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SOME GENERAL KANTOROVICH TYPE OPERATORS

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Abstract. A general class of linear and positive operators of Kantorovich-type is constructed. The operators of this type which preserve exactly two test functions from the set $\{e_0, e_1, e_2\}$ are determined and their approximation properties and convergence theorems are studied.

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1. INTRODUCTION

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In 1930, L. V. Kantorovich in [3] introduced the linear and positive operators $K_m : L_1([0,1]) \to C([0,1])$, $m \in \mathbb{N}_0$, defined for any $f \in L_1([0,1])$, $x \in [0,1]$ and $m \in \mathbb{N}_0$ by

$$(1) \qquad (K_m f)(x) = (m+1) \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)\mathrm{d}t.$$

The operators from (1) are called Kantorovich operators. We remark that these operators preserve only the test function e_0.

The aim of this paper is to construct and study a general class of linear positive operators which preserve exactly two test functions from the set $\{e_0, e_1, e_2\}$.

In [4], J. P. King introduced and studied a Bernstein type operator, which preserves only the test functions e_0 and e_2. Therefore, we say that the operators constructed in this paper are operators of King's type.

In Section 2, we recall some results from [6], which we use for obtaining the main results of this paper.

In Section 3, respectively 4, we determine the unique operators from Section 2, which preserve exactly the test functions e_0 and e_1, respectively e_0 and e_2. In the Sections 4 and 5, we give approximation, convergence and Voronovskaja's type theorems for the operators obtained.

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2. PRELIMINARIES

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In this section we recall some results from [8], which we shall use in the present paper. Let I, J be real intervals with the property $I \cap J \neq \emptyset$. For any $m, k \in \mathbb{N}_0$, $m \neq 0$, we consider the functions $\varphi_{m,k} : J \to \mathbb{R}$, with the property that $\varphi_{m,k}(x) \geq 0$, for any $x \in J$ and the linear positive functionals $A_{m,k} : E(I) \to \mathbb{R}$.

For any $m \in \mathbb{N}$ we define the operator $L_m : E(I) \to F(J)$, by

$$(L_m f)(x) = \sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(f), \tag{2}$$

for any $x \in J$, where $E(I)$ and $F(J)$ are linear subspaces of real valued functions defined on I, resp. J, for which the sequence $(L_n)_{n\geq 0}$ defined above is convergent (in the topology of $F(J)$).

Remark 1. The operators $(L_m)_{m\in\mathbb{N}}$ are linear and positive on $E(I \cap J)$. □

For $m \in \mathbb{N}$ and $i \in \mathbb{N}_0$, we define $(T_{m,i})$ by

$$(T_{m,i} L_m)(x) = m^i (L_m \psi_x^i)(x) = m^i \sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(\psi_x^i) \tag{3}$$

for any $x \in I \cap J$, $\psi_x^i(t) = (t - x)^i$, $t \in I$.

In that follows $s \in \mathbb{N}_0$ is even and we assume that the next two conditions:

- there exist the smallest $\alpha_s, \alpha_{s+2} \in [0, +\infty)$, so that

$$\lim_{m\to\infty} \tfrac{(T_{m,j} L_m)(x)}{m^{\alpha_j}} = B_j(x) \in \mathbb{R} \tag{4}$$

for any $x \in I \cap J$ and $j \in \{s, s+2\}$

$$\alpha_{s+2} < \alpha_s + 2. \tag{5}$$

- $I \cap J$ is an interval.

Theorem 2. (see [8]) *Let $f \in E(I)$ be a function. If $x \in I \cap J$ and f is s times differentiable in a neighborhood of x, $f^{(s)}$ is continuous in x, then*

$$\lim_{m\to\infty} m^{s-\alpha_s} \left((L_m f)(x) - \sum_{i=0}^{s} \tfrac{f^{(i)}(x)}{m^i i!} (T_{m,j} L_m)(x) \right) = 0. \tag{6}$$

Assume that f is s times differentiable on I, $f^{(s)}$ is continuous on I and there exists a compact interval $K \subset I \cap J$, such that there exists $m(s) \in \mathbb{N}$ and constant $k_j \in \mathbb{R}$ depending on K, so for $m \geq m(s)$ and $x \in K$ the following inequalities

$$\tfrac{(T_{m,j} L_m)(x)}{m^{\alpha_j}} \leq k_j, \tag{7}$$

hold for $j \in \{s, s+2\}$.
Then the convergence expressed by (6) *is uniform on* K *and*

$$m^{s-\alpha_s}\left|(L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!}\,(T_{m,i}L_m)(x)\right| \tag{8}$$
$$\leq \frac{1}{s!}\,(k_s + k_{s+2})\omega\left(f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\right),$$

for any $x \in K$, $m \geq m(s)$, *where* $\omega(f;\cdot)$ *denotes the modulus of continuity of the function* f.

Corollary 3. *Let* $f : I \to \mathbb{R}$ *be a* s *times differentiable function on* $I \cap J$ *with* $f^{(s)}$ *continuous on* $I \cap J$. *Then*

$$\lim_{m\to\infty} (L_m f)(x) = B_0(x)f(x) \tag{9}$$

if $s = 0$ *and* $\alpha_0 = 0$, *where* B_0 *is defined by* (4). *If* $s \geq 2$, *then*

$$\lim_{m\to\infty} m^{s-\alpha_s}\left[(L_m f)(x) - \sum_{i=0}^{s-1} \frac{1}{m^i i!}\,(T_{m,i}L_m)(x)f^{(i)}(x)\right] = \frac{1}{s!}\,B_s(x)f^{(s)}(x), \tag{10}$$

where B_s *are defined by* (4).

If f *is a* s *times differentiable function on* $I \cap J$, *with* $f^{(s)}$ *continuous and bounded on* $I \cap J$ *and* (7) *takes place for an interval* $K \subset I \cap J$, *then the convergence in* (9) *and* (10) *are uniform on* K.

3. THE CONSTRUCTION OF A GENERAL LINEAR AND POSITIVE OPERATORS

Let $J \subset R$ be an interval, $m_0 \in \mathbb{N}_0$, $m_0 \geq 2$ given, $\mathbb{N}_1 = \{m \in \mathbb{N} | m \geq m_0\}$, the function $\alpha_m, \beta_m : J \to \mathbb{R}$, $\alpha_m(x) \geq 0$, $\beta_m(x) \geq 0$ for any $x \in J$ and $m \in \mathbb{N}_1$.

Definition 4. For $m \in \mathbb{N}_1$, we define the operator of the following form

$$(K_m^* f)(x) = (m+1)\sum_{k=0}^{m} \binom{m}{k}\alpha_m^k(x)\beta_m^{m-k}(x)\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)\mathrm{d}t \tag{11}$$

for any $f \in L_1([0,1])$ and $x \in J$.

Lemma 5. *The following identities*

$$(K_m^* e_0)(x) = (\alpha_m(x) + \beta_m(x))^m, \tag{12}$$

$$(K_m^* e_1)(x) = \frac{(\alpha_m(x)+\beta_m(x))^{m-1}}{2(m+1)}\,((2m+1)\alpha_m(x) + \beta_m(x)), \tag{13}$$

and

$$(K_m^* e_2)(x) = \frac{(\alpha_m(x)+\beta_m(x))^{m-2}}{3(m+1)^2}\big(3m(m-1)\alpha_m^2(x) \tag{14}$$
$$+ 6m\alpha_m(x)(\alpha_m(x) + \beta_m(x)) + (\alpha_m(x) + \beta_m(x))^2\big),$$

hold, for any $x \in J$ *and any* $m \in \mathbb{N}_1$.

Proof. For $m \in \mathbb{N}_1$ and $k \in \{0, 1, \ldots, m\}$ we have

$$\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} e_0(t)\mathrm{d}t = \tfrac{1}{m+1}\,, \quad \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} e_1(t)\mathrm{d}t = \tfrac{2k+1}{(m+1)^2}$$

$$\text{and} \quad \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} e_2(t)\mathrm{d}t = \tfrac{3k^2+3k+1}{(m+1)^3}\,.$$

Then

$$\begin{aligned}(K_m^* e_0)(x) &= (m+1)\sum_{k=0}^{m} \tbinom{m}{k} \alpha_m^k(x)\beta_m^{m-k}(x) \int_{\frac{k}{k+1}}^{\frac{k+1}{m+1}} e_0(t)\mathrm{d}t \\ &= \sum_{k=0}^{m} \tbinom{m}{k} \alpha_m^k(x)\beta_m^{m-k}(x),\end{aligned}$$

so (12) holds;

$$\begin{aligned}(K_m^* e_1)(x) &= (m+1)\sum_{k=0}^{m} \tbinom{m}{k} \alpha_m^k(x)\beta_m^{m-k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} e_1(t)\mathrm{d}t \\ &= \tfrac{1}{2(m+1)}\Bigg(2m\alpha_m(x)\sum_{k=1}^{m} \tbinom{m-1}{k-1} \alpha_m^{k-1}(x)\beta_m^{m-k}(x) \\ &\qquad + \sum_{k=0}^{m} \tbinom{m}{k} \alpha_m^k(x)\beta_m^{m-1}(x)\Bigg),\end{aligned}$$

so (13) holds and

$$\begin{aligned}(K_m^* e_2)(x) &= (m+1)\sum_{k=0}^{m} \tbinom{m}{k} \alpha_m^k(x)\beta_m^{m-k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} e_2(t)\mathrm{d}t \\ = \tfrac{1}{3(m+1)^2}&\Bigg(3m(m+1)\alpha_m^2(x)\sum_{k=2}^{m} \tbinom{m-2}{k-2} \alpha_m^{k-2}(x)\beta_m^{m-k}(x) \\ +6m\alpha_m(x)&\sum_{k=1}^{m} \tbinom{m-1}{k-1} \alpha_m^{k-1}(x)\beta_m^{m-k}(x) + \sum_{k=0}^{m} \tbinom{m}{k} \alpha_m^k(x)\beta_m^{m-k}(x)\Bigg),\end{aligned}$$

from where (14) follows. □

REMARK 6. In the following, we will use Theorem 2, where $I = [0, 1]$,

$$\varphi_{m,k}(x) = (m+1)\tbinom{m}{k}\alpha_m^k(x)\beta_m^{m-k}(x) \tag{15}$$

and

$$A_{m,k}(x) = \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)\mathrm{d}t \tag{16}$$

for any $x \in J$, $f \in L_1([0,1])$, $m \in \mathbb{N}_1$ and $k \in \{0,1,\dots,m\}$. □

4. KANTOROVICH-TYPE OPERATORS PRESERVING THE TEST FUNCTIONS e_0 AND e_1

In this case, we impose the conditions $(K_m^* e_0)(x) = e_0(x)$ and $(K_m^* e_1)(x) = e_1(x)$, for any $x \in J$ and $m \in \mathbb{N}_1$. From the conditions above, taking (12) and (13) into account, we have

$$(\alpha_m(x) + \beta(x))^m = 1 \text{ and}$$
$$\frac{(\alpha_m(x)+\beta_m(x))^{m-1}}{2(m+1)} \left((2m+1)\alpha_m(x) + \beta_m(x)\right) = x,$$

from where

$$\alpha_m(x) = \frac{2(m+1)x-1}{2m}, \tag{17}$$

$$\beta_m(x) = \frac{2m+1-2(m+1)x}{2m}, \tag{18}$$

for any $x \in [0,1]$ and $m \in \mathbb{N}_1$.
From $\alpha_m(x) \ge 0$ and $\beta_m(x) \ge 0$, for any $m \in \mathbb{N}_1$, we have

$$\frac{1}{2(m+1)} \le x \le \frac{2m+1}{2(m+1)}. \tag{19}$$

Lemma 7. *The following*

$$\left[\frac{1}{2(m_0+1)}; \frac{2m_0+1}{2(m_0+1)}\right] \subset \left[\frac{1}{2(m+1)}; \frac{2m+1}{2(m+1)}\right] \subset [0,1] \tag{20}$$

hold for any $m \in \mathbb{N}_1$.

Proof. Because the function $\frac{1}{2(m+1)}$ is decreasing and the function $\frac{2m+1}{2(m+1)}$ is increasing, relation (20) follows. □

Taking the remarks above, we construct the sequence of operators $(K_{1,m}^*)_{m \ge m_0}$.

Definition 8. If $m \in \mathbb{N}_1$, we define the operator

$$(K_{1,m}^* f)(x) \tag{21}$$
$$= \frac{m+1}{(2m)^m} \sum_{k=0}^{m} \binom{m}{k} (2(m+1)x-1)^k (2m+1-2(m+1)x)^{m-k} \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)\mathrm{d}t$$

for any $f \in L_1([0,1])$ and any $x \in \left[\frac{1}{2(m_0+1)}; \frac{2m_0+1}{2(m_0+1)}\right]$.

Remark 9. In this case, we note $J = \left[\frac{1}{2(m_0+1)}; \frac{2m_0+1}{2(m_0+1)}\right] = I_{(m_0)}^{(1)}$. □

Lemma 10. *We have*

$$(K_{1,m}^* e_0)(x) = 1, \tag{22}$$

$$(K_{1,m}^* e_1)(x) = x \tag{23}$$

and

$$(K^*_{1,m}e_2)(x) = \tfrac{m-1}{m}\, x^2 + \tfrac{1}{m}\, x - \tfrac{5m+3}{12m(m+1)^2} \tag{24}$$

for any $x \in I^{(1)}_{(m_0)}$ *and* $m \in \mathbb{N}_1$*.*

Proof. Results immediately from the condition above and (14). □

Lemma 11. *The following identities*

$$(T_{m,0}K^*_{1,m})(x) = 1, \tag{25}$$

$$(T_{m,1}K^*_{1,m})(x) = 0 \tag{26}$$

and

$$(T_{m,2}K^*_{1,m})(x) = mx(1-x) - \tfrac{m(5m+3)}{12(m+1)^2} \tag{27}$$

hold, for any $x \in I^{(1)}_{(m_0)}$ *and* $m \in \mathbb{N}_1$

Proof. By using Lemma 10 and relation (3), we have

$(T_{m,0}K^*_{1,m})(x) = (K^*_{1,m}e_0)(x) = 1,$
$(T_{m,1}K^*_{1,m})(x) = m(K^*_{1,m}\psi_x)(x) = m((K^*_{1,m}e_1)(x) - x(K^*_{1,m}e_0)(x)) = 0$

and

$$(T_{m,2}K^*_{1,m}(x) = m^2(K^*_{1,m}\psi^2_x)(x) = m^2\big((K^*_{1,m}e_1)(x) + x^2(K^*_{1,m}e_0)(x)\big),$$

from where (27) follows. □

Lemma 12. *We have that*

$$\lim_{m\to\infty}(T_{m,0}K^*_{1,m})(x) = 1, \tag{28}$$

$$\lim_{m\to\infty} \tfrac{(T_{m,2}K^*_{1,m})(x)}{m} = x(1-x) \tag{29}$$

for any $x \in I^{(1)}_{(m_0)}$ *and* $m(0) \in \mathbb{N}$ *exists such that*

$$\tfrac{(T_{m,2}K^*_{1,m})(x)}{m} \leq \tfrac{5}{4} \tag{30}$$

for any $x \in I^{(1)}_{(m_0)}$ *and* $m \in \mathbb{N}_1$*,* $m \geq m(0)$*.*

Proof. The relation (28) and (29) results taking (25) and (28) into account. By using the definition of limit a function and because $x(1-x) \leq \frac{1}{4}$ for any $x \in [0,1]$, from (29) the inequality (30) is obtained. □

Theorem 13. *Let* $f : [0,1] \to \mathbb{R}$ *be a continuous function on* $[0,1]$*. Then*

$$\lim_{m\to\infty} K^*_{1,m}f = f \tag{31}$$

uniformly on $I^{(1)}_{(m_0)}$ *and there exists* $m(0) \in \mathbb{N}_1$ *such that*

$$\left|(K^*_{1,m}f)(x) - f(x)\right| \leq \tfrac{9}{4}\,\omega\left(f; \tfrac{1}{\sqrt{m}}\right) \tag{32}$$

for any $x \in I^{(1)}_{(m_0)}$ *and* $m \in \mathbb{N}_1$, $m \geq m_0$.

Proof. We apply Theorem 2 and Corollary 3 for $s = 0$, $\alpha_0 = 0$, $\alpha_2 = 1$, $k_0 = 1$ and $k_2 = \frac{9}{4}$. □

THEOREM 14. *If* $f \in C([0,1])$, $x \in I^{(1)}_{(m_0)}$, f *is two times differentiable in neighborhood of* x *and* $f^{(2)}$ *is continuous on* x, *then*

$$\lim_{m\to\infty} m\left((K^*_{1,m}f)(x) - f(x)\right) = \tfrac{1}{2}\,x(1-x)f^{(2)}(x). \tag{33}$$

Proof. We use the results from Corollary 3 for $s = 2$. □

5. KANTOROVICH-TYPE OPERATORS PRESERVING THE TEST FUNCTIONS e_0 AND e_2

In this section, we impose the conditions $(K^*_m e_0)(x) = e_0(x)$ and $(K^*_m e_2)(x) = e_2(x)$, for any $x \in J$ and $m \in \mathbb{N}_1$. Then, taking (12) and (14) into account, we have $(\alpha_m(x) + \beta_m(x))^m = 1$ and

$$\tfrac{(\alpha_m(x)+\beta_m(x))^{m-2}}{3(m+1)^2}\left(3m(m-1)\alpha^2_m(x) + 6m\alpha_m(x)(\alpha_m(x)+\beta_m(x)) + (\alpha_m(x)+\beta_m(x))^2\right) = x^2,$$

from where

$$\alpha_m(x) + \beta_m(x) = 1 \tag{34}$$

and

$$3m(m-1)\alpha^2_m(x) + 6m\alpha_m(x) + 1 - 3(m+1)^2x^2 = 0. \tag{35}$$

The discriminant of the equation (35) is

$$\Delta_m = 12m(2m + 1 + 3(m-1)(m+1)^2x^2) \geq 0,$$

for any $x \in J$ and any $m \in \mathbb{N}_1$, $m \geq 2$ and we note

$$\delta_m(x) = 3m(2m + 1 + 3(m-1)(m+1)^2x^2), \tag{36}$$

$x \in J$, $m \in \mathbb{N}_1$, $m \geq 2$.

If $m \in \mathbb{N}_1$, $m \geq 2$, then for

$$x \geq \tfrac{1}{(m+1)\sqrt{3}} \tag{37}$$

the inequality $\frac{1-3(m+1)^2x^2}{3m(m-1)} \leq 0$ is true, so the equation from (35) has exactly one positive solution. This is

$$\alpha_m(x) = \tfrac{-3m+\sqrt{\delta_m(x)}}{3m(m-1)} \tag{38}$$

and then

$$\beta_m(x) = \tfrac{3m^2-\sqrt{\delta_m(x)}}{3m(m-1)} \tag{39}$$

where $x \in J$, and to satisfy (37), $m \in \mathbb{N}_1$, $m \geq 2$.

LEMMA 15. *Let $m \in \mathbb{N}_1$, $m \geq 2$. Then $\beta_m(x) \geq 0$, $x \geq 0$ if and only if*

$$0 \leq x \leq \tfrac{\sqrt{3m^2+3m+1}}{(m+1)\sqrt{3}}\,. \tag{40}$$

Proof. From $\beta_m(x) \geq 0$ we have $3m^2 \geq \sqrt{\delta_m(x)}$, equivalent after calculus to $x^2 \leq \frac{3m^2+3m+1}{3(m+1)^2}$, from where (40) follows. □

LEMMA 16. *Let $m \in \mathbb{N}_1$, $m \geq 2$. If $x \in \left[\frac{1}{(m+1)\sqrt{3}}\,;\frac{\sqrt{3m^2+3m+1}}{(m+1)\sqrt{3}}\right]$, then $\alpha_m(x) \geq 0$ and $\beta_m(x) \geq 0$.*

Proof. Results immediately from (37) and (40). □

LEMMA 17. *The following*

$$\left[\tfrac{1}{(m_0+1)\sqrt{3}}\,;\tfrac{\sqrt{3m_0^2+3m_0+1}}{(m_0+1)\sqrt{3}}\right] \subset \left[\tfrac{1}{(m+1)\sqrt{3}}\,;\tfrac{\sqrt{3m^2+3m+1}}{(m+1)\sqrt{3}}\right] \subset [0,1] \tag{41}$$

hold, for any $m \in \mathbb{N}_1$.

Proof. By using that the functions $\frac{1}{(m+1)\sqrt{3}}$ and $\frac{\sqrt{3m^2+3m+1}}{(m+1)\sqrt{3}}$ are decreasing, relations from (41) follows. □

DEFINITION 18. If $m \in \mathbb{N}_1$, we define the operator $K^*_{2,m}$ by

$$(K^*_{2,m}f)(x) = \tfrac{m+1}{3m(m-1))^m} \sum_{k=0}^{m} \tbinom{m}{k} \left(-3m + \sqrt{\delta_m(x)}\right)^k \times \left(3m^2 - \sqrt{\delta_m(x)}\right)^{m-k} \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)dt \tag{42}$$

for any $f \in L_1([0,1])$ and any $x \in \left[\frac{1}{(m_0+1)\sqrt{3}}\,;\frac{\sqrt{3m_0^2+3m_0+1}}{(m_0+1)\sqrt{3}}\right]$.

REMARK 19. In this section, we note

$$J = \left[\tfrac{1}{(m_0+1)\sqrt{3}}\,;\tfrac{\sqrt{3m_0^2+3m_0+1}}{(m_0+1)\sqrt{3}}\right] = I^{(2)}_{(m_0)}. \quad \square$$

LEMMA 20. *We have*

$$(K^*_{2,m}e_0)(x) = 1, \tag{43}$$

$$(K^*_{2,m}e_1)(x) = \tfrac{2\sqrt{\delta_m(x)}-3m-3}{6(m-1)(m+1)} \tag{44}$$

and

$$(K^*_{1,m}e_2)(x) = x^2 \tag{45}$$

for any $x \in I^{(2)}_{(m_0)}$ *and* $m \in \mathbb{N}_1$.

Proof. It is inferred from the conditions above and (13). $\square$

LEMMA 21. *The following identities*

$$(T_{m,0}K^*_{1,m})(x) = 1, \tag{46}$$

$$(T_{m,1}K^*_{2,m})(x) = m\left(\tfrac{2\sqrt{\delta_m(x)}-3m-3}{6(m-1)(m+1)} - x\right) \tag{47}$$

and

$$(T_{m,2}K^*_{2,m})(x) = 2m^2x\left(x - \tfrac{2\sqrt{\delta_m(x)}-3m-3}{6(m-1)(m+1)}\right). \tag{48}$$

hold, for any $x \in I^{(2)}_{(m_0)}$ *and* $m \in \mathbb{N}_1$

Proof. By using Lemma 12 and (3), we have that

$$(T_{m,0}K^*_{2,m})(x) = (K^*_{2,m}e_0)(x) = 1,$$
$$(T_{m,1}K^*_{2,m})(x) = m(K^*_{2,m}\psi_x)(x) = m\left((K^*_{2,m}e_1)(x) - x(K^*_{2,m}e_0)(x)\right),$$

so (47) holds and

$$\begin{aligned}(T_{m,2}K^*_{2,m})(x) &= m^2(K^*_{2,m}\psi_x^2)(x)\\ &= m^2\left((K^*_{2,m}e_2)(x) - 2x(K^*_{2,m}e_1)(x) + x^2(K_{2,m}e_0)(x)\right),\end{aligned}$$

from where (48) is obtained. $\square$

LEMMA 22. *The following identity*

$$\lim_{m\to\infty} m\left(\tfrac{2\sqrt{\delta_m(x)}-3m-3}{6(m-1)(m+1)} - x\right) = \tfrac{x-1}{2} \tag{49}$$

holds for any $x \in I^{(2)}_{(m_0)}$.

Proof. We have

$$\begin{aligned}\lim_{m\to\infty}&\left(\tfrac{m^2}{(m-1)(m+1)}\cdot\tfrac{\sqrt{\delta_m(x)}-3(m-1)(m+1)x}{3m} - \tfrac{m}{2(m-1)}\right)\\ &= -\tfrac{1}{2} + \lim_{m\to\infty}\tfrac{\sqrt{\delta_m(x)}-3(m-1)(m+1)x}{3m}\\ &= -\tfrac{1}{2} + \lim_{m\to\infty}\tfrac{\delta_m(x)-9(m-1)^2(m+1)^2x^2}{3m\left(\sqrt{\delta_m(x)}+3(m-1)(m+1)x\right)}\end{aligned}$$

and after a few calculations, identity (49) follows. $\square$

Lemma 23. *We have that*

$$\lim_{m\to\infty}(T_{m,0}K^*_{2,m})(x) = 1, \tag{50}$$

$$\lim_{m\to\infty}\frac{(T_{m,2}K^*_{2,m})(x)}{m} = x(1-x) \tag{51}$$

for any $x \in I^{(2)}_{(m_0)}$ *and* $m(0) \in \mathbb{N}$ *exists such that*

$$\frac{(T_{m,2}K^*_{2,m})(x)}{m} \leq \frac{5}{4} \tag{52}$$

for any $x \in I^{(2)}_{(m_0)}$ *and* $m \in \mathbb{N}_1$, $m \geq m(0)$.

Proof. The relations (50) and (51) imply (46), (48) and (49). By using the definition of the limit of a function and because $x(1-x) \leq \frac{1}{4}$ for any $x \in [0,1]$, from (51) the relation (52) is obtained. □

Theorem 24. *Let* $f : [0,1] \to \mathbb{R}$ *be a continuous function on* $[0,1]$. *Then*

$$\lim_{m\to\infty} K^*_{2,m}f = f \tag{53}$$

uniformly on $x \in I^{(2)}_{(m_0)}$ *and there exists* $m(0) \in \mathbb{N}$ *such that*

$$\left|(K^*_{2,m}f)(x) - f(x)\right| \leq \frac{9}{4}\,\omega\left(f;\frac{1}{\sqrt{m}}\right) \tag{54}$$

for any $x \in I^{(2)}_{(m_0)}$ *and any* $m \in \mathbb{N}_1$, $m \geq m(0)$.

Proof. Theorem 24 is a results from Theorem 2 and Corollary 3 for $s = 0$, $\alpha_0 = 0$, $\alpha_2 = 1$, $k_0 = 1$ and $k_2 = \frac{5}{4}$. □

Theorem 25. *If* $f \in C([0,1])$, $x \in I^{(2)}_{(m_0)}$, f *is two times differentiable in a neighborhood of* x, $f^{(2)}$ *is continuous in* x, *then*

$$\lim_{m\to\infty} m\left((K^*_{2,m}f)(x) - f(x)\right) = \frac{x-1}{2}\,f^{(1)}(x) + \frac{x(1-x)}{2}\,f^{(2)}(x). \tag{55}$$

Proof. Taking Lemma 22 into account and applying Theorem 2 for $s = 2$, we obtain the relation (55). □

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