^2}{3}\ s^{2\,m-1} - \frac{a\ \eta^2}{3}\ s^{2\,m-1} \\ & + a\ (1+s+\cdots+s^{m-1}+s^{m-1})\ s^{m-1}\ \eta^2 \\ & - a\ (1+s+\cdots+s^{m-1}+s^{m-1})\ s^{m-1}\ \eta^2 \\ & + a\ (1+s+\cdots+s^{m-1}+s^m)\ s^m\ \eta^2 + c\ s^{m-1}\ \eta - c\ s^{m-1}\ \eta \\ & + c\ s^m\ \eta + 2\ b\ (1+s+\cdots+s^m)\ \eta + 2\ b\ s^{m+1}\ \eta - 2 \\ =\ & f_m(s) + g_m(s) + g(s)\ s^{m-1}\ \eta, \end{aligned}$$

where, the function g is given by (2.11) and

$$g_m(s) = \left(\frac{1}{3}\ s^2 + s + \frac{2}{3}\right) s^{2\,m-1}\ \eta^2\ a \geq 0, \quad (s \geq 0). \tag{2.25}$$

In view of (2.1), (2.24) and (2.25) we have

$$f_{m+1}\left(\frac{\delta}{2}\right) \geq f_m\left(\frac{\delta}{2}\right). \tag{2.26}$$

We shall show, instead of (2.22)

$$f_m\left(\frac{\delta}{2}\right) \leq 0 \qquad (m \geq 1). \tag{2.27}$$

Define the function f_∞ on $[0,1)$ by

$$f_\infty(s)=\lim_{n\to\infty} f_n(s). \tag{2.28}$$

Then, using (2.23), we have:

$$f_\infty(s)=2\left[\frac{b\,\eta}{1-s}-1\right]. \tag{2.29}$$

It also follows from (2.26) that

$$f_\infty\left(\frac{\delta}{2}\right)\geq f_m\left(\frac{\delta}{2}\right). \tag{2.30}$$

In view of (2.27) and (2.30), it is enough to show

$$f_\infty\left(\frac{\delta}{2}\right)\leq 0, \tag{2.31}$$

which is true, since $\eta\leq\delta_3$. This completes the induction.

Therefore, the sequence $\{t_n\}$ is non–decreasing, bounded above by $t^{\star\star}$, given by (2.13) and converges to its unique least upper bound $t^\star$ satisfying (2.14). Finally, estimate (2.16) follows from (2.15) by using standard majorization techniques [4], [10].

That completes the proof of Lemma 1. □

Let us define functions $\overline{f}_m$ by:

$$\begin{aligned}\overline{f}_m(s) &= a\left(\tfrac{1}{3}\,s^{2\,m-1}+(1+s+\cdots+s^{m-1})\,s^{m-1}\right)\eta^2\\ &\quad +c\,s^{m-1}\,\eta+2\,c\,(1+s+\cdots+s^m)\,\eta\\ &\quad +a\,(1+s+\cdots+s^m)^2\,\eta^2-2.\end{aligned} \tag{2.32}$$

Then, we have as in Lemma 1:

$$\overline{f}_{m+1}(s)=\overline{f}_m(s)+\overline{g}_m(s)+\overline{g}(s)\,s^{m-1}\,\eta, \tag{2.33}$$

where,

$$\begin{aligned}\overline{g}_m(s) = &\left(\tfrac{a}{3}\,s^{m+2}+a\,\eta\,s^{m+3}+2\,a\,\eta\,(1+s+\cdots+s^m)\,s^2\right.\\ &\left.+\tfrac{2}{3}\,\eta\,s^m+a\,\eta^{m+1}\right)s^{m-1}\,\eta>0 \qquad (s>0)\end{aligned}$$

and

$$\overline{g}(s)=2\,c\,s^2+c\,s-(c+a\,\eta). \tag{2.34}$$

We also have

$$\overline{p}(s)=\tfrac{a}{6}\,(2+3\,\delta)\,s^2+c\,(1+\delta)\,s-\delta \tag{2.35}$$

and

$$\overline{f}_\infty(s)=\frac{2\,c\,\eta}{1-s}+a\left(\frac{1}{1-s}\right)^2\eta^2-2. \tag{2.36}$$

Moreover, we obtain:

$$\begin{array}{rcl}\overline{\delta}_1 & = & \frac{2\,c}{a},\\ \overline{\delta}_2 & = & \frac{2\,\delta\,(2+3\,\delta)}{c\,(1+\delta)+\sqrt{(c\,(1+\delta))^2+\frac{2}{3}\,a\,\delta\,(2+3\delta)}},\\ \overline{\delta}_3 & = & \frac{2-\delta}{c+\sqrt{c^2+2\,a}} \quad \text{and}\\ \frac{\overline{\delta}}{2} & = & \frac{2\,(c+a\,\eta)}{c+\sqrt{c^2+8\,c\,(c+a\,\eta)}}.\end{array}$$

Set

$$\overline{\eta}_0=\min\{\overline{\delta}_1,\overline{\delta}_2,\overline{\delta}_3\}. \tag{2.37}$$

Then, with the above changes and simply following the proof of Lemma 1, we can provide another majorizing sequence result for Newton's method (1.2):

LEMMA 2. *Let $a>0$, $b>0$, $c>0$ and $\eta>0$ be given constants. Assume:*

$$\eta\le\overline{\eta}_0. \tag{2.38}$$

Inequality (2.38) *is strict if $\overline{\eta}_0=\overline{\delta}_1$.*
Then, scalar sequence $\{v_n\}$ $(n\ge 0)$, given by

$$\begin{array}{l} v_0=0, \qquad v_1=\eta,\\ v_{n+2}=v_{n+1}+\dfrac{a\left(\frac{1}{3}\,(v_{n+1}-v_n)+v_n\right)+c}{2\,(1-c\,v_{n+1}-\frac{a}{2}\,v_{n+1}^2)}\,(v_{n+1}-v_n)^2\end{array} \tag{2.39}$$

is increasing, bounded from above by $t^{\star\star}=\frac{2\,\eta}{2-\delta}$ and converges to its unique least upper bound $v^\star$ satisfying $v^\star\in[0,t^{\star\star}]$.
Moreover the following estimates hold for all $n\ge 0$:

$$0<v_{n+2}-v_{n+1}\le\frac{\overline{\delta}}{2}\,(v_{n+1}-v_n)\le\left(\frac{\overline{\delta}}{2}\right)^{n+1}\eta \tag{2.40}$$

and

$$v^\star-v_n\le\frac{2\,\eta}{2-\delta}\left(\frac{\overline{\delta}}{2}\right)\eta. \tag{2.41}$$

Below is the main semilocal convergence theorem's method involving twice Fréchet differentiable operator and using center-Lipschitz conditions.

THEOREM 3. *Let $F:\mathcal{D}\subseteq\mathcal{X}\longrightarrow\mathcal{Y}$ be a twice Fréchet differentiable operator.*

Assume there exist a point $x_0 \in \mathcal{D}$; constants $\eta > 0$, $a > 0$, $b > 0$, $c > 0$, such that for all $x \in \mathcal{D}$:

$$F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X}), \tag{2.42}$$

$$\| F'(x_0)^{-1} \ F(x_0) \| \leq \eta, \tag{2.43}$$

$$\| F'(x_0)^{-1}[F'(x) - F'(x_0)] \| \leq b \ \| x - x_0 \|, \tag{2.44}$$

$$\| F'(x_0)^{-1} \ F''(x_0) \| \leq c, \tag{2.45}$$

$$\| F'(x_0)^{-1}[F''(x) - F''(x_0)] \| \leq a \ \| x - x_0 \|, \tag{2.46}$$

$$\overline{U}(x_0, t^\star) = \{x \in \mathcal{X}, \ \| x - x_0 \| \leq t^\star\} \subseteq \mathcal{D} \tag{2.47}$$

and hypotheses of Lemma 1 hold.

Then the sequence $\{x_n\}$ defined by Newton's method (1.2) is well defined, remains in $\overline{U}(x_0, t^\star)$ for all $n \geq 0$ and converges to a unique solution $x^\star \in \overline{U}(x_0, t^\star)$ of equation $F(x) = 0$.

Moreover the following estimates hold for all $n \geq 0$:

$$\begin{aligned} \| x_{n+2} - x_{n+1} \| &\leq \frac{a\left(\frac{1}{6} \ \| x_{n+1} - x_n \| + \frac{1}{2} \ \| x_n - x_0 \| \right) + \frac{c}{2}}{1 - b \ \|x_{n+1} - x_0\|} \ \| x_{n+1} - x_n \|^2 \\ &\leq \frac{a\left(\frac{1}{6} \ (t_{n+1} - t_n) + \frac{1}{2} \ (t_n - t_0)\right) + \frac{c}{2}}{1 - b \ (t_{n+1} - t_0)} \ (t_{n+1} - t_n)^2 \qquad (2.48) \\ &= t_{n+2} - t_{n+1} \qquad (2.49) \end{aligned}$$

and

$$\| x_n - x^\star \| \leq t^\star - t_n, \tag{2.50}$$

where, sequence $\{t_n\}$ $(n \geq 0)$ is given by (2.12).

Furthermore, if there exists $R \geq t^\star$, such that

$$U(x_0, R) \subseteq \mathcal{D} \tag{2.51}$$

and

$$b \ (t^\star + R) \leq 2. \tag{2.52}$$

The solution $x^\star$ is unique in $U(x_0, R)$.

Proof. Let us prove that:

$$\| x_{k+1} - x_k \| \leq t_{k+1} - t_k \tag{2.53}$$

and

$$\overline{U}(x_{k+1}, t^\star - t_{k+1}) \subseteq \overline{U}(x_k, t^\star - t_k) \tag{2.54}$$

hold for all $k \geq 0$.

For every $z \in \overline{U}(x_1, t^\star - t_1)$,

$$\begin{aligned} \| z - x_0 \| &\leq \| z - x_1 \| + \| x_1 - x_0 \| \\ &\leq (t^\star - t_1) + (t_1 - t_0) = t^\star - t_0, \end{aligned}$$

implies $z \in \overline{U}(x_0, t^\star - t_0)$. Since, also

$$\| x_1 - x_0 \| = \| F'(x_0)^{-1} F(x_0) \| \leq \eta = t_1 - t_0,$$

estimates (2.53) and (2.54) hold for $k = 0$.

Given they hold for $n = 0, 1, \cdots, k$, then we have:

$$\| x_{k+1} - x_0 \| \leq \sum_{i=1}^{k+1} \| x_i - x_{i-1} \| \leq \sum_{i=1}^{k+1} (t_i - t_{i-1}) = t_{k+1} - t_0 = t_{k+1} \tag{2.55}$$

and

$$\begin{aligned} \| x_k + \theta\,(x_{k+1} - x_k) - x_0 \| &\leq t_k + \theta\,(t_{k+1} - t_k) \\ &\leq t^\star, \end{aligned} \tag{2.56}$$

for all $\theta \in [0, 1]$.

Using (1.2), we obtain the approximation

$$\begin{aligned} F(x_{k+1}) &= F(x_{k+1}) - F(x_k) - F'(x_k)\,(x_{k+1} - x_k) \\ &= \int_0^1 [F'(x_k + \theta\,(x_{k+1} - x_k)) - F'(x_k)]\ (x_{k+1} - x_k)\ \mathrm{d}\theta \\ &= \int_0^1 F''(x_k + \theta\,(x_{k+1} - x_k))\ (1 - \theta)\ (x_{k+1} - x_k)^2\ \mathrm{d}\theta. \end{aligned} \tag{2.57}$$

Then, we get by (2.45), (2.46) and (2.47):

$$\begin{aligned} &\| F'(x_0)^{-1} F(x_{k+1}) \| \\ &\leq \int_0^1 \Big(\| F'(x_0)^{-1}\ [F''(x_k + \theta\,(x_{k+1} - x_k)) - F''(x_0)] \| \\ &\quad + \| F'(x_0)^{-1}\ F''(x_0) \| \Big)\ \| x_{k+1} - x_k \|^2\ (1 - \theta)\ \mathrm{d}\theta \\ &\leq \left\{ a \left(\int_0^1 \| x_k - x_0 \| + \theta\ \| x_{k+1} - x_k \|\ (1 - \theta)\ \mathrm{d}\theta \right) + \tfrac{c}{2} \right\}\ \| x_{k+1} - x_k \|^2 \\ &\leq \tfrac{a}{6}\ \| x_{k+1} - x_k \|^3 + \tfrac{a}{2}\ \| x_k - x_0 \|\, \| x_{k+1} - x_k \|^2 + \tfrac{c}{2}\ \| x_{k+1} - x_k \|^2 \\ &\leq \left\{ a \left(\tfrac{1}{6}\ (t_{k+1} - t_k) + \tfrac{1}{2}\ (t_k - t_0) \right) + \tfrac{c}{2} \right\}\ (t_{k+1} - t_k)^2. \end{aligned} \tag{2.58}$$

Using (2.44), we obtain:

$$\begin{aligned} \| F'(x_0)^{-1}\ (F'(x_{k+1}) - F'(x_0)) \| &\leq b\ \| x_{k+1} - x_0 \| \\ &\leq b\ t_{k+1} \leq b\ t^\star < 1. \end{aligned} \tag{2.59}$$

It follows from the Banach lemma on invertible operators [4], [10] and (2.59) that $F'(x_{k+1})^{-1}$ exists and

$$\begin{aligned} \| F'(x_{k+1})^{-1} F'(x_0) \| &\leq (1 - b \ \| x_{k+1} - x_0 \|)^{-1} \\ &\leq (1 - b\, t_{k+1})^{-1}. \end{aligned} \tag{2.60}$$

Therefore, by (1.2), (2.58) and (2.60), we obtain in turn:

$$\begin{aligned} \| x_{k+2} - x_{k+1} \| &= \| F'(x_{k+1})^{-1} F(x_{k+1}) \| \\ &\leq \| F'(x_{k+1})^{-1} F'(x_0) \| \, \| F'(x_0)^{-1} F(x_{k+1}) \| \\ &\leq t_{k+2} - t_{k+1}. \end{aligned} \tag{2.61}$$

Thus for every $z \in \overline{U}(x_{k+2}, t^\star - t_{k+2})$, we have:

$$\begin{aligned} \| z - x_{k+1} \| &\leq \| z - x_{k+2} \| + \| x_{k+2} - x_{k+1} \| \\ &\leq t^\star - t_{k+2} + t_{k+2} - t_{k+1} = t^\star - t_{k+1}. \end{aligned} \tag{2.62}$$

That is,

$$z \in \overline{U}(x_{k+1}, t^\star - t_{k+1}). \tag{2.63}$$

Estimates (2.60) and (2.63) imply that (2.53) and (2.54) hold for $n = k+1$. The proof of (2.53) and (2.54) is now complete by induction.

Lemma 1 implies that sequence $\{t_n\}$ is a Cauchy sequence. From (2.53) and (2.54) $\{x_n\}$ $(n \geq 0)$ become a Cauchy sequence too and as such it converges to some $x^\star \in \overline{U}(x_0, t^\star)$ (since $\overline{U}(x_0, t^\star)$ is a closed set). Estimate (2.50) follows from (2.49) by using standard majorization techniques [4], [10].

Moreover, by letting $k \to \infty$ in (2.58), we obtain $f(x^\star) = 0$. Finally, to show uniqueness: let $y^\star$ be a solution of equation $F(x) = 0$ in $U(x_0, R)$. It follows from (2.44) for $x = y^\star + \theta\,(x^\star - y^\star)$, $\theta \in [0,1]$, the estimate:

$$\begin{aligned} &\| F'(x_0)^{-1} \int_0^1 (F'(y^\star + \theta\,(x^\star - y^\star)) - F'(x_0)) \ \| \ \mathrm{d}\theta \\ &\leq b \int_0^1 \| y^\star + \theta\,(x^\star - y^\star) - x_0 \| \ \mathrm{d}\theta \\ &\leq b \int_0^1 (\theta \, \| x^\star - x_0 \| + (1-\theta) \ \| y^\star - x_0 \|)\, \mathrm{d}\theta \\ &< \tfrac{b}{2}\,(t^\star + R) \leq 1, \quad \text{(by (2.52))} \end{aligned}$$

and the Banach lemma on invertible operators implies that the linear operator $\mathcal{M} = \int_0^1 F'(y^\star + \theta \ (x^\star - y^\star))\,\mathrm{d}\theta$ is invertible. Using the identity $0 = F(x^\star) - F(y^\star) = \mathcal{M}\,(x^\star - y^\star)$, we deduce $x^\star = y^\star$.

Similarly, we show the uniqueness in $\overline{U}(x_0, t^\star)$ using (2.52).

That completes the proof of Theorem 3. □

REMARK 4. The conclusions of Theorem 3 hold if (2.44) is dropped from the hypotheses and Lemma 1, $\{t_n\}$, $t^\star$ are replaced by Lemma 2, $\{v_n\}$, $v^\star$, respectively. Indeed, we have: □

THEOREM 5. *Let $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be a twice Fréchet differentiable operator.*

Assume hypotheses of Lemma 2 hold and there exist a point $x_0 \in \mathcal{D}$, a constants $\eta > 0$, $a > 0$ and $c > 0$, such that for all $x \in \mathcal{D}$:

$$\begin{aligned}
&F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X}),\\
&\| F'(x_0)^{-1} \ F(x_0) \| \leq \eta,\\
&\| F'(x_0)^{-1} \ F''(x_0) \| \leq c,\\
&\| F'(x_0)^{-1}[F''(x) - F''(x_0)] \| \leq a \ \ \| x - x_0 \|,\\
&\overline{U}(x_0, v^\star) \subseteq \mathcal{D},
\end{aligned}$$

where, $v^\star$ is given in Lemma 2.

Then sequence $\{x_n\}$ defined by Newton's method (1.2) is well defined, remains in $\overline{U}(x_0, v^\star)$ for all $n \geq 0$ and converges to a unique solution $x^\star \in \overline{U}(x_0, v^\star)$ of equation $F(x) = 0$.

Moreover the following estimates hold for all $n \geq 0$:

$$\| x_{n+2} - x_{n+1} \| \leq v_{n+1} - v_n \tag{2.64}$$

and

$$\| x_n - x^\star \| \leq v_{n+2} - v_{n+1}, \tag{2.65}$$

where, sequence $\{v_n\}$ $(n \geq 0)$ is given by (2.39).

Proof. We proceed as in the proof of Theorem 3 until (2.58). Then use (2.45) and (2.46) (instead of (2.44)) to obtain in turn:

$$\begin{aligned}
&\| F'(x_0)^{-1} \ (F'(x_{k+1}) - F'(x_0)) \| \\
&= \| \int_0^1 F'(x_0)^{-1} \ (F''(x_0 + t\,(x_{k+1} - x_0)) - F''(x_0)) \ \mathrm{d}t \ (x_{k+1} - x_0)\\
&\quad + F'(x_0)^{-1} \ F''(x_0) \ (x_{k+1} - x_0) \| \\
&\leq \int_0^1 a \ t \ \mathrm{d}t \ \ \| x_{k+1} - x_0 \|^2 + c \ \ \| x_{k+1} - x_0 \| \\
&\leq \tfrac{a}{2} \ v_{k+1}^2 + c \ v_{k+1} < 1.
\end{aligned} \tag{2.66}$$

It follows from (2.66) and the Banach lemma on invertible operators, that $F'(x_{k+1})^{-1}$ exists and

$$\| F'(x_{k+1})^{-1} \ F'(x_0) \| \leq (1 - c \ v_{k+1} - \tfrac{a}{2} \ v_{k+1}^2)^{-1}. \tag{2.67}$$

The rest of the proof follows as in the proof of Theorem 3, with (2.67) replacing (2.60) until the uniqueness part.

Let $y^\star$ be a solution of equation $F(x)=0$ in $\overline{U}(x_0,t^\star)$. Then, since

(2.68)
$$\begin{aligned}y^\star - x_{k+1} &= y^\star - x_k + F'(x_k)^{-1}\ F(x_k)\\ &= -F'(x_k)^{-1}\ (F(y^\star) - F(x_k) - F'(x_k)\ (y^\star - x_k))\\ &= (-F'(x_0)^{-1}\ F'(x_k))^{-1}\\ &\quad \int_0^1 F'(x_0)^{-1}\ F''(x_k + t\ (y^\star - x_k))\ (1-t)\ \mathrm{d}t\ (y^\star - x_k)^2\end{aligned}$$

and

$$\| y^\star - x_0 \| \leq v^\star - v_0,$$

we obtain:

$$\| x_k - y^\star \| \leq v^\star - v_k, \tag{2.69}$$

which leads to $\lim\limits_{k\longrightarrow\infty} x_k = y^\star$. But, we showed $\lim\limits_{k\longrightarrow\infty} x_k = x^\star$. Hence, we deduce $x^\star = y^\star$.

That completes the proof of Theorem 5. □

We can now compare majorizing sequences $\{t_n\}$ and $\{v_n\}$ $(n \geq 0)$:

Proposition 6. *Assume:*

$$b < \tfrac{a}{2}\ \eta + c, \tag{2.70}$$

hypotheses of Theorems 3 *and* 5 *hold.*

Then, the following estimates hold for all $n \geq 0$*:*

$$\| x_{n+2} - x_{n+1} \| \leq t_{n+2} - t_{n+1} < v_{n+2} - v_{n+1} \tag{2.71}$$

and

$$\| x_n - x^\star \| \leq t^\star - t_n \leq v^\star - v_n. \tag{2.72}$$

Proof. We only need to show using induction on the integer k:

$$t_{k+2} - t_{k+1} < v_{k+2} - v_{k+1}. \tag{2.73}$$

In view of (2.23), (2.52) and (2.70), we obtain for $n=0$:

$$t_2 < v_2 \quad \text{and} \quad t_2 - t_1 < v_2 - v_1.$$

Assuming:

$$t_{k+1} < v_{k+1} \tag{2.74}$$

and

$$t_{k+1} - t_k < v_{k+1} - v_k, \tag{2.75}$$

for $k \leq n+1$, we obtain from (2.17), (2.39), (2.74) and (2.75), that (2.73) holds for all $k \geq 0$.

Estimate (2.72) follows from (2.71) by using standard majorization techniques.

That completes the proof of Proposition 6. □

REMARK 7. If equality holds in (2.70), then $v_n = t_n$ $(n \geq 0)$, whereas if

$$b > \frac{a}{2}\,\eta + c, \tag{2.76}$$

then, the conclusions of Proposition 6 hold with sequence $\{t_n\}$, $t^\star$ switching places with $\{v_n\}$, $v^\star$, respectively in (2.71) and (2.72). □

REMARK 8. We can now compare our results with the ones obtained by Huang [9] and Gutiérrez [8].

Huang [9] used (2.42), (2.45),

$$\| F'(x_0)^{-1}\ (F''(x) - F''(y)) \| \leq \alpha\ \ \| x - y \| \tag{2.77}$$

for all $x, y \in \mathcal{D}$ and

$$3\ \alpha^2\ \eta + 3\ \alpha\ c + c^3 \leq (c^2 + 2\ \alpha)^{3/2} \tag{2.78}$$

and majorizing sequence $\{v_n\}$ to arrive at conclusions (2.64) and (2.65).

Gutiérrez [8] weakened Huang's conditions by using (2.42), (2.45), (2.46) (which is weaker than (2.77), since $a \leq \alpha$) and the condition:

$$3\ a^2\ \eta + 3\ a\ c + c^3 \leq (c^2 + 2\ a)^{3/2} \tag{2.79}$$

and majorizing sequence $\{v_n\}$ to also arrive at conclusions (2.64) and (2.65).

Hypotheses of Lemma 1 use information on a, b, c and η, Huang [9] uses information on α, c and η, whereas, Gutiérrez [8] uses a, c and η.

Therefore, a direct comparison between the sufficient convergence conditions is not possible. However, under (2.70), our majorizing sequence $\{t_n\}$ is finer that $\{v_n\}$ (See Example 2.9).

Note that in this study, we have simplified the sufficient convergence conditions provided by us in [1]. A favorable comparison of our approach with the corresponding one given by the Newton-Kantorovich theorem for solving nonlinear equations was also given in [1]. The same favorable comparison extends in this study.

The results obtained here can be extended for m $(m \geq 2)$ Fréchet differentiable operators [1], [2]. □

Comparison Table 1

n	$t_{n+1} - t_n$	$v_{n+1} - v_n$
0	0.21	0.21
1	0.0775551724	0.0807721314
2	0.0167871035	0.0213332937
3	0.0006952583	0.0018344379
4	0.0000025816	0.0000139339
5	0	0.0000000008
6	0	0

Comparison table 1 justifies the theoretical results of Proposition 6, since the majorizing sequence $\{t_n\}$ is tighter than $\{v_n\}$.

Example 9. Let $a = b = 2$, $c = 1.9$ and $\eta = .21$. Then, condition (2.79) (and (2.78)) is satisfied, since

$$3\ a^2\ \eta + 3\ a\ c + c^3 = 20.779 < 20.99311985 = (c^2 + 2\ a)^{3/2}$$

However, using (2.1)–(2.10), we get $\delta_1 = 2$, $\delta_2 = .259636075$, $\delta_3 = .220086356$, $\delta = 1.119654576$ and $\eta_0 = \delta_3 > \eta = .21$. Note also that (2.70) holds, since

$$b = 2 < 2.11 = \frac{a}{2}\eta + c.$$

We can compare the error estimates using (2.12) and (2.39). □

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